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THE INTERIOR GRADIENT ESTIMATE FOR A CLASS OF MIXED HESSIAN CURVATURE EQUATIONS

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Abstract. In this paper, we are concerned with a class of mixed Hessian curvature equations with non-degeneration. By using the maximum principle and constructing an auxiliary function, we obtain the interior gradient estimate of $(k-1)$-admissible solutions.

1. Introduction

For a function $u \in C^2(B_r(0))$, the principal curvatures of the graph of u are the eigenvalues of Weingarten curvature matrix $A = \{a_{ij}\}_{n\times n}$ (see [22]), where

$$a_{ij} = \frac{1}{W}\left(u_{ij} - \frac{u_i u_k u_{kj}}{W(W+1)} - \frac{u_j u_k u_{ik}}{W(W+1)} + \frac{u_i u_j u_k u_l u_{kl}}{W(W+1)}\right), \tag{1.1}$$

$W = \sqrt{1+|\nabla u|^2}$. In this paper, we study the interior gradient estimate for a class of mixed Hessian curvature equations

$$\sigma_k(A) + \alpha(x)\sigma_{k-1}(A) = \sum_{l=0}^{k-2} \alpha_l(x)\sigma_l(A),\ x \in B_r(0) \subset \mathbb{R}^n, \tag{1.2}$$

where

$$\sigma_k(A) = \sigma_k(\lambda(A)) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \quad 1 \le k \le n,$$

$\lambda(A)$ denote eigenvalues of the matrix $A = \{a_{ij}\}_{n\times n}$.

Equation (1.2) are in the form of the linear combinations of the elementary symmetric functions and include the k-Hessian equations and quotient equations as special cases. The equations have naturally appeared in many important geometric problems, for example, so-called Fu-Yau equation [7, 8]

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motivated from the study of the Hull-Strominger system in theoretical physics [17], the special Lagrangian equation introduced by Harvey and Lawson [10]. Indeed, equation (1.2) was already considered by Krylov as an important example of an application of the general notion of fully nonlinear elliptic equations developed by himself in [13]. By using certain concave structure of the elliptic operator, Krylov reduced this equation to the Bellman-type equations and then applied the general theorems on the Bellman equations to obtain the crucial $C^{1,1}$ a priori estimates and proved the existence of smooth solutions for $\alpha(x) < 0$ and $\alpha_l(x) > 0,\ 0 \le l \le k-2$. Recently, Guan-Zhang [9] studied equation (1.2) in the problem of prescribing convex combination of area measures, which can be viewed as generalization of the equation for Christoffel-Minkowski problem in convex geometry. In comparison with Krylov's equations, a key new feature is that there is no sign requirement for the coefficient function of σ_{k-1}. Li-Ren-Wang [16] considered a closed convex hypersurface that satisfies equation (1.2) for $\alpha(x) \le 0$ and $\alpha_l(x) \ge 0,\ 0 \le l \le k-2$, established the curvature estimates of convex solutions for the equations. Later, equation (1.2) has been extensively studied (see [2–4], etc).

When $\alpha(x) = \alpha_l(x) = 0,\ 1 \le l \le k-2$, equation (1.2) are the so-called mean curvature equations. The interior gradient estimate of mean curvature equation has been studied extensively. In the case of two variables, Finn [6] obtained the interior gradient estimate of the minimal surface equation, while the high dimensional situation was studied by Bombieri-De Giorgi-Miranda [1]. For the general mean curvature equations, the estimations were acquired by Ladyzhenskaya-Ural'tseva [14], Trudinger [20] and Simon [18], respectively. All their methods were based on test function technique and Sobolev inequality on the graph. Using the invariance of the equations under rigid motion, the interior gradient estimates for high order mean curvature equation in Euclidean space have been gained in [12, 15, 21]. Applying standard Bernstein technique, Wang [22] gave a new proof of the interior gradient estimate of mean curvature equations. Following his idea, Chen-Xu-Zhang [5] established the interior gradient estimate of Hessian quotient curvature equations and Weng [23] obtained the same result for the Hessian curvature equation in hyperbolic space.

To my best knowledge, there is no priori interior gradient estimate for the mixed Hessian curvature equations. Naturally, we want to know how about the interior gradient estimate of equation (1.2). In this paper, we adapt the method used in [22] to obtain the interior gradient estimate for equation (1.2). Recall that the Garding's cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_i(\lambda) > 0, \forall\ 1 \le i \le k\}.$$

A function $u \in C^2(B_r)$ is called $(k-1)$-admissible if $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n) \in \Gamma_{k-1}$, where $\kappa_1, \kappa_2, \dots, \kappa_n$ are the principal curvatures of the graph of u. In this paper, we establish the interior gradient estimate as follow.

Theorem 1.1. *Suppose* $u \in C^3(B_r(0))$ *is a* $(k-1)$*-admissible solution of equation* (1.2)*, with*

$$\alpha(x) \in C^1(B_r(0)),$$
$$0 < \alpha_{k-2}(x) \in C^1(B_r(0)),$$
$$0 \le \alpha_l(x) \in C^1(B_r(0)),\ 0 \le l \le k-3.$$

Then

$$|\nabla u(0)| \le \exp\big[C\big(\frac{M}{r} + M + 1\big)^2\big],$$

where $M = \underset{B_r}{osc}\, u$ *and* C *is a positive constant depending only on* n, k, $\|\alpha\|_{C^1(B_r)}$, $\|\alpha_l\|_{C^1(B_r)}$ *and* $\inf\limits_{B_r} \alpha_{k-2}$.

Remark 1.2. In [5], for Hessian quotient curvature equations

$$\frac{\sigma_k(A)}{\sigma_l(A)} = \alpha_l(x), \tag{1.3}$$

the interior gradient estimate was obtained in the admissible set Γ_k. In this paper, the admissible set is Γ_{k-1}. Obviously, $\Gamma_k \subset \Gamma_{k-1}$. There are some terms in the proof of Theorem 1.1, sign of which is not easy to be determined in Γ_{k-1}. By concavity property of operator, sign of the terms is determined (see Proposition 2.5 below). Moreover, compared with the proof in [5], there are some new terms to deal with in the proving of Theorem 1.1 (see Lemma 3.1 below).

