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LIPSCHITZ TYPE CHARACTERIZATION OF FOCK TYPE SPACES

Hong Rae Cho and Jeong Min Ha

Abstract. For setting a general weight function on n dimensional complex space $\mathbb{C}^n$, we expand the classical Fock space. We define Fock type space $F^{p,q}_{\phi,t}(\mathbb{C}^n)$ of entire functions with a mixed norm, where $0 < p, q < \infty$ and $t \in \mathbb{R}$ and prove that the mixed norm of an entire function is equivalent to the mixed norm of its radial derivative on $F^{p,q}_{\phi,t}(\mathbb{C}^n)$. As a result of this application, the space $F^{p,p}_{\phi,t}(\mathbb{C}^n)$ is especially characterized by a Lipschitz type condition.

1. Introduction

Equivalent norms of an analytic function and its derivative have been extensively proved. It is known as a Littlewood-Paley formula, which is useful to analyze an analytic function space in the study of questions related to a derivative. In [8] and [9], mixed norm equivalence for an analytic function and its higher order derivative and mixed norm equivalence for an analytic function and derivative on the unit disc are proved, respectively. In [4], equivalence of mixed norms of an entire function and its derivative on a weighted Fock space on complex plane $\mathbb{C}$ is researched and Condition $\mathcal{L}$ for a weight function class is introduced. A Littlewood-Paley formula plays a key role in the proof of boundedness and compactness of an integral operator such as Volterra type operator. In [1] and [5], it is verified that two mixed norms of an entire function and its radial derivative are equivalent on n dimensional complex space $\mathbb{C}^n$. In addition, research has been conducted on Lipschitz type characterization for the Bergman space with a standard weight in terms of the Euclidean, hyperbolic, and pseudohyperbolic metrics on the unit disc in [10]. These results are generalized to the unit ball of $\mathbb{C}^n$ in [6] and [7]. Lipschitz type characterization of the classical and weighted Fock spaces on $\mathbb{C}^n$ have been proved in [2] and [3], respectively. For positive p, q, and real t, we define Fock type space $F^{p,q}_{\phi,t}(\mathbb{C}^n)$

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and find an equivalent mixed norm on $F^{p,q}_{\phi,t}(\mathbb{C}^n)$ in terms of a radial derivative. We then apply it to a Lipschitz type characterization of the space $F^{p,p}_{\phi,t}(\mathbb{C}^n)$.

Let dS be the surface measure on the unit sphere $\mathbb{S}^n$ of $\mathbb{C}^n$ with $S(\mathbb{S}^n) = 1$, and let the Lebesgue space $L^{p,q}_{\phi,t}(\mathbb{C}^n)$ consist of complex-valued measurable functions on $\mathbb{C}^n$, where

$$\|f\|_{p,q,\phi,t} = \left[\int_0^\infty M_q^p(r,f)\frac{r^{2n-1}e^{-p\phi(r)}}{(1+\phi'(r))^t}dr\right]^{1/p} < \infty,$$

and

$$M_q(r,f) = \left[\int_{\mathbb{S}^n} |f(r\zeta)|^q \, dS(\zeta)\right]^{1/q},$$

which is called the surface area integral mean of f. Let $H(\mathbb{C}^n)$ be the space of all entire functions on $\mathbb{C}^n$. We define Fock type space $F^{p,q}_{\phi,t}(\mathbb{C}^n)$ such that $F^{p,q}_{\phi,t}(\mathbb{C}^n) = L^{p,q}_{\phi,t}(\mathbb{C}^n) \cap H(\mathbb{C}^n)$. Now, we suggest Condition $\mathcal{L}$ which is slightly different from the condition for a radial weight in [4].

Definition (Condition $\mathcal{L}$)**.** Let $\phi : [0,\infty) \to \mathbb{R}^+$ be a twice continuously differentiable and increasing convex weight function. Let $C \in \mathbb{R}$. Then there exists $r_0 > 0$ such that

$$\phi'(r) \geq 1 \tag{1}$$

for $r \geq r_0$ and

$$\lim_{r\to\infty} \frac{1}{r}\left(\frac{r}{\phi'(r)}\right)' = 0, \tag{2}$$

$$\lim_{r\to\infty} \frac{\phi'(r + \frac{C}{1+\phi'(r)})}{\phi'(r)} = 1. \tag{3}$$

Remark 1.1. In this paper, we consider a convex weight function. In the case of a concave weight function, Lemma 2.3 and Theorem 1.5 are similarly proved.

Example 1.2. Functions satisfying Condition $\mathcal{L}$ are as follows:

- $\phi(r) = r^\alpha$ for $\alpha \geq 1$.
- $\phi(r) = e^{\beta r}$ for $\beta > 0$.
- $\phi(r) = e^{e^r}$.
- $\phi(r) = e^r r^\theta$ for $\theta \geq 0$.
- $\phi(r) = e^r \log(1+r)$.
- $\phi(r) = \int_0^r \frac{e^x}{x+1}dx$.

