J. Korean Math. Soc. **59** (2022), No. 1, pp. 129–150
https://doi.org/10.4134/JKMS.j210224
pISSN: 0304-9914 / eISSN: 2234-3008

REGULARITY OF THE GENERALIZED POISSON OPERATOR

Pengtao Li, Zhiyong Wang, and Kai Zhao

Abstract. Let $L = -\Delta + V$ be a Schrödinger operator, where the potential V belongs to the reverse Hölder class. In this paper, by the subordinative formula, we investigate the generalized Poisson operator $P^L_{t,\sigma}$, $0 < \sigma < 1$, associated with L. We estimate the gradient and the time-fractional derivatives of the kernel of $P^L_{t,\sigma}$, respectively. As an application, we establish a Carleson measure characterization of the Campanato type space $\mathcal{C}^{\gamma}_L(\mathbb{R}^n)$ via $P^L_{t,\sigma}$.

1. Introduction

The fractional powers of the Laplace operator $(-\Delta)^{\alpha}, 0 < \alpha < 1$, which is defined via the Fourier transform as

$$\widehat{(-\Delta)^{\alpha}}u(t,\xi) = |\xi|^{2\beta}\widehat{u}(t,\xi),$$

play a significant roles in many areas of mathematics, such as, harmonic analysis and PDEs. Due to the salient significance and backgrounds in mathematical physics, the fractional Laplacian $(-\Delta)^{\alpha}$ has also been applied to study a wide class of physical systems and engineering problems, including Lévy flights, stochastic interfaces and anomalous diffusion problems. It is well known that the fractional Laplacian is a nonlocal operator and local PDE techniques can not be applied to deal with nonlinear problems for $(-\Delta)^{\sigma}$. To overcome this difficulty, Caffarelli and Silvestre showed in [2] that if $u(x,y) : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is a solution to the boundary value problem:

$$\begin{cases} \Delta_x u + \frac{1-2\sigma}{y}u_y + u_{yy} = 0, & (x,y) \in \mathbb{R}^n \times (0,\infty); \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

Received April 3, 2021; Accepted September 24, 2021.
2010 *Mathematics Subject Classification.* Primary 42B35, 35J10, 42B38.
Key words and phrases. Regularities, Schrödinger operators, the generalized Poisson operators, Campanato type spaces.
This work was financially supported by the National Natural Science Foundation of China (No. 12071272) and Shandong Natural Science Foundation of China (Nos. ZR2020MA004, ZR2017JL008).

then up to a multiplicative constant depending only on σ,

$$-\lim_{y\to 0+} y^{1-2\sigma}u_y(x,y) = (-\Delta)^\sigma f(x).$$

This characterization of $(-\Delta)^\sigma f$ via the above local (degenerate) PDE was used for the first time in [1] to get regularity estimates for the obstacle problem for the fractional Laplacian. Caffarelli and Silvestre noted that the above equation can be thought as the harmonic extension of f in $2-2\sigma$ dimensions more (see [1]). From there, they established the fundamental solution and a Poisson formula for u via a conjugate equation. Furthermore, taking advantage of the general theory of degenerate elliptic equations developed by Fabes et al. in 1982-1983, Harnack's estimates for u and for f were proved, respectively.

Recently, the results of [1, 2] were further generalized to other differential operators. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, $n \geq 1$ and $\sigma \in (0,1)$. In [18], P. Stinga and J. Torrea investigated the following extension problem:

$$\begin{cases} u(x,0) = f(x), \ x \in \mathcal{O}; \\ -L_x u + \dfrac{1-2\sigma}{y} u_y + u_{yy} = 0, \ (x,y) \in \mathcal{O}\times(0,\infty). \end{cases} \tag{1}$$

The authors proved that any fractional power L^σ can be described as an operator that maps a Dirichlet condition to a Neumann-type condition via an extension problem as in [2]. Also, if u is a solution to (1), then u can be represented via the following Poisson formula:

$$u(x,y) = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-r}e^{-\frac{y^2}{4r}L}(f)(x)\frac{dr}{r^{1-\sigma}} := P^L_{t,\sigma}(f). \tag{2}$$

Specially, for $\sigma = 1/2$, (2) is exactly the Poisson semigroup related with L, i.e., $u(x,y) = e^{-y\sqrt{L}}(f)(x)$. Based on this understanding, we call $P^L_{t,\sigma}$ the generalized Poisson operator related with L.

In this paper, our purpose is to investigate the regularities of generalized Poisson operators related with Schrödinger operators which is defined as follows.

$$L(f)(x) := (-\Delta f)(x) + V(x)f(x),$$

where $-\Delta$ is the Laplace operator: $-\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $V(x)$ is a nonnegative potential belonging to the reverse Hölder class B_q, i.e., for every ball B,

$$\Big(\frac{1}{|B|}\int_B V^q(x)dx\Big)^{1/q} \leq \frac{C}{|B|}\int_B V(x)dx.$$

In recent years, due to the background of the quantum mechanics, in the fields of partial differential equations and mathematical physics, more and more scholars are interested in the study of nonlinear problems involving fractional powers of the Schrödinger operator L^σ, $0 < \sigma < 1$, we refer the reader to [5, 9].

In order to estimate the kernel of $P^L_{t,\sigma}$, we notice that the integral kernel of the Poisson semigroup associated with L can be expressed as

$$p^L_t(x,y)=\int_0^\infty \frac{e^{-u}}{\sqrt{u}}K^L_{t^2/4u}(x,y)du,$$

where $K^L_t(\cdot,\cdot)$ denotes the integral kernel of e^{-tL}, i.e.,

$$e^{-tL}(f)(x):=\int_{\mathbb{R}^n}K^L_t(x,y)f(y)dy.$$

In this manner, it can be deduced from (2) that the kernel $P^L_{t,\sigma}$, denoted by $p^L_{t,\sigma}(\cdot,\cdot)$, can be represented via the subordinate formula:

$$p^L_{t,\sigma}(x,y)=\frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-r}K^L_{t^2/4r}(x,y)\frac{dr}{r^{1-\sigma}}, \tag{3}$$

which enables us to deduce the regularity properties of $p^L_{t,\sigma}(\cdot,\cdot)$ via the heat kernel $K^L_t(\cdot,\cdot)$, see Proposition 3.3. Let ∇ denote the gradient operator on $\mathbb{R}^n$, that is, $\nabla=(\nabla_x,\partial_t)$, where $\nabla_x=(\partial/\partial x_1,\dots,\partial/\partial x_n)$. By the aid of the results obtained in [5], we use the subordinative formula (3) and a direct calculation to obtain the regularity of $\nabla_x p^L_{t,\sigma}(\cdot,\cdot)$, see Proposition 3.6.

In Section 3.2, we investigate the time-fractional derivatives of $p^L_{t,\sigma}(\cdot,\cdot)$, since time-fractional operators are proved to be very useful for modeling purpose. Let $\mathbb{Z}_+$ denote the set of all positive integers. For $\alpha>0$ and $k\in\mathbb{Z}_+$, define

$$D^{L,\sigma}_{t,\alpha}(x,y):=t^\alpha\partial^\alpha_t p^L_{t,\sigma}(x,y)=\frac{e^{-i\pi(k-\alpha)}}{\Gamma(k-\alpha)}\int_0^\infty \partial^k_t p^L_{t+u,\sigma}(x,y)u^{k-\alpha}\frac{du}{u},\ \alpha>0.$$

In order to estimate $D^{L,\sigma}_{t,\alpha}(\cdot,\cdot)$, in Proposition 3.4, we first investigate the regularities of $t^k\partial^k_t p^L_{t,\sigma}(\cdot,\cdot)$, $k\in\mathbb{Z}_+$. The regularity estimates of $D^{L,\sigma}_{t,\alpha}(\cdot,\cdot)$ can be deduced from Proposition 3.4 and the functional calculus, see Proposition 3.7.

As an application, in Section 4, we use the generalized Poisson semigroup $\{P^L_{t,\sigma}\}_{t>0}$ to characterize the Campanato type spaces associated to L denoted by $\mathcal{C}^\gamma_L(\mathbb{R}^n)$. In the last decades, the study of function spaces associated with Schrödinger operators has inspired great interest, for example, square functions characterizations for Hardy spaces [8], Carleson measure characterizations for BMO spaces [7] and Morry-Campanato spaces [5]. For further information on this topic, we refer to [3,4,6,9,11,12,14,17,19–21] and the references therein. By the regularity estimates obtained in Section 3, we establish the following equivalent characterizations: for $0<\gamma<\min\{2\sigma,2\alpha\sigma\}$,

$$\begin{aligned} f\in\mathcal{C}^\gamma_L(\mathbb{R}^n) &\sim \sup_B \frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t^\alpha\partial^\alpha_t P^L_{t,\sigma}(f)(x)|^2\frac{dxdt}{t}<\infty\\ &\sim \sup_B \frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t\nabla P^L_{t,\sigma}(f)(x)|^2\frac{dxdt}{t}<\infty, \end{aligned}$$

see Theorems 4.6 and 4.7, respectively.

Remark 1.1. We point out that our regularity results obtained in Section 3 are new and generalize the former results obtained by several authors. Specially, if $\sigma = 1/2$, the estimation of $t^\alpha \partial_t^\alpha e^{-t\sqrt{L}}(\cdot,\cdot)$ is consistent with the results of Ma-Stinga-Torrea-Zhang [15, Proposition 3.6], and the estimation of $t\nabla_x e^{-t\sqrt{L}}(\cdot,\cdot)$ comes back to [5, Lemma 3.9].