2. Preliminary

Let $\sigma_k(\lambda|i)$ denote the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$. Also denote by $\sigma_k(A|i)$ the symmetric function with A deleting the i-th row and i-th column, and $\sigma_k(A|ij)$ the symmetric function with A deleting the i-th, j-th rows and i-th, j-th columns, for all $1 \le i, j \le n$. For completeness, we define $\sigma_0(\lambda) = 1$ and $\sigma_{-1}(\lambda) = \sigma_{-2}(\lambda) = 0$.

Proposition 2.1 ([11, 15])**.** *Let* $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_k$ *and* $k \in \{1, 2, \dots, n\}$. *Suppose that*

$$\lambda_1 \ge \cdots \ge \lambda_k \ge \cdots \ge \lambda_n.$$

Then

$$\sigma_{k-1}(\lambda|n) \ge \sigma_{k-1}(\lambda|n-1) \ge \cdots \ge \sigma_{k-1}(\lambda|1), \tag{2.1}$$

$$C_n^k \lambda_1 \cdots \lambda_k \ge \sigma_k(\lambda), \quad \lambda_k > 0, \tag{2.2}$$

$$\sigma_{k-1}(\lambda) \ge \lambda_1 \lambda_2 \cdots \lambda_{k-1}, \quad k \ge 2, \tag{2.3}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

The generalized Newton-MacLaurin inequality is as follow, which will be used throughout this paper.

Proposition 2.2 ([19]). *For $\lambda \in \Gamma_k$ and $k > l \geq 0$, $r > s \geq 0$, $k \geq r$, $l \geq s$, we have*

$$\left[\frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{k-l}} \leq \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}. \tag{2.4}$$

For simplicity, we denote

$$\begin{aligned} G_k(\lambda) &= \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)}, \\ G_l(\lambda) &= -\frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}, \\ G(\lambda) &= \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} - \sum_{l=0}^{k-2} \alpha_l(x)\frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}. \end{aligned} \tag{2.5}$$

Then equation (1.2) can be rewritten as

$$\begin{aligned} G(A) &= \frac{\sigma_k(A)}{\sigma_{k-1}(A)} - \sum_{l=0}^{k-2} \alpha_l(x)\frac{\sigma_l(A)}{\sigma_{k-1}(A)} \\ &= G_k(A) + \sum_{l=0}^{k-2} \alpha_l(x) G_l(A) = -\alpha(x). \end{aligned} \tag{2.6}$$

Proposition 2.3 ([9]). *For $\lambda \in \Gamma_{k-1}$, $\alpha_l(x) > 0$, $0 \leq l \leq k-2$, the operator*

$$G(\lambda) = G_k(\lambda) + \sum_{l=0}^{k-2} \alpha_l(x) G_l(\lambda)$$

is elliptic and concave.

The following proposition plays an important role in our proof. In [2], there is a similar result, but the admissible set is Γ_k.

Proposition 2.4. *Let $0 < \alpha_{k-2}(x) \in C(\Omega)$, $0 \leq \alpha_l(x) \in C(\Omega)$, $0 \leq l \leq k-3$. Assume symmetric matrix $A = \{a_{ij}\}_{n\times n}$ satisfies $\lambda(A) \in \Gamma_{k-1}$, where*

$$a_{11} < 0, \ \{a_{ij}\}_{2\leq i,j\leq n} \text{ is diagonal.}$$

Then

$$\frac{\partial G(A)}{\partial a_{11}} \geq c_0 \sum_{i=1}^{n} \frac{\partial G(A)}{\partial a_{ii}},$$

where c_0 depends on n, k, $|\alpha|_{C^0(\Omega)}$ and $\inf_{\Omega} \alpha_{k-2}$.

Proof. By direct computations, we get

$$\sum_{i=1}^{n} \frac{\partial G}{\partial a_{ii}} = \frac{(n-k+1)\sigma_{k-1}(A)\sigma_{k-1}(A) - (n-k+2)\sigma_{k-2}(A)\sigma_k(A)}{\sigma_{k-1}^2(A)}$$

$$
\begin{aligned}
&+\sum_{l=0}^{k-2}\alpha_l\frac{(n-k+2)\sigma_{k-2}(A)\sigma_l(A)-(n-l+1)\sigma_{l-1}(A)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)}\\
&\leq n-k+1-(n-k+2)\frac{\sigma_{k-2}(A)}{\sigma_{k-1}(A)}\Big[\frac{\sigma_k(A)}{\sigma_{k-1}(A)}-\sum_{l=0}^{k-2}\alpha_l\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big]\\
&\leq n-k+1+(n-k+2)\alpha(x)\frac{\sigma_{k-2}(A)}{\sigma_{k-1}(A)}\\
&\leq C(n,k,|\alpha|_{C^0(\Omega)})+C(n,k,|\alpha|_{C^0(\Omega)})\Big[\frac{\sigma_{k-2}(A)}{\sigma_{k-1}(A)}\Big]^2. \qquad (2.7)
\end{aligned}
$$

Let $\widetilde{\lambda}=(a_{22},\dots,a_{nn})$. Since $a_{11}<0$, then $\sigma_{k-1}(A|1)=\sigma_{k-1}(\widetilde{\lambda})\geq\sigma_{k-1}(A)>0$. Notice that

$$
\sigma_k(A)=a_{11}\sigma_{k-1}(A|1)-\sum_{j=2}^{n}a_{1j}^2\sigma_{k-2}(A|1j)+\sigma_k(A|1).
$$