Given a function $f \in H(\mathbb{C}^n)$, we define radial derivative $\mathcal{R}f$ of f as follows:

$$\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

Theorem 1.3. *Let $0 < p, q < \infty$ and let $t \in \mathbb{R}$. If ϕ satisfies Condition $\mathcal{L}$, then*

$$\int_0^\infty M_q^p(r, f) \frac{r^{2n-1} e^{-p\phi(r)}}{(1+\phi'(r))^t} dr$$

is equivalent to

$$|f(0)|^p + \int_0^\infty \frac{M_q^p(r, \mathcal{R}f)}{(1+r\phi'(r))^p} \frac{r^{2n-1} e^{-p\phi(r)}}{(1+\phi'(r))^t} dr$$

for all $f \in H(\mathbb{C}^n)$.

When $t = 0$, we have the same result in [1]. Note that when $p = q$, in polar coordinates,

$$\int_{\mathbb{C}^n} \left| f(z) e^{-\phi(|z|)} \right|^p \frac{dV(z)}{(1+\phi'(|z|))^t} = 2n \int_0^\infty M_p^p(r, f) \frac{r^{2n-1} e^{-p\phi(r)}}{(1+\phi'(r))^t} dr.$$

We define $dG_{p,\phi,t}$ as t-weighted ϕ-Gaussian measure on $\mathbb{C}^n$ as follows:

$$dG_{p,\phi,t}(z) = C_{p,\phi,t}\, e^{-p\phi(|z|)} \frac{dV(z)}{(1+\phi'(|z|))^t},$$

where dV is the volume measure on $\mathbb{C}^n$ and $C_{p,\phi,t}$ is the normalizing constant. Fock type space $F_{\phi,t}^{p,p}(\mathbb{C}^n)$ norm is defined as

$$\|f\|_{F_{\phi,t}^{p,p}}^p = C_{p,\phi,t} \int_{\mathbb{C}^n} \left| f(z) e^{-\phi(|z|)} \right|^p \frac{dV(z)}{(1+\phi'(|z|))^t}.$$

The complex gradient of f at z is defined as

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right).$$

Then the mixed norm equivalence in Theorem 1.3 induces the following results.

Corollary 1.4. *Let $0 < p < \infty$ and $t \in \mathbb{R}$. If ϕ satisfies Condition $\mathcal{L}$, then the following norms*

$$\|f - f(0)\|_{F_{\phi,t}^{p,p}}, \qquad \left\| \frac{\nabla f(z)}{1+\phi'(|z|)} \right\|_{L^p(dG_{p,\phi,t})}$$

are comparable to each another for $f \in H(\mathbb{C}^n)$.

By using the norm equivalence in Corollary 1.4, we characterize the Fock type space $F_{\phi,t}^{p,p}(\mathbb{C}^n)$ with respect to the Lipschitz type condition as follows:

Theorem 1.5. *Let $0 < p < \infty$ and $s, t \in \mathbb{R}$. If ϕ satisfies Condition $\mathcal{L}$, then the following statements are equivalent for entire functions f on $\mathbb{C}^n$:*

(a) *$f \in F_{\phi,t}^{p,p}(\mathbb{C}^n)$.*

(b) *There exists a nonnegative continuous function* $g \in L^p(dG_{p,\phi,-sp+t})$ *such that*

$$\frac{|f(z)-f(w)|}{|z-w|} \le (1+\phi'(|z|)+\phi'(|w|))^{1+s}(g(z)+g(w))$$

for each $z, w \in \mathbb{C}^n$ *with* $z \neq w$.

We use notation $X \lesssim Y$ or $Y \gtrsim X$, respectively, for nonnegative quantities X and Y to indicate $X \le CY$ or $Y \le CX$ for some inessential constant $C > 0$. Similarly, we use notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries

The next lemma shows properties of a weight function ϕ from (2) of Condition $\mathcal{L}$.

Lemma 2.1 ([1], Lemma 2.3)**.** *Let* ϕ *be a function satisfying Condition* $\mathcal{L}$. *Then we have*

(a)

$$\lim_{r\to\infty} \frac{1}{r\phi'(r)} = 0.$$

(b)

$$\lim_{r\to\infty} \frac{\phi''(r)}{(\phi'(r))^2} = 0.$$

The following lemma is derived from Lemma 2.1.