Notations. In the sequel, we always assume that $\delta_0 = 2 - n/q$ and $\delta = \min\{2\alpha\sigma, 2\sigma, \delta_0\}$. We will use c and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $C^{-1} \leq B_1/B_2 \leq C$.

2. Preliminaries

Let the auxiliary function $\rho(x, V) = \rho(x)$ (cf. [16]) is defined as

$$\rho(x) := \sup\Big\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,y)} V(y)dy \leq 1\Big\}.$$

Lemma 2.1 ([16, Lemma 1.2]).

(i) *For* $0 < r < R < \infty$,

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C\Big(\frac{r}{R}\Big)^\delta \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

(ii) *If* $r = \rho(x)$, *then* $r^{2-n} \int_{B(x,r)} V(y)dy = 1$. *Moreover,* $r \sim \rho(x)$ *if and only if* $r^{2-n} \int_{B(x,r)} V(y)dy \sim 1$.

Lemma 2.2 ([16, Lemma 1.4]).

(i) *There exist* $C > 0$ *and* $k_0 \geq 1$ *such that for all* $x, y \in \mathbb{R}^n$,

$$C^{-1}\rho(x)\Big(1 + |x - y|/\rho(x)\Big)^{-k_0} \leq \rho(y) \leq C\rho(x)\Big(1 + |x - y|/\rho(x)\Big)^{k_0/(1+k_0)}.$$

In particular, $\rho(y) \sim \rho(x)$ *if* $|x - y| < C\rho(x)$.

(ii) *There exists* $l_0 > 1$ *such that*

$$\int_{B(x,R)} \frac{V(y)}{|x - y|^{n-2}} dy \leq \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)dy \leq C\Big(1 + \frac{R}{\rho(x)}\Big)^{l_0}.$$

Let f be a locally integrable function on $\mathbb{R}^n$ and $B = B(x, r)$ be a ball. Denote by f_B the mean of f on B, i.e., $f_B = |B|^{-1} \int_B f(y)dy$. Let

$$f(B, V) := \begin{cases} f_B, & r < \rho(x); \\ 0, & r \geq \rho(x). \end{cases}$$

Definition. Let $0 < \gamma \leq 1$. The Campanato type space $\mathcal{C}_L^\gamma(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f satisfying

$$\|f\|_{\mathcal{C}_L^\gamma} := \sup_{B \subset \mathbb{R}^n} \Big\{\frac{1}{|B|^{1+\gamma/n}} \int_B |f(x) - f(B, V)|dx\Big\} < \infty.$$

Proposition 2.3 ([15, Proposition 4.3]). *Let $B = B(x,r)$ with $r < \rho(x)$. If $f \in \mathcal{C}_L^\gamma(\mathbb{R}^n)$, $0 < \gamma \leq 1$, then there exists a constant C such that $|f_B| \leq C(\rho(x))^\gamma \|f\|_{\mathcal{C}_L^\gamma}$.*

The space $\mathcal{C}_L^\gamma(\mathbb{R}^n)$ is equivalent to the following Lipschitz type space related to L.

Definition. For $0 < \gamma \leq 1$, a continuous function f defined on $\mathbb{R}^n$ belongs to the space $\mathcal{C}_L^{0,\gamma}(\mathbb{R}^n)$ if

$$\sup_{x,y\in\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \frac{|f(x)|}{\rho(x)^\gamma} < \infty.$$

Proposition 2.4 ([15, Proposition 4.6]). *If $0 < \gamma \leq 1$, then the space $\mathcal{C}_L^\gamma(\mathbb{R}^n)$ equals to $\mathcal{C}_L^{0,\gamma}(\mathbb{R}^n)$ and their norms are equivalent.*

The Hardy type spaces $H_L^p(\mathbb{R}^n)$, $0 < p \leq 1$, are defined by the maximal function generated by the semigroup $\{e^{-tL}\}_{t>0}$. Let $\mathcal{M}_L$ denote the semigroup maximal function, i.e.,

$$\mathcal{M}_L(f)(x) := \sup_{t>0} |T_t^L f(x)|, \ x \in \mathbb{R}^n.$$

An integrable function f is an element of the Hardy type space $H_L^p(\mathbb{R}^n)$ if the maximal function $M_L(f)(x)$ belongs to $L^p(\mathbb{R}^n)$. The quasi-norm in $H_L^p(\mathbb{R}^n)$ is defined by

$$\|f\|_{H_L^p} := \|M_L(f)\|_{L^p}.$$

Similar to the classical Hardy spaces, the atom of $H_L^p(\mathbb{R}^n)$ is defined as follows.

Definition. Let $n/(n+\delta) < p \leq 1 \leq q \leq \infty$ with $p \neq q$. A function a is called an $H_L^{p,q}$-atom related to a ball $B(x_0, r_B)$ if

(i) supp $a \subset B(x_0, r_B)$;
(ii) $\|a\|_{L^q} \leq |B(x_0, r_B)|^{1/q-1/p}$;
(iii) if $r_B < \rho(x_0)/4$, then $\int_{B(x_0,r_B)} a(y)dy = 0$.

The atomic norm of $H_L^p(\mathbb{R}^n)$ is defined by $\|f\|_{H_L^{p,q}\text{-atom}} := \inf\{(\sum |c_j|^p)^{1/p}\}$, where the infimum is taken over all decompositions $f = \Sigma c_j a_j$, and a_j are $H_L^{p,q}$-atoms.

Proposition 2.5 ([15, Theorem 4.5]). *Let $0 \leq \gamma < 1$. The dual space of $H_L^{n/(n+\gamma)}(\mathbb{R}^n)$ is $\mathcal{C}_L^\gamma(\mathbb{R}^n)$. More precisely, any continuous linear functional Φ over $H_L^{n/(n+\gamma)}(\mathbb{R}^n)$ can be represented as*

$$\Phi(a) = \int_{\mathbb{R}^n} f(x)a(x)dx$$

for some function $f \in \mathcal{C}_L^\gamma(\mathbb{R}^n)$ and all $H_L^{n/(n+\gamma)}$-atoms a. Moreover, the operator norm $\|\Phi\|_{op} \sim \|f\|_{\mathcal{C}_L^\gamma}$.

3. Regularities of the generalized Poisson semigroups

By the fundamental solution of Schrödinger operators, J. Dziubański and J. Zienkiewicz in [7] obtained the following estimates of $K_t^L(\cdot,\cdot)$.

Proposition 3.1 ([7, Theorems 4.10 & 4.11]).

(i) *For any $M>0$, there exist constants $C_M, c>0$ such that*

$$|K_t^L(x,y)|\le C_M t^{-n/2}e^{-c|x-y|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-M}.$$

(ii) *Assume that $0<\delta\le\min\{1,\delta_0\}$. For any $M>0$, there exist constants $C_M, c>0$ such that for all $|h|<\sqrt{t}$,*

$$\Big|K_t^L(x+h,y)-K_t^L(x,y)\Big|\le C_M\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta}t^{-n/2}e^{-c|x-y|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-M}.$$

For $k\in\mathbb{Z}_+$, define

$$Q_{k,t}^L(x,y):=t^{2k}\partial_s^kK_s^L(x,y)|_{s=t^2}.$$

In [10], Huang-Li-Liu proved the following estimates about the kernel $Q_{k,t}^L(\cdot,\cdot)$.

Proposition 3.2 ([10, Proposition 3.3]).

(i) *For any $M>0$, there exist constants $C_M, c>0$ such that*

$$\Big|Q_{k,t}^L(x,y)\Big|\le C_M t^{-n}e^{-c|x-y|^2/t^2}\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-M}.$$

(ii) *Assume that $0<\delta_1\le\min\{1,\delta_0\}$. For any $M>0$, there exist constants $C_M, c>0$ such that for all $|h|<\sqrt{t}$*

$$\Big|Q_{k,t}^L(x+h,y)-Q_{k,t}^L(x,y)\Big|\le C_M\Big(\frac{|h|}{t}\Big)^{\delta}t^{-n}e^{-c|x-y|^2/t^2}\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-M}.$$

(iii) *For any $M>0$, there exists a constant $C_M>0$ such that*

$$\Big|\int_{\mathbb{R}^n}Q_{k,t}^L(x,y)dy\Big|\le\frac{C_M(t/\rho(x))^{\delta_1}}{(1+t/\rho(x))^M}.$$

Now we investigate the regularities of $p_{t,\sigma}^L(\cdot,\cdot)$. At first, we get the following estimates:

Proposition 3.3. *Let $0<\sigma<1$.*

(i) *For $M>0$, there exists a constant $C_M>0$ such that*

$$|p_{t,\sigma}^L(x,y)|\le\frac{C_Mt^{2\sigma}}{(t^2+|x-y|^2)^{n/2+\sigma}}\Big(1+\frac{\sqrt{|x-y|^2+t^2}}{\rho(x)}+\frac{\sqrt{|x-y|^2+t^2}}{\rho(y)}\Big)^{-M}.$$