Hence, from (2.7) and Proposition 2.2, we obtain

$$
\begin{aligned}
\frac{\partial G}{\partial a_{11}}&=\frac{\sigma_{k-1}^2(\widetilde{\lambda})-\sigma_{k-2}(\widetilde{\lambda})\sigma_k(\widetilde{\lambda})+\sum_{j=2}^{n}a_{1j}^2[\sigma_{k-2}^2(\widetilde{\lambda}|j)-\sigma_{k-1}(\widetilde{\lambda}|j)\sigma_{k-2}(\widetilde{\lambda}|j)]}{\sigma_{k-1}^2(A)}\\
&\quad+\sum_{l=0}^{k-2}\alpha_l\frac{\sigma_{k-2}(\widetilde{\lambda})\sigma_l(\widetilde{\lambda})-\sigma_{l-1}(\widetilde{\lambda})\sigma_{k-1}(\widetilde{\lambda})+\sum_{j=2}^{n}a_{1j}^2[\sigma_{l-1}(\widetilde{\lambda}|j)\sigma_{k-3}(\widetilde{\lambda}|j)-\sigma_{k-2}(\widetilde{\lambda}|j)\sigma_{l-2}(\widetilde{\lambda}|j)]}{\sigma_{k-1}^2(A)}\\
&\geq\frac{\sigma_{k-1}^2(\widetilde{\lambda})-\sigma_{k-2}(\widetilde{\lambda})\sigma_k(\widetilde{\lambda})}{\sigma_{k-1}^2(A)}+\sum_{l=0}^{k-2}\alpha_l\frac{\sigma_{k-2}(\widetilde{\lambda})\sigma_l(\widetilde{\lambda})-\sigma_{l-1}(\widetilde{\lambda})\sigma_{k-1}(\widetilde{\lambda})}{\sigma_{k-1}^2(A)}\\
&\geq C(n,k)\frac{\sigma_{k-1}^2(\widetilde{\lambda})}{\sigma_{k-1}^2(A)}+\alpha_{k-2}\frac{\sigma_{k-2}^2(\widetilde{\lambda})-\sigma_{k-3}(\widetilde{\lambda})\sigma_{k-1}(\widetilde{\lambda})}{\sigma_{k-1}^2(A)}\\
&\geq C(n,k)+\inf_{\Omega}\alpha_{k-2}C(n,k)\frac{\sigma_{k-2}^2(A)}{\sigma_{k-1}^2(A)}\\
&\geq c_0\sum_{i=1}^{n}\frac{\partial G(A)}{\partial a_{ii}}. \qquad (2.8)
\end{aligned}
$$

This ends the proof. □

We obtain the following proposition by concavity property of operator.

Proposition 2.5. *Let symmetric matrix $A=\{a_{ij}\}_{n\times n}$ satisfy $\lambda(A)\in\Gamma_{k-1}$, where*

$$
a_{11}<0,\ \{a_{ij}\}_{2\leq i,j\leq n}\ \textit{is diagonal.}
$$

Then

$$
\sum_{i=2}^{n}\frac{\partial G(A)}{\partial a_{1i}}a_{1i}\leq 0.
$$

Proof. Let

$$A=\{a_{ij}\}_{n\times n},\ B=\{b_{ij}\}_{n\times n},\ C=\{c_{ij}\}_{n\times n},$$

where

$$b_{ij}=\begin{cases}0, & i=1, j\geq 2,\\ 0, & i\geq 2, j=1,\\ a_{ij}, & \text{otherwise},\end{cases}\quad c_{ij}=\begin{cases}a_{1j}, & i=1, j\geq 2,\\ a_{i1}, & i\geq 2, j=1,\\ 0, & \text{otherwise}.\end{cases}$$

For $2\leq m\leq k-1$, we can compute

$$\sigma_m(B+tC)=\sigma_m(A)+(1-t^2)\sum_{i=2}^{n}a_{1i}^2\sigma_{m-2}(A|1i)\geq\sigma_m(A)>0,\ t\in[-1,1],$$

which implies

$$\lambda(B+tC)\in\Gamma_{k-1}.$$

Considering

$$f(t)=G(B+tC),\ t\in[-1,1],$$

we have $f(-1)=f(1)$. It follows from Proposition 2.3 that $f(t)=G(B+tC)$ is concave with regard to $t\in[-1,1]$. Then

$$f'(-1)\geq 0,\ f'(1)\leq 0.$$

Direct calculations yield

$$\begin{aligned}0\geq f'(1)=&-2\Bigg[\frac{\sigma_{k-2}(A|1i)\sigma_{k-1}(A)-\sigma_{k-3}(A|1i)\sigma_k(A)}{\sigma_{k-1}^2(A)}\\&+\sum_{l=1}^{k-2}\alpha_l(x)\frac{\sigma_{k-3}(A|1i)\sigma_l(A)-\sigma_{l-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)}\Bigg]a_{1i}^2\\&=2\sum_{i=2}^{n}\frac{\partial(\frac{\sigma_k(A)}{\sigma_{k-1}(A)})}{\partial a_{1i}}a_{1i}-2\sum_{l=0}^{k-2}\sum_{i=2}^{n}\alpha_l(x)\frac{\partial(\frac{\sigma_l(A)}{\sigma_{k-1}(A)})}{\partial a_{1i}}a_{1i}\\&=2\sum_{i=2}^{n}\frac{\partial G(A)}{\partial a_{1i}}a_{1i}.\end{aligned}\tag{2.9}$$

This ends the proof. □

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1, which follows the idea of [19]. In fact, the calculations are more complicated.

Proof. Consider the auxiliary function

$$\phi(x,\xi)=\rho(x)g(u)\log(u_\xi(x)),$$

where

$$\rho(x)=r^2-|x|^2,\quad g(u)=\frac{1}{M}(M+u-\inf_{\mathrm{B_r}}u),\ M=\underset{B_r}{osc}\,u.$$

Suppose ϕ attains its maximum at $x = x_0 \in B_r$ and $\xi = e_1$. By rotating the coordinate $e_2, \ldots, e_n$, we can assume that

$$u_1(x_0) = |\nabla u|(x_0) > e,\ \{u_{ij}(x_0)\}_{2\leq i,j\leq n} \text{ is diagonal,} \tag{3.1}$$

and

$$u_{22} \geq u_{33} \geq \cdots \geq u_{nn}.$$

Then the function

$$\varphi(x) = \log\, \rho(x) + \log\, g(u) + \log\, \log\, u_1 \tag{3.2}$$

attains its local maximum at $x_0 \in B_r$. In the following, all the calculations are done at x_0. Direct calculation gives

$$A = \{a_{ij}\}_{n\times n} = \begin{pmatrix} \frac{1}{W^3}u_{11} & \frac{1}{W^2}u_{12} & \cdots & \frac{1}{W^2}u_{1n} \\ \frac{1}{W^2}u_{21} & \frac{1}{W}u_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{W^2}u_{n1} & 0 & \cdots & \frac{1}{W}u_{nn} \end{pmatrix}. \tag{3.3}$$

Hence

$$G^{ij} = \frac{\partial G}{\partial u_{ij}} = \frac{\partial G}{\partial a_{kl}}\frac{\partial a_{kl}}{\partial u_{ij}} = \begin{cases} \dfrac{1}{W^3}\dfrac{\partial G}{\partial a_{11}}, & i = j = 1; \\ \dfrac{1}{W^2}\dfrac{\partial G}{\partial a_{1i}}, & i \geq 2,\ j = 1; \\ \dfrac{1}{W}\dfrac{\partial G}{\partial a_{ii}}, & i = j \geq 2; \\ 0 \quad, & \text{otherwise.} \end{cases} \tag{3.4}$$