Lemma 2.2 ([1], Lemma 2.4)**.** *Let* ϕ *be a function satisfying Condition* $\mathcal{L}$. *We define a sequence* $\{r_k\}_{k=-1}^{\infty}$ *as*

$$\phi(r_k) = k, \quad k \ge 0, \tag{4}$$

and $r_{-1} := 0$. *Then we have the following*:

(a)

$$\lim_{k\to\infty} \frac{r_{k+1}}{r_k} = 1.$$

(b) *For* $\varepsilon > 0$, *there exists* $k_0 \in \mathbb{N}$ *such that*

$$\left(\frac{e^{\phi(y)}}{e^{\phi(x)}}\right)^{-\varepsilon} \le \frac{\phi'(y)}{\phi'(x)} \le \left(\frac{e^{\phi(y)}}{e^{\phi(x)}}\right)^{\varepsilon}$$

for all $r_k \le x < y \le r_{k+1}$ *and for all* $k \ge k_0$,

For $\delta > 0$, a ball $E_\delta(z)$ is defined by

$$E_\delta(z) = \left\{ w \in \mathbb{C}^n : |w-z| < \frac{\delta}{1+\phi'(|z|)} \right\}. \tag{5}$$

We characterize some functions in terms of a weight function ϕ in $E_\delta(z)$.

Lemma 2.3. *Let $\delta > 0$ and $|z| = r$ for $z \in \mathbb{C}^n$. Then there exist $C_1, C_2 > 0$ and $r_0 > 0$ such that for $r \geq r_0$,*

$$C_1^{-1} \leq \frac{1+\phi'(|w|)}{1+\phi'(|z|)} \leq C_1, \tag{6}$$

and

$$C_2^{-1} \leq \frac{e^{\phi(|w|)}}{e^{\phi(|z|)}} \leq C_2 \tag{7}$$

for any $w \in E_\delta(z)$. In particular, the constants C_1 and C_2 are independent of z and w.

Proof. Let $w \in E_\delta(z)$. We assume that $z = r\zeta$ and $w = a\eta$ in $\mathbb{C}^n$, where $0 \leq r, a < \infty$ and $\zeta, \eta \in \mathbb{S}^n$. Then we need to show that there exists $C > 0$ such that

$$C^{-1} \leq \frac{\phi'(a)}{\phi'(r)} \leq C$$

for

$$r - \frac{\delta}{1+\phi'(r)} < a < r + \frac{\delta}{1+\phi'(r)}.$$

Since a given weight ϕ is under Condition $\mathcal{L}$, it follows that

$$\frac{\phi'(r - \frac{\delta}{1+\phi'(r)})}{\phi'(r)} \leq \frac{\phi'(a)}{\phi'(r)} \leq \frac{\phi'(r + \frac{\delta}{1+\phi'(r)})}{\phi'(r)}.$$

Thus, there exists $C > 0$ such that

$$C^{-1} \leq \frac{\phi'(a)}{\phi'(r)} \leq C$$

for

$$r - \frac{\delta}{1+\phi'(r)} < a < r + \frac{\delta}{1+\phi'(r)},$$

and for all $r \geq r_0$ by (3). Hence, the inequality (6) is proved. Moreover, for a weight ϕ satisfying Condition $\mathcal{L}$, the mean value theorem yields

$$|\phi(r) - \phi(a)| = |r - a|\phi'(x)$$

for some

$$r - \frac{\delta}{1+\phi'(r)} < x < r + \frac{\delta}{1+\phi'(r)}.$$

From (5) and (6), it follows that

$$\begin{aligned} |\phi(r) - \phi(a)| &\leq \frac{\delta\phi'(x)}{1+\phi'(r)} \\ &\leq C_1\delta, \end{aligned}$$

with the same constant C_1 in (6). Thus, we have

$$e^{-C_1\delta} \le \frac{e^{\phi(a)}}{e^{\phi(r)}} \le e^{C_1\delta}.$$

This gives (7), and the proof is complete. □

The following two inequalities are required to obtain the mixed norm equivalence of an entire function and its radial derivative.

Lemma 2.4 ([5], Lemma 2.1)**.** *If $f \in H(\mathbb{C}^n)$, $0 < q < \infty$, then there is a constant C_q such that*

$$M_q(r, \mathcal{R}f) \le C_q \frac{r}{\rho - r} M_q(\rho, f),$$

where $0 < r < \rho < \infty$.

Lemma 2.5 ([5], Lemma 2.2)**.** *If $f \in H(\mathbb{C}^n)$, $0 < q < \infty$ and $s = \min\{q, 1\}$, then there is a constant C such that*

$$M_q^s(\rho, f) - M_q^s(r, f) \le C \left(\frac{\rho - r}{r}\right)^s M_q^s(\rho, \mathcal{R}f),$$

where $0 < r < \rho < \infty$.