(ii) *Let $0 < \delta_1 \le \min(2\sigma, \delta_0)$. For any $M > 0$ there exists a constant $C_M > 0$ such that for all $|h| \le t$,*

$$|p^L_{t,\sigma}(x+h,y) - p^L_{t,\sigma}(x,y)| \le C_M\Big(\frac{|h|}{\sqrt{t^2+|x-y|^2}}\Big)^{\delta_1}\frac{t^{2\sigma}}{(t^2+|x-y|^2)^{n/2+\sigma}} \times\Big(1+\frac{\sqrt{|x-y|^2+t^2}}{\rho(x)}+\frac{\sqrt{|x-y|^2+t^2}}{\rho(y)}\Big)^{-M}.$$

Proof. For (i), by the subordinative formula (3) and (i) of Proposition 3.1, we can get

$$\begin{aligned}
&|p^L_{t,\sigma}(x,y)|\\
&\le C\int_0^\infty e^{-r}|K^L_{t^2/4r}(x,y)|\frac{dr}{r^{1-\sigma}}\\
&\le C\int_0^\infty e^{-r}(t^2/4r)^{-n/2}e^{-c|x-y|^2r/t^2}\Big(1+\frac{\sqrt{t^2/4r}}{\rho(x)}+\frac{\sqrt{t^2/4r}}{\rho(y)}\Big)^{-M}\frac{dr}{r^{1-\sigma}}\\
&\le Ct^{-n-2M}\rho(x)^M\rho(y)^M\int_0^\infty e^{-cr(1+|x-y|^2/t^2)}r^{n/2+M+\sigma-1}dr.
\end{aligned}$$

Letting $r(1+\frac{|x-y|^2}{t^2}) = u$, we obtain

$$\begin{aligned}
&|p^L_{t,\sigma}(x,y)|\\
&\le Ct^{-n-2M}\rho(x)^M\rho(y)^M(1+|x-y|^2/t^2)^{-n/2-M-\sigma}\int_0^\infty e^{-cu}u^{n/2+M+\sigma-1}du\\
&\le C\frac{t^{2\sigma}}{(t^2+|x-y|^2)^{n/2+\sigma}}\Big(1+\frac{\sqrt{|x-y|^2+t^2}}{\rho(x)}+\frac{\sqrt{|x-y|^2+t^2}}{\rho(y)}\Big)^{-M}.
\end{aligned}$$

For (ii), similar to the proof of (i), we can use (ii) of Proposition 3.1 to obtain the desired result. So the details are omitted. □

For $k \in \mathbb{Z}_+$ and $t > 0$, define $D^{L,\sigma}_{t,k}(x,y) = t^k\partial^k_t p^L_{t,\sigma}(x,y)$. We can get the following estimates about the kernel $D^{L,\sigma}_{t,k}(\cdot,\cdot)$.

Proposition 3.4.

(i) *For $M > 0$, there exists a constant $C_M > 0$ such that*

$$|D^{L,\sigma}_{t,k}(x,y)| \le \frac{C_M t^{2\sigma}}{(t^2+|x-y|^2)^{n/2+\sigma}}\Big(1+\frac{\sqrt{t^2+|x-y|^2}}{\rho(x)}+\frac{\sqrt{t^2+|x-y|^2}}{\rho(y)}\Big)^{-M}.$$

(ii) *Let $0 < \delta \le \min(2\sigma, \delta_0)$. For any $M > 0$, there exists a constant $C_M > 0$ such that, for all $|h| \le t$,*

$$\Big|D^{L,\sigma}_{t,k}(x+h,y) - D^{L,\sigma}_{t,k}(x,y)\Big| \le C_M\Big(\frac{|h|}{\sqrt{t^2+|x-y|^2}}\Big)^{\delta}\frac{t^{2\sigma}}{(t^2+|x-y|^2)^{n/2+\sigma}} \times\Big(1+\frac{\sqrt{t^2+|x-y|^2}}{\rho(x)}+\frac{\sqrt{t^2+|x-y|^2}}{\rho(y)}\Big)^{-M}.$$

(iii) *For any $M>0$, there exists a constant $C_M>0$ such that*

$$\Big|\int_{\mathbb{R}^n} D_{t,k}^{L,\sigma}(x,y)dy\Big| \le C_M \frac{(t/\rho(x))^{\delta}}{(1+t/\rho(x))^M}.$$

Proof. For (i), it is easy to see that

$$\Big|\frac{\partial^k}{\partial t^k} p_{t,\sigma}^L(x,y)\Big| = \Big|C\int_0^\infty e^{-r}\frac{\partial^k}{\partial t^k}K_{t^2/4r}^L(x,y)\frac{dr}{r^{1-\sigma}}\Big|.$$

Then we first recall that the higher-order derivative formula of the composite function: if $y=f(u)$ and $u=\varphi(x)$, then

$$\frac{\partial^k y}{\partial x^k} = \sum_{i=1}^k \frac{\mathbf{p}_{k.i}(x)}{i!} f^{(i)}(u),$$

where $\mathbf{p}_{k,i}(x) = \sum_{s=0}^{i-1}(-1)^s C_i^s u^s \frac{\partial^k}{\partial x^k} u^{i-s}$.

So, let $f(u)=e^{-uL}$ and $u=t^2/4r$, we obtain

$$\frac{\partial^k}{\partial t^k}K_{t^2/4r}^L(x,y) = C\sum_{i=1}^k t^{2i-k}(4r)^{-i}\frac{\partial^k}{\partial t^k}K_s^L(x,y)\Big|_{s=t^2/4r}.$$

Then, we use Proposition 3.2(i) to deduce that

$$\begin{aligned}
&\Big|\frac{\partial^k}{\partial t^k} p_{t,\sigma}^L(x,y)\Big| \\
&\le C_k\sum_{i=1}^k \Big|\int_0^\infty e^{-r}t^{2i-k}(4r)^{-i}\frac{\partial^i}{\partial s^i}K_s^L(x,y)\Big|_{s=t^2/4r}\frac{dr}{r^{1-\sigma}}\Big| \\
&\le C_k t^{-k}\sum_{i=1}^k\int_0^\infty e^{-r}\Big|(\frac{t^2}{4r})^i L^i K_{t^2/4r}^L(x,y)\Big|\frac{dr}{r^{1-\sigma}} \\
&\le C_k t^{-k}\int_0^\infty e^{-r}(t^2/4r)^{-n/2}e^{-c|x-y|^2r/t^2} \\
&\quad\times\Big(1+\frac{t}{\rho(x)\sqrt{r}}\Big)^{-M}\Big(1+\frac{t}{\rho(y)\sqrt{r}}\Big)^{-M}\frac{dr}{r^{1-\sigma}} \\
&\le C_k t^{-k-n-2M}\rho(x)^M\rho(y)^M\int_0^\infty e^{-cr(1+|x-y|^2/t^2)}r^{n/2+M+\sigma-1}dr.
\end{aligned}$$

Letting $r(1+|x-y|^2/t^2)=u$, we can get

$$\begin{aligned}
&\Big|\frac{\partial^k}{\partial t^k} p_{t,\sigma}^L(x,y)\Big| \\
&\le Ct^{-k-n-2M}\rho(x)^M\rho(y)^M\int_0^\infty e^{-cu}\Big(\frac{u}{1+|x-y|^2/t^2}\Big)^{n/2+M+\sigma-1} \\
&\quad\times\frac{du}{(1+|x-y|^2/t^2)}
\end{aligned}$$

$$\le Ct^{2\sigma}t^{-k}t^{-2\sigma-n-2M}\Big(1+|x-y|^2/t^2)^{-n/2-M-\sigma}\rho(x)^M\rho(y)^M$$
$$\le \frac{Ct^{2\sigma-k}}{(t^2+|x-y|^2)^{n/2+\sigma}}\Big(1+\frac{\sqrt{t^2+|x-y|^2}}{\rho(x)}+\frac{\sqrt{t^2+|x-y|^2}}{\rho(y)}\Big)^{-M}.$$

So we can obtain

$$\Big|t^k\partial_t^k p^L_{t,\sigma}(x,y)\Big|$$
$$\le \frac{C_M t^{2\sigma}}{(t^2+|x-y|^2)^{n/2+\sigma}}\Big(1+\frac{\sqrt{t^2+|x-y|^2}}{\rho(x)}+\frac{\sqrt{t^2+|x-y|^2}}{\rho(y)}\Big)^{-M}.$$

(ii) Similar to (i), the statement (ii) can be deduced from (ii) of Proposition 3.2, we omit the details.