Differentiating the auxiliary function φ, we get

$$0 = \varphi_i = \frac{\rho_i}{\rho} + \frac{g_i}{g} + \frac{u_{1i}}{u_1 \log u_1}, \tag{3.5}$$

which shows

$$\frac{u_{11}}{u_1 \log u_1} = -(\frac{\rho_1}{\rho} + \frac{g_1}{g}), \tag{3.6}$$

$$\frac{u_{1i}}{u_1 \log u_1} = -\frac{\rho_i}{\rho},\ i \geq 2. \tag{3.7}$$

In the following, we always assume

$$u_1(x_0) \geq e^{16\frac{M}{r}},$$

otherwise the proof is done. Then we get

$$\begin{aligned} u_1(x_0)\rho(x_0) &\geq \frac{g(u(0))}{g(u(x_0))} \cdot \log|u_1(0)| \cdot \rho(0) \\ &\geq \frac{1}{2} \cdot 16\frac{M}{r} \cdot r^2 \\ &\geq 2 \cdot \frac{g(u(x_0))}{g'(u(x_0))} \cdot |\rho_1(x_0)|. \end{aligned} \tag{3.8}$$

Therefore, by (3.3), (3.6) and (3.8), we have

$$a_{11}=\frac{1}{W^3}u_{11}=\frac{1}{W^3}(u_1\mathrm{log}u_1)\big(-\frac{2g\rho_1+\rho g'u_1}{2\rho g}-\frac{g'}{2g}u_1\big)$$

$$\le -\frac{1}{W^3}(u_1\mathrm{log}u_1)\frac{g'}{2g}u_1<0. \tag{3.9}$$

Taking the second derivative to the auxiliary function φ gives

$$\varphi_{ij}=\frac{\rho_{ij}}{\rho}-\frac{\rho_i\rho_j}{\rho^2}+\frac{g_{ij}}{g}-\frac{g_ig_j}{g^2}+\frac{u_{1ij}}{u_1\mathrm{log}u_1}-\big(1+\frac{1}{\mathrm{log}u_1}\big)\frac{u_{1i}u_{1j}}{u_1^2\mathrm{log}u_1}. \tag{3.10}$$

Since G is elliptic, it follows from (3.5) that

$$0\ge G^{ij}\varphi_{ij}$$

$$=G^{ij}\Big[\frac{\rho_{ij}}{\rho}+2\frac{\rho_i}{\rho}\frac{g_j}{g}+\frac{g_{ij}}{g}\Big]+G^{ij}\Big[\frac{u_{1ij}}{u_1\mathrm{log}u_1}-\big(1+\frac{2}{\mathrm{log}u_1}\big)\frac{u_{1i}u_{1j}}{u_1^2\mathrm{log}u_1}\Big]$$

$$:=\mathcal{A}+\mathcal{B}. \tag{3.11}$$

We divide the lower estimation of (3.11) into two steps.

Step 1. Prove

$$\mathcal{A}\ge\big(-\frac{2}{W\rho}-\frac{4r}{M\rho}\frac{u_1}{W^3}\big)\sum_{i=1}^n\frac{\partial G}{\partial a_{ii}}+\frac{g'}{g}\Big[-\alpha(x)+\sum_{l=0}^{k-2}(k-l)\alpha_l(x)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big]. \tag{3.12}$$

Observe that

$$\mathcal{A}=-\frac{2}{\rho}\sum_{i=1}^nG^{ii}+2G^{11}\frac{\rho_1}{\rho}\frac{g'}{g}u_1+2\sum_{i=2}^nG^{1i}\frac{\rho_i}{\rho}\frac{g'}{g}u_1+\frac{g'}{g}G^{ij}u_{ij}. \tag{3.13}$$

To estimate the term $\mathcal{A}$, we note that

$$-\frac{2}{\rho}\sum_{i=1}^nG^{ii}=-\frac{2}{\rho}\big(\frac{\partial G}{\partial a_{11}}\frac{1}{W^3}+\sum_{i=2}^n\frac{\partial G}{\partial a_{ii}}\frac{1}{W}\big)\ge-\frac{2}{W\rho}\sum_{i=1}^n\frac{\partial G}{\partial a_{ii}}, \tag{3.14}$$

$$2G^{11}\frac{\rho_1}{\rho}\frac{g'}{g}u_1\ge-2\frac{\partial G}{\partial a_{11}}\frac{1}{W^3}\frac{|\rho_1|}{\rho}\frac{g'}{g}u_1\ge-\frac{4r}{M\rho}\frac{u_1}{W^3}\sum_{i=1}^n\frac{\partial G}{\partial a_{ii}}. \tag{3.15}$$

By Proposition 2.5 and (3.7), we deduce

$$2\sum_{i=2}^nG^{1i}\frac{\rho_i}{\rho}\frac{g'}{g}u_1=-2\sum_{i=2}^n\frac{\partial G}{\partial a_{1i}}\frac{1}{W^2}\frac{u_{1i}}{u_1\mathrm{log}u_1}\frac{g'}{g}u_1$$

$$=-2\sum_{i=2}^n\frac{\partial G}{\partial a_{1i}}a_{1i}\frac{1}{\mathrm{log}u_1}\frac{g'}{g}\ge 0. \tag{3.16}$$

By direct computation, we have

$$G^{ij}u_{ij}=\frac{\partial G}{\partial a_{ij}}a_{ij}=G_k(A)-\sum_{l=0}^{k-2}(k-1-l)G_l$$

$$= -\alpha(x) + \sum_{l=0}^{k-2}(k-l)\alpha_l(x)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}, \tag{3.17}$$

which implies

$$\frac{g'}{g}G^{ij}u_{ij} = -\frac{g'}{g}\alpha(x) + \frac{g'}{g}\sum_{l=0}^{k-2}(k-l)\alpha_l(x)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}. \tag{3.18}$$

Putting (3.14), (3.15), (3.16) and (3.18) into (3.13), we find (3.12), which finishes the proof of Step 1.