For (iii), since

$$\begin{aligned} t^k\frac{\partial^k}{\partial t^k}p^L_{t,\sigma}(x,y) &= t^k\frac{\partial^k}{\partial t^k}\Big(\int_0^\infty e^{-r}K^L_{t^2/4r}(x,y)\frac{dr}{r^{1-\sigma}}\Big)\\ &= C_M\sum_{i=1}^k\int_0^\infty e^{-r}Q^L_{i,\sqrt{t^2/4r}}(x,y)\frac{dr}{r^{1-\sigma}}, \end{aligned}$$

using (iii) of Proposition 3.2, we can get

$$\begin{aligned} \Big|\int_{\mathbb{R}^n}D^{L,\sigma}_{t,k}(x,y)dy\Big| &\le C\int_0^\infty e^{-r}\Big|\int_{\mathbb{R}^n}Q^L_{i,\sqrt{t^2/4r}}(x,y)dy\Big|\frac{dr}{r^{1-\sigma}}.\\ &\le C\int_0^\infty e^{-r}\frac{(\sqrt{t^2}/\sqrt{4r}\rho(x))^\delta}{(1+\sqrt{t^2}/\sqrt{4r}\rho(y))^M}\frac{dr}{r^{1-\sigma}}. \end{aligned}$$

Case 1: $t>\rho(x)$. We obtain

$$\begin{aligned} \Big|\int_{\mathbb{R}^n}D^{L,\sigma}_{t,k}(x,y)dy\Big| &\le Ct^{\delta-M}\rho(x)^{M-\delta}\int_0^\infty e^{-r}r^{-\delta+M+\sigma-1}dr\\ &\le C\frac{(t/\rho(x))^\delta}{(1+t/\rho(x))^M}. \end{aligned}$$

Case 2: $t\le\rho(x)$. We can get

$$\begin{aligned} \Big|\int_{\mathbb{R}^n}D^{L,\sigma}_{t,k}(x,y)dy\Big| &\le C\int_0^\infty e^{-r}\Big(\frac{\sqrt{t^2/4r}}{\rho(x)}\Big)^\delta\frac{dr}{r^{1-\sigma}}\\ &\le C\Big(\frac{t}{\rho(x)}\Big)^\delta\int_0^\infty e^{-r}r^{-\delta+\sigma-1}dr\\ &\le C\frac{(t/\rho(x))^\delta}{(1+t/\rho(x))^M}. \end{aligned}$$

□

3.1. Estimation on the spatial gradient

In this section, we investigate the spatial gradient of $p^L_{t,\sigma}(\cdot,\cdot)$. For the special case $\sigma=1/2$, i.e., the Poisson kernel, the regularity estimates have been obtained in [5, Lemma 3.9].

Lemma 3.5. *Suppose $V \in B_q$ for some $q > n$. Let $\delta_1 = 1 - n/q$. For every $M > 0$, there exist constants $C_M > 0$ and $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$, the semigroup kernels $K_t^L(x,y)$ associated to e^{-tL}, satisfy the following estimates*:

$$|\nabla_x K_t^L(x,y)|+|\sqrt{t}\nabla_x K_t^L(x,y)| \leq C_M t^{-(n+1)/2} e^{-c|x-y|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-M},$$

and for $|h| < |x-y|/4$,

$$|\nabla_x K_t^L(x+h,y) - \nabla_x K_t^L(x,y)| \leq C_M \Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta_1} t^{-(n+1)/2} e^{-c|x-y|^2/t}.$$

Then we give the gradient estimate of $p_{t,\sigma}^L(\cdot,\cdot)$.

Proposition 3.6. *Suppose that $\sigma > 0$ and $V \in B_q$ for some $q > n$.*

(i) *For every $M > 0$, there exists a constant $C_M > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$\begin{aligned}&|\nabla_x p_{t,\sigma}^L(x,y)| \\ &\leq \frac{C_M t^{2\sigma}}{(t^2+|x-y|^2)^{\sigma+(n+1)/2}}\Big(1+\frac{\sqrt{|x-y|^2+t^2}}{\rho(x)}+\frac{\sqrt{|x-y|^2+t^2}}{\rho(y)}\Big)^{-M}.\end{aligned}$$

(ii) *Let $\delta_1 = 1 - n/q$. For every $M > 0$, there exist constants $C_M > 0$ and $c > 0$ such that for all $x, y \in \mathbb{R}^n$, $t > 0$ and $|h| < |x-y|/4$,*

$$|\nabla_x p_{t,\sigma}^L(x+h,y) - \nabla_x p_{t,\sigma}^L(x,y)| \leq \frac{C_M t^{2\sigma}}{(t^2+|x-y|^2)^{\sigma+(n+1)/2}}\Big(\frac{|h|}{\sqrt{t^2+|x-y|^2}}\Big)^{\delta_1}.$$

(iii) *Let $\sigma \in (0, 1/2 - n/2q)$. For every $M > 0$,*

$$\Big|t\nabla_x P_{t,\sigma}^L(1)(x)\Big| \leq C\min\{(t/\rho(x))^{1+2\sigma}, (t/\rho(x))^{-M}\}.$$

Proof. For (i), since

$$\nabla_x p_{t,\sigma}^L(x,y) = C\int_0^\infty e^{-r}\nabla_x K_{t^2/4r}^L(x,y)\frac{dr}{r^{1-\sigma}},$$

we apply Lemma 3.5 to obtain

$$\begin{aligned}&|\nabla_x p_{t,\sigma}^L(x,y)| \\ &\leq C\int_0^\infty e^{-r}\Big(\frac{t^2}{4r}\Big)^{-(n+1)/2} e^{-c|x-y|^2 r/t^2}\Big(\frac{t}{\sqrt{r}\rho(x)}\Big)^{-M}\Big(\frac{t}{\sqrt{r}\rho(y)}\Big)^{-M}\frac{dr}{r^{1-\sigma}} \\ &\leq Ct^{-(n+1)-2M}\rho(x)^M\rho(y)^M\int_0^\infty e^{-cr(1+|x-y|^2/t^2)} r^{M+\sigma+(n+1)/2-1}dr.\end{aligned}$$

Let $r(1+|x-y|^2/t^2) = u$. We can get

$$\begin{aligned}&|\nabla_x p_{t,\sigma}^L(x,y)| \\ &\leq Ct^{-n-2M-1}\rho(x)^M\rho(y)^M\end{aligned}$$

$$\times \int_0^\infty e^{-cu}\Big(\frac{u}{1+|x-y|^2/t^2}\Big)^{M+\sigma+(n+1)/2-1}(1+|x-y|^2/t^2)^{-1}du$$
$$\leq \frac{Ct^{2\sigma}}{(t^2+|x-y|^2)^{\sigma+(n+1)/2}}\Big(1+\frac{\sqrt{|x-y|^2+t^2}}{\rho(x)}+\frac{\sqrt{|x-y|^2+t^2}}{\rho(y)}\Big)^{-M}.$$

For (ii), by the subordinative formula and Lemma 3.5, We can prove that (ii) holds.

For (iii), we divide the proof into two cases.

Case 1: $t>\rho(x)$. Using a direct computation, we obtain

$$\begin{aligned}
&|t\nabla_x P^L_{t,\sigma}(1)(x)|\\
&\leq C\int_{\mathbb{R}^n}\frac{t^{2\sigma+1}}{(t^2+|x-y|^2)^{\sigma+(n+1)/2}}\Big(1+\frac{t}{\rho(x)}\Big)^{-M}\Big(1+\frac{t}{\rho(y)}\Big)^{-M}dy\\
&\leq C\Big(\frac{t}{\rho(x)}\Big)^{-M}\int_{\mathbb{R}^n}\frac{t^{2\sigma+1}}{(t+|x-y|)^{2\sigma+n+1}}dy\\
&\leq C\Big(\frac{t}{\rho(x)}\Big)^{-M}.
\end{aligned}$$

Case 2: $t\leq\rho(x)$. In this case, it follows from the subordination formula (3) that

$$|t\nabla_x P^L_{t,\sigma}(1)(x)|\leq Ct\int_0^\infty e^{-r}\Big(\int_{\mathbb{R}^n}|\nabla_x K^L_{t^2/4r}(x,y)|dy\Big)\frac{dr}{r^{1-\sigma}}=I_1+I_2,$$

where

$$\begin{cases}I_1:=t\displaystyle\int_0^{t^2/4\rho(x)^2}e^{-r}\Big(\int_{\mathbb{R}^n}|\nabla_x K^L_{t^2/4r}(x,y)|dy\Big)\frac{dr}{r^{1-\sigma}};\\ I_2:=t\displaystyle\int_{t^2/4\rho(x)^2}^\infty e^{-r}\Big(\int_{\mathbb{R}^n}|\nabla_x K^L_{t^2/4r}(x,y)|dy\Big)\frac{dr}{r^{1-\sigma}}.\end{cases}$$

By Lemma 3.5, we obtain

$$\int_{\mathbb{R}^n}|\nabla_x K^L_{t^2/4r}(x,y)|dy\leq C\sqrt{r}/t. \tag{4}$$

Then we can deduce from (4) that

$$I_1\leq C\int_0^{t^2/4\rho(x)^2}e^{-r}r^{\sigma-1/2}dr\leq C(t/\rho(x))^{1+2\sigma}.$$

For I_2, since $q>n$, it follows form the formula

$$h_u(x-y)-K^L_u(x,y)=\int_0^u\int_{\mathbb{R}^n}h_s(x-z)V(z)K_{u-s}(z,y)dzds$$

that for $\delta_0=2-n/q>1$,

$$\sqrt{u}\nabla_x e^{-uL}(1)(x)|\leq C\int_0^u\sqrt{\frac{u}{s}}(\frac{\sqrt{s}}{\rho(x)})^{\delta_0}\frac{ds}{s}\leq C(\frac{\sqrt{u}}{\rho(x)})^{\delta_0}.$$

Therefore,

$$I_2 \leq C \int_{t^2/4\rho(x)^2}^{\infty} e^{-r} r^{\sigma-1/2} \Big(\frac{t}{\rho(x)\sqrt{r}}\Big)^{\delta_0} dr \leq C\Big(\frac{t}{\rho(x)}\Big)^{1+2\sigma}.$$

□

3.2. Estimation on time-fractional derivatives

For $\alpha > 0$ and $0 < \sigma < 1$, define $D^{L,\sigma}_{t,\alpha}(f)(x) := t^\alpha \partial_t^\alpha P^L_{t,\sigma}(f)(x)$. Denote by $D^{L,\sigma}_{t,\alpha}(\cdot,\cdot)$ the integral kernel of $D^{L,\sigma}_{t,\alpha}$.