Step 2. Prove

$$\begin{aligned}\mathcal{B} \geq & \sum_{l=0}^{k-2}\frac{(\alpha_l)_1}{u_1\log u_1}\frac{\sigma_l(A)}{\sigma_{k-1}(A)} \\ & - \frac{(\alpha)_1}{u_1\log u_1} - \frac{u_1}{W^2}\Big(\frac{\rho_1}{\rho} + \frac{g_1}{g}\Big)\Big[-\alpha(x) + \sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big] \\ & + \frac{c_0}{128M^2}\frac{u_1^2\log u_1}{W^3}\sum_{i=1}^{n}\frac{\partial G}{\partial a_{ii}}.\end{aligned} \tag{3.19}$$

Differentiating equation (2.6) in the direction e_1 shows

$$\begin{aligned}\frac{\partial G}{\partial a_{ij}}\frac{\partial a_{ij}}{\partial x_1} = & \frac{\partial G}{\partial a_{11}}\frac{\partial a_{11}}{\partial x_1} + 2\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}\frac{\partial a_{1i}}{\partial x_1} + \sum_{i=2}^{n}\frac{\partial G}{\partial a_{ii}}\frac{\partial a_{ii}}{\partial x_1} \\ & - \sum_{l=0}^{k-2}(\alpha_l)_1\frac{\sigma_l(A)}{\sigma_{k-1}(A)} \\ = & -(\alpha)_1.\end{aligned} \tag{3.20}$$

From (3.3), we obtain (as in [5]),

$$\frac{\partial a_{11}}{\partial x_1} = \frac{1}{W^3}u_{111} - \frac{3u_1}{W^5}u_{11}^2 - \frac{2u_1}{W^3(1+W)}\sum_{k=2}^{n}u_{k1}^2, \tag{3.21}$$

$$\frac{\partial a_{1i}}{\partial x_1} = \frac{1}{W^2}u_{11i} - \frac{2u_1}{W^4}u_{11}u_{1i} - \frac{u_1}{W^2(1+W)}u_{1i}u_{ii} - \frac{u_1}{W^3(1+W)}u_{11}u_{i1}, \tag{3.22}$$

$$\frac{\partial a_{ii}}{\partial x_1} = \frac{1}{W}u_{ii1} - \frac{u_1}{W^3}u_{11}u_{ii} - \frac{2u_1}{W^2(1+W)}u_{1i}^2. \tag{3.23}$$

Therefore,

$$\begin{aligned}G^{ij}u_{ij1} = & \frac{\partial G}{\partial a_{11}}\frac{1}{W^3}u_{111} + 2\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}\frac{1}{W^2}u_{1i1} + \sum_{i=2}^{n}\frac{\partial G}{\partial a_{ii}}\frac{1}{W}u_{ii1} \\ = & \frac{\partial G}{\partial a_{11}}\frac{\partial a_{11}}{\partial x_1} + \frac{\partial G}{\partial a_{11}}\Big[\frac{3u_1}{W^5}u_{11}^2 + \frac{2u_1}{W^3(1+W)}\sum_{k=2}^{n}u_{k1}^2\Big] + 2\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}\frac{\partial a_{1i}}{\partial x_1}\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}\Big[\frac{2u_1}{W^4}u_{11}u_{1i}+\frac{u_1}{W^2(1+W)}u_{1i}u_{ii}+\frac{u_1}{W^3(1+W)}u_{11}u_{i1}\Big]\\
&+\sum_{i=2}^{n}\frac{\partial G}{\partial a_{ii}}\frac{\partial a_{ii}}{\partial x_1}+\sum_{i=2}^{n}\frac{\partial G}{\partial a_{ii}}\Big[\frac{u_1}{W^3}u_{11}u_{ii}+\frac{2u_1}{W^2(1+W)}u_{1i}^2\Big]. \qquad (3.24)
\end{aligned}
$$

Using (3.17), (3.20) and (3.24), we have

$$
\begin{aligned}
G^{ij}u_{ij1}=&\sum_{l=0}^{k-2}(\alpha_l)_1\frac{\sigma_l(A)}{\sigma_{k-1}(A)}-(\alpha)_1+\frac{u_{11}u_1}{W^2}\Big[-\alpha(x)+\sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big]\\
&+\frac{\partial G}{\partial a_{11}}\Big[\frac{2u_1}{W^5}u_{11}^2+\frac{2u_1}{W^3(1+W)}\sum_{k=2}^{n}u_{k1}^2\Big]+\sum_{i=2}^{n}\frac{\partial G}{\partial a_{ii}}\frac{2u_1}{W^2(1+W)}u_{1i}^2\\
&+2\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}\Big[\frac{u_1}{W^4}u_{11}u_{1i}+\frac{u_1}{W^2(1+W)}u_{1i}u_{ii}+\frac{u_1}{W^3(1+W)}u_{11}u_{i1}\Big]. \qquad (3.25)
\end{aligned}
$$

By direct computation, we get

$$
\begin{aligned}
&-\big(1+\frac{2}{\log u_1}\big)\frac{G^{ij}u_{1i}u_{1j}}{u_1^2\log u_1}\\
&=-\frac{1+\frac{2}{\log u_1}}{u_1^2\log u_1}\Big[\frac{\partial G}{\partial a_{11}}\frac{u_{11}^2}{W^3}+2\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}\frac{u_{11}u_{1i}}{W^2}+\sum_{i=2}^{n}\frac{\partial G}{\partial a_{ii}}\frac{u_{1i}^2}{W}\Big]. \qquad (3.26)
\end{aligned}
$$

Putting (3.25) and (3.26) into $\mathcal{B}$, one obtains

$$
\begin{aligned}
\mathcal{B}=&\sum_{l=0}^{k-2}\frac{(\alpha_l)_1}{u_1\log u_1}\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\\
&-\frac{(\alpha)_1}{u_1\log u_1}-\frac{u_1}{W^2}\big(\frac{\rho_1}{\rho}+\frac{g_1}{g}\big)\Big[-\alpha(x)+\sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big]\\
&+\frac{\partial G}{\partial a_{11}}\Big[\frac{2u_1^2}{W^5}u_{11}^2+\frac{2u_1^2}{W^3(1+W)}\sum_{k=2}^{n}u_{k1}^2-\big(1+\frac{2}{\log u_1}\big)\frac{u_{11}^2}{W^3}\Big]\frac{1}{u_1^2\log u_1}\\
&+2\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}\Big[\frac{u_1^2(2W+1)}{W^4(1+W)}u_{11}u_{1i}+\frac{u_1^2}{W^2(1+W)}u_{1i}u_{ii}\\
&\qquad-\big(1+\frac{2}{\log u_1}\big)\frac{u_{11}u_{1i}}{W^2}\Big]\frac{1}{u_1^2\log u_1}\\
&+\sum_{i=2}^{n}\frac{\partial G}{\partial a_{ii}}u_{1i}^2\Big[\frac{2u_1^2}{W^2(1+W)}-\big(1+\frac{2}{\log u_1}\big)\frac{1}{W}\Big]\frac{1}{u_1^2\log u_1}. \qquad (3.27)
\end{aligned}
$$