Proposition 3.7. *Let $\alpha > 0$ and $\sigma \in (0,1)$.*

(i) *For any $M > 0$, there exists a constant $C > 0$ such that*

$$|D^{L,\sigma}_{t,\alpha}(x,y)| \leq \frac{C_M t^\alpha}{(t+|x-y|)^{n+\alpha}} \Big(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\Big)^{-M}.$$

(ii) *Let $0 < \delta_1 \leq \delta = \min(2\alpha\sigma, 2\sigma, \delta_0)$. For any $M > 0$, there exists a constant $C > 0$ such that for all $|h| \leq t$,*

$$|D^{L,\sigma}_{t,\alpha}(x+h,y) - D^{L,\sigma}_{t,\alpha}(x,y)| \leq \frac{Ct^\alpha}{(t+|x-y|)^{n+\alpha}} \Big(\frac{|h|}{t}\Big)^{\delta_1} \Big(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\Big)^{-M}.$$

(iii) *For any $M > 0$, there exists a constant $C > 0$ such that*

$$\Big| \int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy \Big| \leq \frac{C(t/\rho(x))^{\delta_1}}{(1+t/\rho(x))^M}.$$

Proof. At first, we prove (i). The following two cases are considered.

Case 1: $\alpha \in (0,1)$. By the functional calculus, we get

$$t^\alpha \partial_t^\alpha p^L_{t,\sigma}(x,y) = C_\alpha t^\alpha \int_0^\infty \int_0^s \partial_r p^L_{t+r,\sigma}(x,y) \frac{drds}{s^{1+\alpha}}.$$

By (i) of Proposition 3.4, we obtain

$$\begin{aligned}
\Big| D^{L,\sigma}_{t,\alpha}(x,y) \Big| &\leq Ct^\alpha \int_0^\infty \int_0^s |D^{L,\sigma}_{t+r,1}(x,y)| \frac{dr}{t+r} \frac{ds}{s^{1+\alpha}} \\
&\leq Ct^\alpha \int_0^\infty \int_0^s \frac{(t+r)^{2\sigma}}{((t+r)^2+|x-y|^2)^{n/2+\sigma}} \\
&\quad \times \Big(1 + \frac{(t+r)}{\rho(x)}\Big)^{-M} \Big(1 + \frac{(t+r)}{\rho(y)}\Big)^{-M} \frac{dr}{t+r} \frac{ds}{s^{1+\alpha}}.
\end{aligned}$$

On the one hand, a direct computation gives

$$\begin{aligned}
\Big| D^{L,\sigma}_{t,\alpha}(x,y) \Big| &\leq Ct^\alpha \int_0^\infty \int_0^s (t+r)^{-n} \Big(\frac{(t+r)}{\rho(x)}\Big)^{-M} \Big(\frac{(t+r)}{\rho(y)}\Big)^{-M} \frac{dr}{t+r} \frac{ds}{s^{1+\alpha}} \\
&\leq Ct^\alpha \rho(x)^M \rho(y)^M \int_0^\infty \int_0^s (t+r)^{-n-2M-1} \frac{drds}{s^{1+\alpha}} \\
&\leq C\rho(x)^M \rho(y)^M t^{-n-2M-1} \int_0^\infty (r/t)^{-\alpha} (1+r/t)^{-n-2M-1} dr.
\end{aligned}$$

Taking the change of variable $u = r/t$, we obtain

$$\begin{aligned}\left|D_{t,\alpha}^{L,\sigma}(x,y)\right| &\le C\rho(x)^M\rho(y)^M t^{-n-2M}\int_0^\infty (u)^{-\alpha}(1+u)^{-n-2M-1}du\\ &\le Ct^{-n}\Big(\frac{t}{\rho(x)}\Big)^{-M}\Big(\frac{t}{\rho(y)}\Big)^{-M}.\end{aligned}$$

One the other hand, since

$$p_{t,\sigma}^L(x,y) = C\int_0^\infty e^{-w}K_{(t+r)^2/4w}^L(x,y)\frac{dw}{w^{1-\sigma}},$$

we can get

$$\begin{aligned}&t^\alpha\partial_t^\alpha p_{t,\sigma}^L(x,y)\\ &= C_\alpha t^\alpha\int_0^\infty \partial_r\Big(\int_0^\infty e^{-w}K_{(t+r)^2/4w}^L(x,y)\frac{dw}{w^{1-\sigma}}\Big)\frac{dr}{r^\alpha}\\ &= C_\alpha t^\alpha\int_0^\infty \Big(\int_0^\infty e^{-w}\frac{(t+r)}{2w}LK_{(t+r)^2/4w}^L(x,y)\frac{dw}{w^{1-\sigma}}\Big)\frac{dr}{r^\alpha}\\ &= C_\alpha t^\alpha\int_0^\infty \Big(\int_0^\infty \frac{(t+r)^2}{4w}LK_{(t+r)^2/4w}^L(x,y)\frac{dr}{r^\alpha(t+r)}\Big)e^{-w}\frac{dw}{w^{1-\sigma}},\end{aligned}$$

which, together with Proposition 3.2, yields

$$\begin{aligned}&\left|D_{t,\alpha}^{L,\sigma}(x,y)\right|\\ &\le Ct^\alpha\rho(x)^M\rho(y)^M\int_0^\infty e^{-w}\Big(\int_0^\infty \frac{e^{-c|x-y|^2w/(t+r)^2}}{(t+r)^{n+2M+1}}\frac{dr}{r^\alpha}\Big)w^{n/2+M+\sigma-1}dw\\ &\le Ct^\alpha\rho(x)^M\rho(y)^M\int_0^\infty e^{-w}\Big(\int_0^\infty \Big(\frac{|x-y|^2w}{(t+r)^2}\Big)^{-(n+\alpha)/2}\\ &\quad\times\frac{1}{(t+r)^{n+2M+1}}\frac{dr}{r^\alpha}\Big)w^{n/2+M+\sigma-1}dw\\ &\le \frac{Ct^\alpha\rho(x)^M\rho(y)^M}{|x-y|^{n+\alpha}}\int_0^\infty e^{-w}\Big(\int_0^\infty (t+r)^{\alpha-2M-1}\frac{dr}{r^\alpha}\Big)w^{M+\sigma-1-\alpha/2}dw\\ &\le \frac{Ct^\alpha}{|x-y|^{n+\alpha}}\Big(\frac{t}{\rho(x)}\Big)^{-M}\Big(\frac{t}{\rho(y)}\Big)^{-M}.\end{aligned}$$

If $t > |x-y|$, then

$$\Big(\frac{t}{\rho(x)}\Big)^M\Big(\frac{t}{\rho(y)}\Big)^M\left|D_{t,\alpha}^{L,\sigma}(x,y)\right| \le \frac{Ct^\alpha}{t^{n+\alpha}} \le \frac{Ct^\alpha}{(|x-y|+t)^{n+\alpha}}.$$

If $t \le |x-y|$, we can also get

$$\Big(\frac{t}{\rho(x)}\Big)^M\Big(\frac{t}{\rho(y)}\Big)^M\left|D_{t,\alpha}^{L,\sigma}(x,y)\right| \le \frac{Ct^\alpha}{(|x-y|+t)^{n+\alpha}}.$$

The arbitrariness of M implies that

$$|D_{t,\alpha}^{L,\sigma}(x,y)| \le \frac{Ct^\alpha}{(t+|x-y|)^{n+\alpha}}\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-M}.$$

Case 2: $\alpha \ge 1$. Let $v=\alpha-[\alpha]$ and $k=[\alpha]+1$. We can get

$$\begin{aligned} t^\alpha \partial_t^\alpha p_{t,\sigma}^L(x,y) &= t^\alpha L^{[\alpha]\sigma}\int_0^\infty \int_0^s \partial_r p_{t+r,\sigma}^L(x,y)\frac{drds}{s^{1+v}} \\ &= t^\alpha \int_0^\infty \int_0^s L^{k\sigma} p_{t+r,\sigma}^L(x,y)\frac{drds}{s^{2+\alpha-k}} \\ &= t^\alpha \int_0^\infty \int_0^s D_{t+r,k}^{L,\sigma}(x,y)\frac{drds}{(t+r)^k s^{2+\alpha-k}}. \end{aligned}$$

One the one hand, by Proposition 3.4,

$$\begin{aligned} \Big|D_{t,\alpha}^{L,\sigma}(x,y)\Big| &\le Ct^\alpha \int_0^\infty \int_0^s (t+r)^{-n}\Big(\frac{(t+r)}{\rho(x)}\Big)^{-M}\Big(\frac{(t+r)}{\rho(y)}\Big)^{-M}\frac{dr}{(t+r)^k}\frac{ds}{s^{2+\alpha-k}} \\ &\le Ct^\alpha \rho(x)^M\rho(y)^M \int_0^\infty r^{k-\alpha-1}(t+r)^{-n-2M-k}dr \\ &\le Ct^{-n}\Big(1+t/\rho(x)\Big)^{-M}\Big(1+t/\rho(y)\Big)^{-M}. \end{aligned}$$

On the other hand, we apply the change of variables to obtain

$$\begin{aligned} \Big|D_{t,\alpha}^{L,\sigma}(x,y)\Big| &= \Big|Ct^\alpha \int_0^\infty e^{-w}\Big(\int_0^\infty \partial_r^k K_{(t+r)^2/4w}^L(x,y)\frac{dr}{r^{\alpha+1-k}}\Big)\frac{dw}{w^{1-\sigma}}\Big| \\ &\le Ct^\alpha \int_0^\infty e^{-w}\Big\{\int_0^\infty (t+r)^{-k}((t+r)^2/4w)^{-n/2}e^{-c|x-y|^2 w/(t+r)^2} \\ &\quad \times \Big(\frac{(t+r)}{\rho(x)\sqrt{w}}\Big)^{-M}\Big(\frac{(t+r)}{\rho(y)\sqrt{w}}\Big)^{-M}\frac{dr}{r^{\alpha+1-k}}\Big\}\frac{dw}{w^{1-\sigma}} \\ &\le Ct^\alpha \rho(x)^M\rho(y)^M\int_0^\infty e^{-w}\Big\{\int_0^\infty (t+r)^{-k-n-2M} \\ &\quad \times \Big(\frac{|x-y|^2 w}{(t+r)^2}\Big)^{-(n+\alpha)/2} r^{k-\alpha-1}dr\Big\}w^{-\alpha/2+M+\sigma-1}dw \\ &\le \frac{Ct^\alpha\rho(x)^M\rho(y)^M}{|x-y|^{n+\alpha}}t^{-2M}\int_0^\infty e^{-w} \\ &\quad \times \Big(\int_0^\infty (1+u)^{-k+\alpha-2M}u^{k-\alpha-1}du\Big)w^{-\alpha/2+M+\sigma-1}dw \\ &\le \frac{Ct^\alpha}{|x-y|^{n+\alpha}}\Big(1+\frac{t}{\rho(x)}\Big)^{-M}\Big(1+\frac{t}{\rho(y)}\Big)^{-M}. \end{aligned}$$

The arbitrariness of M implies that

$$|D_{t,\alpha}^{L,\sigma}(x,y)| \le \frac{Ct^\alpha}{(t+|x-y|)^{n+\alpha}}\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-M}.$$

Similar to (i), the proof of (ii) can be completed by Propositions 3.4 & 3.2. So we omit the details.

For (iii), when $\alpha \in (0,1)$, by (iii) of Proposition 3.4, we change the order of integration to obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| &\le Ct^\alpha \int_0^\infty \int_0^s \Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t+r,1}(x,y)dy\Big| \frac{dr}{t+r}\frac{ds}{s^{1+\alpha}}\\ &\le Ct^\alpha \int_0^\infty \int_0^s \frac{((t+r)/\rho(x))^{\delta_1}}{(1+(t+r)/\rho(x))^M}\frac{dr}{t+r}\frac{ds}{s^{1+\alpha}}.\end{aligned}$$

If $t>\rho(x)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| &\le Ct^\alpha \int_0^\infty \int_0^s \Big(\frac{(t+r)}{\rho(x)}\Big)^{\delta_1-M}\frac{dr}{t+r}\frac{ds}{s^{1+\alpha}}\\ &\le Ct^\alpha\rho(x)^{M-\delta_1}t^{-\alpha+\delta_1-M}\int_0^\infty u^{-\alpha}(1+u)^{\delta_1-M-1}du\\ &\le \frac{C(t/\rho(x))^{\delta_1}}{(1+t/\rho(x))^M}.\end{aligned}$$

If $t\le\rho(x)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| &\le Ct^\alpha \int_0^\infty \Big(\frac{(t+r)}{\rho(x)}\Big)^{\delta_1}\frac{1}{(1+t/\rho(x))^M}\frac{dr}{(t+r)r^\alpha}\\ &\le C\Big(\frac{t}{\rho(x)}\Big)^{\delta_1} \le \frac{C(t/\rho(x))^{\delta_1}}{(1+t/\rho(x))^M}.\end{aligned}$$

For $\alpha\ge 1$, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| &\le Ct^\alpha \int_0^\infty \int_0^s \Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t+r,k}(x,y)dy\Big| \frac{dr}{(t+r)^k}\frac{ds}{s^{2+\alpha-k}}\\ &\le Ct^\alpha \int_0^\infty \int_0^s \frac{((t+r)/\rho(x))^{\delta_1}}{(1+(t+r)/\rho(x))^M}\frac{dr}{(t+r)^k}\frac{ds}{s^{2+\alpha-k}}.\end{aligned}$$

If $t>\rho(x)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| &\le Ct^\alpha\rho(x)^{M-\delta_1}\int_0^\infty (t+r)^{\delta_1-M-k}\Big(\int_r^\infty \frac{ds}{s^{2+\alpha-k}}\Big)dr\\ &\le Ct^\alpha\rho(x)^{M-\delta_1}\int_0^\infty (t+r)^{\delta_1-M-k}r^{k-\alpha-1}dr\\ &\le \frac{C(t/\rho(x))^{\delta_1}}{(1+t/\rho(x))^M}.\end{aligned}$$

If $t\le\rho(x)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| &\le Ct^\alpha \int_0^\infty \Big(\frac{(t+r)}{\rho(x)}\Big)^{\delta_1}(1+(t+r)/\rho(x))^{-M}\frac{dr}{(t+r)^k r^{\alpha+1-k}}\\ &\le Ct^\alpha\rho(x)^{-\delta_1}\int_0^\infty (t+r)^{\delta_1-k}\frac{dr}{r^{\alpha+1-k}}\end{aligned}$$

$$\leq \frac{C(t/\rho(x))^{\delta_1}}{(1+t/\rho(x))^M}.$$

This completes the proof of Proposition 3.7. □

4. Characterization of Campanato spaces associated with L

In order to characterize $\mathcal{C}_L^\gamma(\mathbb{R}^n)$, we need the following lemmas whose proofs are quite similar to those in [13].

Lemma 4.1. *Let $\sigma \in (0,1)$ and $\alpha > 0$. The operator $t^\alpha \partial_t^\alpha P_{t,\sigma}^L$ defines an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}_+^{n+1}, dxdt/t)$. Moreover, in the sense of $L^2(\mathbb{R}^n)$, it holds*

$$f(x) = C \lim_{N\to\infty} \lim_{\varepsilon\to 0} \int_\varepsilon^N (t^\alpha \partial_t^\alpha P_{t,\sigma}^L)^2(f)(x)\frac{dt}{t}.$$

For $\sigma \in (0,1)$ and $\alpha > 0$, define an area function $S_{\alpha,\sigma}^L$ as follows:

$$S_{\alpha,\sigma}^L(f)(x) := \Big(\int\int_{\Gamma(x)} |t^\alpha \partial_t^\alpha P_{t,\sigma}^L(f)(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\Gamma(x)$ denotes the cone $\{(y,t) : |x-y| < t\}$.

Lemma 4.2. *Let $\sigma \in (0,1)$ and $\alpha > 0$. The area function $S_{\alpha,\sigma}^L$ is bounded on $L^2(\mathbb{R}^n)$.*

Lemma 4.3. *Let $\sigma \in (0,1)$, $\alpha > 0$ and $0 < \gamma \leq \min\{2\alpha, 2\alpha\sigma\}$. Let f be a linear combination of $H_L^{n/(n+\gamma)}$-atoms. There exists a constant C such that*

$$\|S_{\alpha,\sigma}^L(f)\|_{L^{n/(n+\gamma)}} \leq C\|f\|_{H_L^{n/(n+\gamma)}}.$$

Lemma 4.4. *Let $0 < \gamma \leq 1$. For any pair of measurable functions on $\mathbb{R}_+^{n+1}$, we have*

$$\begin{aligned}
&\int\int_{\mathbb{R}_+^{n+1}} |t^\alpha \partial_t^\alpha P_{t,\sigma}^L(f)(x)| \cdot |t^\alpha \partial_t^\alpha P_{t,\sigma}^L(g)(x)|\frac{dxdt}{t}\\
&\leq C \sup_B \Big(\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B} \int_B |t^\alpha \partial_t^\alpha P_{t,\sigma}^L(f)(x)|^2 \frac{dxdt}{t}\Big)^{1/2}\\
&\quad \times \left\{\int_{\mathbb{R}^n} \Big(\int\int_{|x-y|<t} |t^\alpha \partial_t^\alpha P_{t,\sigma}^L(g)(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{n/2(n+\gamma)} dx\right\}^{1+\gamma/n}.
\end{aligned}$$

Lemma 4.5. *Let $\sigma \in (0,1)$, $\alpha > 0$ and $0 < \gamma \leq \min\{2\alpha, 2\alpha\beta\}$. Let $f \in L^1(\mathbb{R}^n, (1+|x|)^{-(n+\gamma+\varepsilon)}dx)$ for any $\varepsilon > 0$ and let a be an $H_L^{n/(n+\gamma)}$-atom. Then for*

$$\begin{cases} F(x,t) := t^\alpha \partial_t^\alpha P_{t,\sigma}^L(f)(x),\\ G(x,t) := t^\alpha \partial_t^\alpha P_{t,\sigma}^L(a)(x),\end{cases}$$

there exists a constant C such that

$$C\int_{\mathbb{R}^n} f(x)\overline{a(x)}dx = \int\int_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)}\frac{dt}{t}.$$

Finally, we can obtain the following characterization of $\mathcal{C}^\gamma_L(\mathbb{R}^n)$ corresponding to the time-fractional derivative of $p^L_{\sigma,t}(\cdot,\cdot)$.