Providing that $u_1(x_0)$ is large enough, we can get

$$
\frac{2u_1^2}{W(1+W)}-\big(1+\frac{2}{\log u_1}\big)>\frac{1}{4}. \qquad (3.28)
$$

It follows from (3.28) that

$$
\begin{aligned}
\mathcal{B} \geq & \sum_{l=0}^{k-2} \frac{(\alpha_l)_1}{u_1 \log u_1} \frac{\sigma_l(A)}{\sigma_{k-1}(A)} \\
& - \frac{(\alpha)_1}{u_1 \log u_1} - \frac{u_1}{W^2}\big(\frac{\rho_1}{\rho} + \frac{g_1}{g}\big)\Big[- \alpha(x) + \sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big] \\
& + \frac{2}{\log u_1} \sum_{i=2}^{n} \frac{\partial G}{\partial a_{1i}} \frac{u_{1i}u_{ii}}{W^2(1+W)} \\
& + \Big[\frac{2u_1^2}{W^2} - \big(1 + \frac{2}{\log u_1}\big)\Big] \frac{u_{11}^2}{W^3 u_1^2 \log u_1} \frac{\partial G}{\partial a_{11}}.
\end{aligned} \tag{3.29}
$$

Let $\Upsilon := \{2 \leq i \leq n \,|\, a_{ii} \geq 0\}$. Recall that $a_{11} < 0$ and $\lambda(A) \in \Gamma_{k-1}$. Therefore, either $\sigma_k(A) \geq 0$ or $\sigma_k(A) < 0$ at point x_0. We will discuss into two cases.

Case 1. If $\sigma_k \geq 0$, then using Proposition 2.1 and Proposition 2.5, we estimate

(3.30)

$$
\begin{aligned}
& \sum_{i=2}^{n} \frac{\partial G}{\partial a_{1i}} u_{1i}u_{ii} \\
= & -\sum_{i=2}^{n} \Big[\frac{\sigma_{k-2}(A|1i)\sigma_{k-1}(A) - \sigma_{k-3}(A|1i)\sigma_k(A)}{\sigma_{k-1}^2(A)} \\
& + \sum_{l=1}^{k-2} \alpha_l(x) \frac{\sigma_{k-3}(A|1i)\sigma_l(A) - \sigma_{l-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)}\Big] a_{i1}u_{1i}u_{ii} \\
\geq & -\sum_{i\in\Upsilon}^{n} \Big[\frac{\sigma_{k-2}(A|1i)\sigma_{k-1}(A) - \sigma_{k-3}(A|1i)\sigma_k(A)}{\sigma_{k-1}^2(A)} \\
& + \sum_{l=1}^{k-2} \alpha_l(x) \frac{\sigma_{k-3}(A|1i)\sigma_l(A) - \sigma_{l-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)}\Big] a_{ii}\frac{u_{1i}^2}{W} \\
\geq & -\sum_{i\in\Upsilon} \Big[\frac{a_{ii}\sigma_{k-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} + \sum_{l=1}^{k-2}\alpha_l(x)\frac{a_{ii}\sigma_{k-3}(A|1i)\sigma_l(A)}{\sigma_{k-1}^2(A)}\Big]\frac{u_{1i}^2}{W} \\
\geq & -\sum_{i\in\Upsilon} \Big[\frac{C_{n-2}^{k-2}a_{22}\cdots a_{kk}\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} + \sum_{l=1}^{k-2}\alpha_l(x)\frac{C_{n-2}^{k-3}a_{22}\cdots a_{k-1k-1}\sigma_l(A)}{\sigma_{k-1}^2(A)}\Big]\frac{u_{1i}^2}{W} \\
\geq & -\Big[\frac{C_{n-2}^{k-2}\sigma_{k-1}(A|1)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} + \sum_{l=1}^{k-2}\alpha_l(x)\frac{C_{n-2}^{k-3}\sigma_{k-2}(A|1)\sigma_l(A)}{\sigma_{k-1}^2(A)}\Big]\sum_{i=2}^{n}\frac{u_{1i}^2}{W} \\
\geq & -\Big[C_{n-2}^{k-2}\frac{k(n-k+1)}{n}\frac{\sigma_{k-1}(A|1)\sigma_{k-1}(A) - \sigma_{k-2}(A|1)\sigma_k(A)}{\sigma_{k-1}^2(A)}
\end{aligned}
$$

$$+\sum_{l=1}^{k-2}\alpha_l(x)C_{n-2}^{k-3}\frac{(k-1)(n-l)}{n(k-l-1)}\frac{\sigma_{k-2}(A|1)\sigma_l(A)-\sigma_{l-1}(A|1)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)}\Big]\sum_{i=2}^{n}\frac{u_{1i}^2}{W}$$
$$\geq -C(n,k)\sum_{i=2}^{n}\frac{u_{1i}^2}{W}\frac{\partial G}{\partial a_{11}}$$
$$\geq -C(n,k)\frac{u_1^2\log^2 u_1}{W}\frac{|\nabla\rho|^2}{\rho^2}\frac{\partial G}{\partial a_{11}}.$$

Case 2. If $\sigma_k(A)<0$, then using the equation (2.6), we have

$$\frac{\sigma_{k-2}(A)}{\sigma_{k-1}(A)}\leq\frac{1}{\inf_{\mathrm{B_r}}\alpha_{k-2}}|\alpha(x_0)|. \tag{3.31}$$

From (2.7), (2.8) and (3.31), we obtain

$$C(n,k,|\alpha|_{C^0(B_r(0))})\geq\sum_{i=1}^{n}\frac{\partial G}{\partial a_{ii}}\geq\frac{\partial G}{\partial a_{11}}$$
$$\geq C(n,k,\inf_{\mathrm{B_r}}\alpha_{k-2})\big[\frac{\sigma_{k-1}^2(\widetilde{\lambda})}{\sigma_{k-1}^2(A)}+\frac{\sigma_{k-2}^2(\widetilde{\lambda})}{\sigma_{k-1}^2(A)}\big].$$

Hence,

$$\sum_{i=2}^{n}\frac{\partial G}{\partial a_{1i}}u_{1i}u_{ii}= -\sum_{i=2}^{n}\Big[\frac{\sigma_{k-2}(A|1i)\sigma_{k-1}(A)-\sigma_{k-3}(A|1i)\sigma_k(A)}{\sigma_{k-1}^2(A)}$$
$$+\sum_{l=1}^{k-2}\alpha_l(x)\frac{\sigma_{k-3}(A|1i)\sigma_l(A)-\sigma_{l-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)}\Big]a_{i1}u_{1i}u_{ii}$$
$$\geq -\sum_{i\in\Upsilon}^{n}\Big[\frac{\sigma_{k-2}(A|1i)\sigma_{k-1}(A)-\sigma_{k-3}(A|1i)\sigma_k(A)}{\sigma_{k-1}^2(A)}$$
$$+\sum_{l=1}^{k-2}\alpha_l(x)\frac{\sigma_{k-3}(A|1i)\sigma_l(A)-\sigma_{l-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)}\Big]a_{ii}\frac{u_{1i}^2}{W}$$
$$\geq -\sum_{i\in\Upsilon}\Big[\frac{a_{ii}\sigma_{k-2}(A|1i)}{\sigma_{k-1}(A)}+\alpha(x)\frac{a_{ii}\sigma_{k-3}(A|1i)}{\sigma_{k-1}(A)}\Big]\frac{u_{1i}^2}{W}$$
$$\geq -\sum_{i\in\Upsilon}\Big[\frac{C_{n-2}^{k-2}a_{22}\cdots a_{kk}}{\sigma_{k-1}(A)}+|\alpha(x)|\frac{C_{n-2}^{k-3}a_{22}\cdots a_{k-1k-1}}{\sigma_{k-1}(A)}\Big]\frac{u_{1i}^2}{W}$$
$$\geq -C(n,k,|\alpha|_{C^0(B_r)})\Big[\frac{\sigma_{k-1}(\widetilde{\lambda})}{\sigma_{k-1}(A)}+\frac{\sigma_{k-2}(\widetilde{\lambda})}{\sigma_{k-1}(A)}\Big]\sum_{i=2}^{n}\frac{u_{1i}^2}{W}$$
$$\geq -C(n,k,|\alpha|_{C^0(B_r)},\inf_{\mathrm{B_r}}\alpha_{k-2})\frac{u_1^2\log^2 u_1}{W}\frac{|\nabla\rho|^2}{\rho^2}\frac{\partial G}{\partial a_{11}}. \tag{3.32}$$

By using (3.9) and (3.28), we have

$$\left[\frac{2u_1^2}{W^2}-\big(1+\frac{2}{\log u_1}\big)\right]\frac{u_{11}^2}{W^3u_1^2\log u_1}\frac{\partial G}{\partial a_{11}}\geq\frac{1}{64M^2}\frac{u_1^2\log u_1}{W^3}\frac{\partial G}{\partial a_{11}}. \tag{3.33}$$

Suppose $\phi(x_0)$ is large enough so that

$$\frac{1}{128M^2}\frac{u_1^2\log u_1}{W^3}\geq 2C(n,k,|\alpha|_{C^0(B_r)},\inf_{\mathrm{B_r}}\alpha_{k-2})\frac{u_1^2\log u_1}{W^3(1+W)}\frac{|\nabla\rho|^2}{\rho^2}. \tag{3.34}$$

Otherwise, if

$$\frac{1}{128M^2}\frac{u_1^2\log u_1}{W^3}\leq 2C(n,k,|\alpha|_{C^0(B_r)},\inf_{\mathrm{B_r}}\alpha_{k-2})\frac{u_1^2\log u_1}{W^3(1+W)}\frac{|\nabla\rho|^2}{\rho^2},$$

we have

$$\rho^2\log^2u_1(x_0)\leq\rho^2u_1(x_0)\leq\rho^2W\leq\widetilde{C}^2M^2r^2.$$

Then

$$\log|\nabla u|(0)\leq\frac{g(x_0)}{\rho(0)g(0)}\rho(x_0)\log|\nabla u|(x_0)\leq C\frac{M}{r},$$

and the proof is complete. Inserting (3.30), (3.32), (3.33) and (3.34) into (3.29) and using Proposition 2.4, we obtain (3.19), which finishes the proof of Step 2.

It follows from (3.12) and (3.19) that

$$\begin{aligned}0\geq&\sum_{l=0}^{k-2}\frac{(\alpha_l)_1}{u_1\log u_1}\frac{\sigma_l(A)}{\sigma_{k-1}(A)}-\frac{u_1}{W^2}\frac{\rho_1}{\rho}\left[-\alpha(x)+\sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\right]\\&+\frac{1}{W^2}\frac{g'}{g}\left[-\alpha(x)+\sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\right]-\frac{(\alpha)_1}{u_1\log u_1}\\&+\left[\frac{c_0}{128M^2}\frac{u_1^2\log u_1}{W^3}-\frac{2}{W\rho}-\frac{4r}{M\rho}\frac{u_1}{W^3}\right]\sum_{i=1}^{n}\frac{\partial G}{\partial a_{ii}}.\end{aligned} \tag{3.35}$$

Using (3.35) and Lemma 3.1 below, we see that

$$0\geq\frac{c_0}{128M^2}\frac{u_1^2\log u_1}{W^3}-\frac{2}{W\rho}-\frac{4r}{M\rho}\frac{u_1}{W^3}-C\left[\frac{1}{W^2M}+\frac{1}{u_1\log u_1}+\frac{u_1}{W^2}\frac{r}{\rho}\right].$$

Then

$$\rho(x_0)\log u_1(x_0)\leq C(M^2+Mr+Mr^2+M^2r^2+M^2r)\leq C\big(M+Mr+r\big)^2.$$

Hence

$$\log|\nabla u|(0)\leq\frac{g(x_0)}{\rho(0)g(0)}\rho(x_0)\log|\nabla u|(x_0)\leq C\big(\frac{M}{r}+M+1\big)^2.$$