Theorem 4.6. *Let $V \in B_q$, $q > n$. Assume that $\sigma \in (0,1)$, $\alpha > 0$, $0 < \gamma \leq 1$ with $0 < \gamma < \min\{2\sigma, 2\alpha\sigma\}$. Let f be a function such that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+\gamma+\varepsilon}}dx < \infty \tag{5}$$

for some $\varepsilon > 0$. The following statements are equivalent:

(i) $f \in \mathcal{C}^\gamma_L(\mathbb{R}^n)$;

(ii) *There exists C such that $\|D^{L,\sigma}_{t,\alpha}(f)\|_\infty \leq Ct^\gamma$;*

(iii) *For all $B = B(x_B, r_B) \subset \mathbb{R}^n$,*

$$\Big(\frac{1}{|B|}\int_0^{r_B}\int_B |D^{L,\sigma}_{t,\alpha}(f)(x)|^2\frac{dxdt}{t}\Big)^{1/2} \leq C|B|^{\gamma/n}. \tag{6}$$

Proof. (i)⇒(ii). If $f \in \mathcal{C}^\gamma_L(\mathbb{R}^n)$, then $|t^\alpha\partial^\alpha_t P^L_{t,\sigma}f(x)| \leq I + II$, where

$$\begin{cases} I := \Big|\displaystyle\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)(f(y)-f(x))dy\Big|; \\ II := \Big|f(x)\displaystyle\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big|. \end{cases}$$

For I, we have

$$I \leq C\|f\|_{\mathcal{C}^\gamma_L}\int_{\mathbb{R}^n}\frac{t^\alpha|x-y|^\gamma}{(t+|x-y|)^{n+\alpha}}dy \leq Ct^\gamma\|f\|_{\mathcal{C}^\gamma_L}.$$

We further divide the estimate of II into the following two cases.

Case 1: $\rho(x) \leq t$. By Proposition 3.7,

$$\begin{aligned} II &\leq \|f\|_{\mathcal{C}^\gamma_L}\rho(x)^\gamma\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| \\ &\leq \|f\|_{\mathcal{C}^\gamma_L}t^\gamma\int_{\mathbb{R}^n}\frac{t^\alpha}{(t+|x-y|)^{n+\alpha}}dy \\ &\leq \|f\|_{\mathcal{C}^\gamma_L}t^\gamma. \end{aligned}$$

Case 2: $\rho(x) > t$. We use Proposition 3.7 again to obtain that there exists $\delta_1 > \gamma$ such that

$$\begin{aligned} II &\leq \|f\|_{\mathcal{C}^\gamma_L}\rho(x)^\gamma\Big|\int_{\mathbb{R}^n} D^{L,\sigma}_{t,\alpha}(x,y)dy\Big| \\ &\leq \|f\|_{\mathcal{C}^\gamma_L}\rho(x)^\gamma\frac{(t/\rho(x))^{\delta_1}}{(1+t/\rho(x))^M} \end{aligned}$$

$$\begin{aligned}&\leq \|f\|_{\mathcal{C}_L^\gamma} t^\gamma \frac{(t/\rho(x))^{\delta_1-\gamma}}{(1+t/\rho(x))^M}\\ &\leq \|f\|_{\mathcal{C}_L^\gamma} t^\gamma.\end{aligned}$$

(ii)$\Rightarrow$(iii). Assume that (ii) holds. Then

$$\begin{aligned}\Big(\frac{1}{|B|}\int_0^{r_B}\int_B |D_{t,\alpha}^{L,\sigma}(f)(x)|^2\frac{dxdt}{t}\Big)^{1/2} &\leq \|f\|_{\mathcal{C}_L^\gamma}\Big(\frac{1}{|B|}\int_0^{r_B}\int_B t^{2\gamma}\frac{dxdt}{t}\Big)^{1/2}\\ &\leq \|f\|_{\mathcal{C}_L^\gamma}|B|^{\gamma/n}.\end{aligned}$$

(iii)$\Rightarrow$(i). Assume that (6) holds. Let a be an $H_L^{n/(n+\gamma)}$-atom associated with $B = B(x_B, r_B)$. Then by Lemma 4.5,

$$\int_{\mathbb{R}^n} f(x)\overline{a(x)}dx = C\int\int_{\mathbb{R}^{n+1}_+} t^\alpha\partial_t^\alpha P_{t,\sigma}^L(f)(x)\overline{t^\alpha\partial_t^\alpha P_{t,\sigma}^L(a)(x)}\frac{dxdt}{t},$$

which, together with Lemmas 4.4 & 4.3, gives

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} f(x)\overline{a(x)}dx\Big|\\ &\leq \sup_B\Big(\frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t^\alpha\partial_t^\alpha P_{t,\sigma}^L(f)(x)|^2\frac{dxdt}{t}\Big)^{1/2}\\ &\quad\times\Big\{\int_{\mathbb{R}^n}\Big(\int\int_{|x-y|<t}|t^\alpha\partial_t^\alpha P_{t,\sigma}^L(a)(y)|^2\frac{dydt}{t^{n+1}}\Big)^{n/2(n+\gamma)}dx\Big\}^{1+\gamma/n}\\ &\leq \|S_{\alpha,\sigma}^L(a)\|_{L^{n/(n+\gamma)}}\sup_B\Big(\frac{1}{|B|^{1+2\gamma/n}}\int_0^{r_B}\int_B |t^\alpha\partial_t^\alpha P_{t,\sigma}^L(f)(x)|^2\frac{dxdt}{t}\Big)^{1/2}\\ &\leq \|a\|_{H_L^{n/(n+\gamma)}}.\end{aligned}$$

Hence

$$T(g) := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx, \quad g\in H_L^{n/(n+\gamma)}(\mathbb{R}^n)$$

is a bounded linear functional on $H_L^{n/(n+\gamma)}(\mathbb{R}^n)$, equivalently,

$$f \in (H_L^{n/(n+\gamma)}(\mathbb{R}^n))^* = \mathcal{C}_L^\gamma(\mathbb{R}^n).$$

□

Below we consider the characterization of $\mathcal{C}_L^\gamma(\mathbb{R}^n)$ via the spatial gradient. Define a general gradient as $\nabla := (\nabla_x, \partial_t)$.

Theorem 4.7. *Let $V \in B_q$, $q > n$. Assume that $\sigma \in (0, 1/2 - n/2q)$, $\alpha > 0$ and $0 < \gamma < 1$ with $0 < \gamma < \min\{2\sigma, 2\alpha\sigma\}$. Let f be a function satisfying* (5). *The following statements are equivalent*:

(i) $f \in \mathcal{C}_L^\gamma(\mathbb{R}^n)$;

(ii) *There exists a constant $C > 0$ such that*

$$\|t\nabla P_{t,\sigma}^L f\|_\infty \leq Ct^\gamma;$$

(iii) $u(x,t) = P^L_{t,\sigma} f(x)$ *satisfies that for any balls* $B = B(x_B, r_B)$,

$$\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B} \int_B |t\nabla P^L_{t,\sigma}(f)(x)|^2 \frac{dxdt}{t} \le C. \tag{7}$$

Proof. (i)⇒(ii). Let $f \in \mathcal{C}^\gamma_L(\mathbb{R}^n)$. By Theorem 4.6, $\|t\partial_t P^L_{t,\sigma}(f)\|_\infty \le Ct^\gamma$. One writes $t\nabla_x P^L_{t,\sigma} f(x) = I + II$, where

$$\begin{cases} I := \displaystyle\int_{\mathbb{R}^n} t\nabla_x P^L_{t,\sigma}(f(z) - f(x))dz; \\ II := f(x) t\nabla_x P^L_{t,\sigma}(1)(x). \end{cases}$$

We first estimate the term I. Because $f \in \mathcal{C}^\gamma_L(\mathbb{R}^n)$, then

$$|f(x) - f(z)| \le \|f\|_{\mathcal{C}^\gamma_L} |x - z|^\gamma.$$

Since

$$|t\nabla_x P^L_{t,\sigma}(x,z)| \le \frac{Ct^{2\sigma+1}}{(t^2 + |x-y|^2)^{\sigma+(n+1)/2}},$$

a direct computation gives

$$\begin{aligned} |I| &\le \|f\|_{\mathcal{C}^\gamma_L} \int_{\mathbb{R}^n} |t\nabla_x P^L_{t,\sigma}(x,z)| \cdot |x-z|^\gamma dz \\ &\le \|f\|_{\mathcal{C}^\gamma_L} \int_{\mathbb{R}^n} \frac{t^{2\sigma+1}|x-z|^\gamma}{(t^2 + |x-y|^2)^{\sigma+(n+1)/2}} \\ &\le t^\gamma \|f\|_{\mathcal{C}^\gamma_L}. \end{aligned}$$

By Proposition 3.6, we have

$$|t\nabla_x P^L_{t,\sigma}(1)| \le \min\{(t/\rho(x))^{1+2\sigma}, (t/\rho(x))^{-M}\}.$$

The estimate of II is divided into two cases.