We thus complete the proof. □

Lemma 3.1. *Let*

$$\begin{aligned}&\alpha(x)\in C^1(B_r(0)),\\&0<\alpha_{k-2}(x)\in C^1(B_r(0)),\end{aligned}$$

$$0 \le \alpha_l(x) \in C^1(B_r(0)),\ 0 \le l \le k-3.$$

Then

$$\sum_{l=0}^{k-2} \frac{(\alpha_l)_1}{u_1 \log u_1} \frac{\sigma_l(A)}{\sigma_{k-1}(A)} - \frac{u_1}{W^2}\frac{\rho_1}{\rho}\Big[-\alpha(x) + \sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big]$$
$$+ \frac{1}{W^2}\frac{g'}{g}\Big[-\alpha(x) + \sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big] - \frac{(\alpha)_1}{u_1 \log u_1}$$
$$\ge -C\Big[\frac{1}{W^2 M} + \frac{1}{u_1 \log u_1} + \frac{u_1}{W^2}\frac{r}{\rho}\Big]\sum_{i=1}^n \frac{\partial G}{\partial a_{ii}}, \tag{3.36}$$

where C depends on $n,\ k,\ \|\alpha\|_{C^0(B_r)},\ \|\alpha_l\|_{C^1(B_r)}$ and $\inf_{B_r} \alpha_{k-2}$.

Proof. From direct computation, we obtain

$$\sum_{l=0}^{k-2} \frac{(\alpha_l)_1}{u_1 \log u_1} \frac{\sigma_l(A)}{\sigma_{k-1}(A)} - \frac{u_1}{W^2}\frac{\rho_1}{\rho}\Big[-\alpha(x) + \sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big]$$
$$+ \frac{1}{W^2}\frac{g'}{g}\Big[-\alpha(x) + \sum_{l=0}^{k-2}(k-l)(\alpha_l)\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big] - \frac{(\alpha)_1}{u_1 \log u_1}$$
$$\ge -C(n,\ k,\ \|\alpha\|_{C^0(B_r)},\ \|\alpha_l\|_{C^1(B_r)})\Big[\frac{1}{W^2 M} + \frac{1}{u_1 \log u_1} + \frac{u_1}{W^2}\frac{r}{\rho}$$
$$+ \frac{1}{u_1 \log u_1}\sum_{l=0}^{k-2}\frac{\sigma_l(A)}{\sigma_{k-1}(A)} + \frac{u_1}{W^2}\frac{r}{\rho}\sum_{l=0}^{k-2}\frac{\sigma_l(A)}{\sigma_{k-1}(A)}\Big]. \tag{3.37}$$

To get the desired estimate, we need to get a lower bound for $\sum_{i=1}^n \frac{\partial G}{\partial a_{ii}}$. By Proposition 2.2 and the straightforward computation, we have

$$\sum_{i=1}^n \frac{\partial G}{\partial a_{ii}} = \sum_{i=1}^n \frac{\sigma_{k-1}(A|i)\sigma_{k-1} - \sigma_{k-2}(A|i)\sigma_k}{\sigma_{k-1}^2}$$
$$+ \sum_{l=0}^{k-2}\sum_{i=1}^n \alpha_l \frac{\sigma_{k-2}(A|i)\sigma_l - \sigma_{l-1}(A|i)\sigma_{k-1}}{\sigma_{k-1}^2}$$
$$\ge \frac{n-k+1}{k} + \sum_{l=0}^{k-2}\alpha_l\Big(1 - \frac{l}{k-1}\Big)(n-k+2)\frac{\sigma_{k-2}\sigma_l}{\sigma_{k-1}^2}$$
$$\ge \frac{n-k+1}{k}. \tag{3.38}$$

We will discuss into two cases.

Case 1. If $\frac{\sigma_k}{\sigma_{k-1}} \leq \big(\sum\limits_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}\big)^{\frac{1}{k-1}}$, then we have

$$\frac{\sigma_{k-2}}{\sigma_{k-1}} \leq \frac{1}{\inf\limits_{\mathrm{B_r}} \alpha_{k-2}} \big(\frac{\sigma_k}{\sigma_{k-1}} + \alpha\big) \leq C(|\ \alpha\ |_{C^0(B_r)}, \inf\limits_{\mathrm{B_r}} \alpha_{k-2}) \big(\sum_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}\big)^{\frac{1}{k-1}}.$$

Case 2. If $\frac{\sigma_k}{\sigma_{k-1}} > \big(\sum\limits_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}\big)^{\frac{1}{k-1}}$, then, it follows from Proposition 2.2 that

$$\frac{\sigma_{k-2}}{\sigma_{k-1}} \leq C(n,k) \frac{\sigma_{k-1}}{\sigma_k} \leq C(n,k) \big(\sum_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}\big)^{\frac{-1}{k-1}}.$$

For $0 \leq l \leq k-2$, it follows from Proposition 2.2 that

$$\frac{\sigma_{k-2}}{\sigma_{k-1}} \geq \frac{k-1}{n-k+2} \big(\frac{C_n^{k-1}}{C_n^l}\big)^{\frac{1}{k-l-1}} \big(\frac{\sigma_l}{\sigma_{k-1}}\big)^{\frac{1}{k-l-1}}.$$

Hence, we have, in Case 1,

$$\begin{aligned} \frac{\sigma_l}{\sigma_{k-1}} &\leq C(n,k) \big(\frac{\sigma_{k-2}}{\sigma_{k-1}}\big)^{k-l-1} \\ &\leq C(n,k,|\ \alpha\ |_{C^0(B_r)}, \inf_{\mathrm{B_r}} \alpha_{k-2}) \big(\sum_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}\big)^{\frac{k-l-1}{k-1}} \\ &\leq C(n,k,|\ \alpha\ |_{C^0(B_r)}, \inf_{\mathrm{B_r}} \alpha_{k-2}) \sum_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}; \end{aligned} \tag{3.39}$$

and in Case 2,

$$\begin{aligned} \frac{\sigma_l}{\sigma_{k-1}} &\leq C(n,k) \big(\frac{\sigma_{k-2}}{\sigma_{k-1}}\big)^{k-l-1} \\ &\leq C(n,k) \big(\sum_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}\big)^{-\frac{k-l-1}{k-1}} \\ &\leq C(n,k) \sum_{i=1}^{n} \frac{\partial G}{\partial a_{ii}}. \end{aligned} \tag{3.40}$$

By (3.37), (3.39) and (3.40), we get (3.36). The proof is complete. □

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