Case 1: $\rho(x) \le t$. $f \in \mathcal{C}^\gamma_L(\mathbb{R}^n)$ implies that $|f(x)| \le \rho(x)^\gamma \|f\|_{\mathcal{C}^\gamma_L}$. Then

$$\begin{aligned} II &\le C\|f\|_{\mathcal{C}^\gamma_L} \rho(x)^\gamma |t\nabla_x P^L_{t,\sigma}(1)(x)| \\ &\le C\|f\|_{\mathcal{C}^\gamma_L} \rho(x)^\gamma \Big(\frac{t}{\rho(x)}\Big)^{-M} \\ &\le C\|f\|_{\mathcal{C}^\gamma_L} t^\gamma \Big(\frac{t}{\rho(x)}\Big)^{-M-\gamma} \\ &\le C\|f\|_{\mathcal{C}^\gamma_L} t^\gamma. \end{aligned}$$

Case 2: $\rho(x) > t$. We can get

$$\begin{aligned} II &\le C\|f\|_{\mathcal{C}^\gamma_L} \rho(x)^\gamma |t\nabla_x P^L_{t,\sigma}(1)(x)| \\ &\le C\|f\|_{\mathcal{C}^\gamma_L} \rho(x)^\gamma \Big(\frac{t}{\rho(x)}\Big)^{1+2\sigma} \\ &\le C\|f\|_{\mathcal{C}^\gamma_L} t^\gamma \Big(\frac{t}{\rho(x)}\Big)^{1+2\sigma-\gamma} \\ &\le C\|f\|_{\mathcal{C}^\gamma_L} t^\gamma. \end{aligned}$$

(ii)⇒(iii). For every ball $B = B(x_B, r_B)$,

$$\int_0^{r_B}\int_B |t\nabla P_{t,\sigma}^L f(x)|^2 \frac{dxdt}{t} \le \int_0^{r_B}\int_B t^{2\gamma}\frac{dxdt}{t} \le r_B^{n+2\gamma},$$

which implies that (7) holds.

(iii)⇒(i). Assume that (7) holds. For any ball $B = B(x_B, r_B)$, it holds

$$\sup_B r_B^{-(n+2\gamma)} \int_0^{r_B}\int_B |t\partial_t P_{t,\sigma}^L(f)(x)|^2 \frac{dxdt}{t} < \infty.$$

It is a corollary of Theorem 4.6 that $f \in \mathcal{C}_L^\gamma(\mathbb{R}^n)$ with

$$\|f\|_{\mathcal{C}_L^\gamma} \le \sup_B r_B^{-(n+2\gamma)} \int_0^{r_B}\int_B |t\partial_t P_{t,\sigma}^L(f)(x)|^2 \frac{dxdt}{t} < \infty. \qquad \square$$

A positive measure ν on $\mathbb{R}_+^{n+1}$ is called a κ-Carleson measure if

$$\|\nu\|_{\mathcal{C}} := \sup_{x\in\mathbb{R}^n, r>0} \frac{\nu(B(x,r)\times(0,r))}{|B(x,r)|^\kappa} < \infty.$$

The following result can be deduced from Theorems 4.6 & 4.7 immediately.

Theorem 4.8. *Let $V \in B_q$, $q > n$. Assume that $\sigma \in (0, 1/2 - n/2q)$, $\alpha > 0$ and $0 < \gamma < 1$ with $0 < \gamma < \min\{2\sigma, 2\alpha\sigma\}$. Let $d\nu$ be a measure defined by*

$$d\nu(x,t) := |t\nabla P_{t,\sigma}^L(f)(x)|^2 dxdt/t, \quad (x,t) \in \mathbb{R}_+^{n+1}.$$

(i) *If $f \in \mathcal{C}_L^\gamma(\mathbb{R}^n)$, then $d\nu$ is a $(1 + 2\gamma/n)$-Carleson measure;*

(ii) *Conversely, if $f \in L^1((1 + |x|)^{-n-\gamma-\varepsilon}dx)$ for some $\varepsilon > 0$ and $d\nu$ is a $(1+2\gamma/n)$-Carleson measure, then $f \in \mathcal{C}_L^\gamma(\mathbb{R}^n)$. Moreover, in any case, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|^2_{\mathcal{C}_L^\gamma} \le \|d\nu\|_{\mathcal{C}} \le C\|f\|^2_{\mathcal{C}_L^\gamma}.$$

References

[1] L. A. Caffarelli, S. Salsa, and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), no. 2, 425–461. https://doi.org/10.1007/s00222-007-0086-6

[2] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260. https://doi.org/10.1080/03605300600987306

[3] R. R. Coifman, Y. Meyer, and E. M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. **62** (1985), no. 2, 304–335. https://doi.org/10.1016/0022-1236(85)90007-2

[4] D. Deng, X. T. Duong, L. Song, C. Tan, and L. Yan, *Functions of vanishing mean oscillation associated with operators and applications*, Michigan Math. J. **56** (2008), no. 3, 529–550. https://doi.org/10.1307/mmj/1231770358

[5] X. T. Duong, L. Yan, and C. Zhang, *On characterization of Poisson integrals of Schrödinger operators with BMO traces*, J. Funct. Anal. **266** (2014), no. 4, 2053–2085. https://doi.org/10.1016/j.jfa.2013.09.008

[6] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. **249** (2005), no. 2, 329–356. https://doi.org/10.1007/s00209-004-0701-9

[7] J. Dziubański and J. Zienkiewicz, *H^p spaces for Schrödinger operators*, in Fourier analysis and related topics, 45–53, Banach Center Publ., 56, Polish Acad. Sci. Inst. Math., Warsaw, 2000. https://doi.org/10.4064/bc56-0-4

[8] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193. https://doi.org/10.1007/BF02392215

[9] J. Huang, M. Duan, Y. Wang, and W. Li, *Fractional Carleson measure associated with Hermite operator*, Anal. Math. Phys. **9** (2019), no. 4, 2075–2097. https://doi.org/10.1007/s13324-019-00300-2

[10] J. Huang, P. Li, and Y. Liu, *Regularity properties of the heat kernel and area integral characterization of Hardy space $H^1_{\mathcal{L}}$ related to degenerate Schrödinger operators*, J. Math. Anal. Appl. **466** (2018), no. 1, 447–470. https://doi.org/10.1016/j.jmaa.2018.06.008

[11] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, Mem. Amer. Math. Soc. **214** (2011), no. 1007, vi+78 pp. https://doi.org/10.1090/S0065-9266-2011-00624-6

[12] P. Li, *Riesz potentials of Hardy-Hausdorff spaces and Q-type spaces*, Sci. China Math. **63** (2020), no. 10, 2017–2036. https://doi.org/10.1007/s11425-018-9443-7

[13] P. Li, Z. Wang, T. Qian, and C. Zhang, *Regularity of fractional heat semigroup associated with Schrödinger operators*, preprint available at arXiv:2012.07234.

[14] C.-C. Lin and H. Liu, $\mathrm{BMO}_L(\mathbb{H}^n)$ *spaces and Carleson measures for Schrödinger operators*, Adv. Math. **228** (2011), no. 3, 1631–1688. https://doi.org/10.1016/j.aim.2011.06.024

[15] T. Ma, P. R. Stinga, J. L. Torrea, and C. Zhang, *Regularity properties of Schrödinger operators*, J. Math. Anal. Appl. **388** (2012), no. 2, 817–837. https://doi.org/10.1016/j.jmaa.2011.10.006

[16] Z. W. Shen, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, 513–546.

[17] L. Song, X. X. Tian, and L. X. Yan, *On characterization of Poisson integrals of Schrödinger operators with Morrey traces*, Acta Math. Sin. (Engl. Ser.) **34** (2018), no. 4, 787–800. https://doi.org/10.1007/s10114-018-7368-3

[18] P. R. Stinga and J. L. Torrea, *Extension problem and Harnack's inequality for some fractional operators*, Comm. Partial Differential Equations **35** (2010), no. 11, 2092–2122. https://doi.org/10.1080/03605301003735680

[19] Y. Wang, Y. Liu, C. Sun, and P. Li, *Carleson measure characterizations of the Campanato type space associated with Schrödinger operators on stratified Lie groups*, Forum Math. **32** (2020), no. 5, 1337–1373. https://doi.org/10.1515/forum-2019-0224

[20] D. Yang, D. Yang, and Y. Zhou, *Localized Morrey-Campanato spaces on metric measure spaces and applications to Schrödinger operators*, Nagoya Math. J. **198** (2010), 77–119. https://doi.org/10.1215/00277630-2009-008

[21] D. Yang, D. Yang, and Y. Zhou, *Localized BMO and BLO spaces on RD-spaces and applications to Schrödinger operators*, Commun. Pure Appl. Anal. **9** (2010), no. 3, 779–812. https://doi.org/10.3934/cpaa.2010.9.779

Pengtao Li
School of Mathematics and Statistics
Qingdao University
Qingdao 266071, P. R. China
Email address: ptli@qdu.edu.cn

Zhiyong Wang
School of Mathematics and Statistics
Qingdao University
Qingdao 266071, P. R. China
Email address: zywang199703@163.com

Kai Zhao
School of Mathematics and Statistics
Qingdao University
Qingdao 266071, P. R. China
Email address: zhkzhc@aliyun.com