Lemma 2.6. *Let $\{A_k\}_{k=0}^{\infty}$ be a sequence of complex numbers and $0 < \gamma < \infty$. Set*

$$Q_1 = \sum_{k=0}^{\infty} |A_k|^\gamma r_k^{2n-1} (\phi'(r_k))^{-t-1} e^{-kp},$$

$$Q_2 = |A_0|^\gamma + \sum_{k=0}^{\infty} |A_{k+1} - A_k|^\gamma r_k^{2n-1} (\phi'(r_k))^{-t-1} e^{-kp}.$$

Then there is a positive constant $C = C(p)$ independent of $\{A_k\}_{k=0}^{\infty}$ such that $C^{-1} Q_1 \le Q_2 \le C Q_1$.

Proof. We prove this substituting $-t$ for Np of Lemma 2.5 in [1]. □

3. Equivalence of norms

We prove equivalence of mixed norms of an entire function and its radial derivative for a radial weight function ϕ under Condition $\mathcal{L}$.

Proposition 3.1. *Let $0 < p, q < \infty$ and $t \in \mathbb{R}$. If ϕ satisfies Condition $\mathcal{L}$, then*

$$\int_0^\infty M_q^p(r, f) \frac{r^{2n-1} e^{-p\phi(r)}}{(1 + \phi'(r))^t} dr$$

is equivalent to

$$M_q^p(r_0, f) + \int_0^\infty \frac{M_q^p(r, \mathcal{R}f)}{(1 + r\phi'(r))^p} \frac{r^{2n-1} e^{-p\phi(r)}}{(1 + \phi'(r))^t} dr$$

for all $f \in H(\mathbb{C}^n)$.

Proof. Let $\{r_k\}_{k=-1}^{\infty}$ be the sequence defined by (4). First, we suppose that

$$\int_0^\infty M_q^p(r,f)\frac{r^{2n-1}e^{-p\phi(r)}}{(1+\phi'(r))^t}dr<\infty.$$

We note (1). Then

$$\begin{aligned}L&:=\int_0^\infty M_q^p(r,f)\frac{r^{2n-1}e^{-p\phi(r)}}{(1+\phi'(r))^t}dr\\&\approx\sum_{k=0}^\infty\int_{r_k}^{r_{k+1}}M_q^p(r,f)\frac{r^{2n-1}e^{-p\phi(r)}}{(\phi'(r))^t}dr.\end{aligned}$$

The fact that $M_q^p(r,\mathcal{R}f)$ and r_k^{2n-1} are monotonically increasing yields

$$L\gtrsim\sum_{k=0}^\infty M_q^p(r_k,f)r_k^{2n-1}\int_{r_k}^{r_{k+1}}\frac{e^{-p\phi(r)}}{(\phi'(r))^t}dr.$$

By the mean value theorem for definite integrals, we have

$$\int_{r_k}^{r_{k+1}}\frac{e^{-p\phi(r)}}{(\phi'(r))^t}dr=\frac{1}{-p(\phi'(x_k))^{t+1}}\int_{r_k}^{r_{k+1}}(-p\phi'(r))e^{-p\phi(r)}dr \tag{8}$$

for some $r_k<x_k<r_{k+1}$. The definition of a function $\phi(r)$ yields

$$\int_{r_k}^{r_{k+1}}(-p\phi'(r))e^{-p\phi(r)}dr=e^{-kp}(e^{-p}-1). \tag{9}$$

By (6), we know

$$L\gtrsim\sum_{k=0}^\infty M_q^p(r_k,f)\frac{r_k^{2n-1}e^{-kp}}{(\phi'(r_k))^{t+1}}.$$

In addition, based on Lemma 2.1(a), we have $r\phi'(r)\to\infty$ as $r\to\infty$. Thus, we may assume $r\phi'(r)\ge 1$ for all $r\ge r_0$. It follows that

$$\begin{aligned}R&:=\int_0^\infty\frac{M_q^p(r,\mathcal{R}f)}{(1+r\phi'(r))^p}\frac{r^{2n-1}e^{-p\phi(r)}}{(1+\phi'(r))^t}dr\\&\approx\sum_{k=0}^\infty\int_{r_k}^{r_{k+1}}\frac{M_q^p(r,\mathcal{R}f)}{(r\phi'(r))^p}\frac{r^{2n-1}e^{-p\phi(r)}}{(\phi'(r))^t}dr.\end{aligned}$$

By considering the monotonicity of $M_q^p(r,f)$ and r_k^{2n-1} again, we have

$$R\lesssim\sum_{k=0}^\infty M_q^p(r_{k+1},\mathcal{R}f)\frac{r_{k+1}^{2n-1}}{r_k^p}\int_{r_k}^{r_{k+1}}\frac{e^{-p\phi(r)}}{(\phi'(r))^{p+t}}dr.$$

By applying the mean value theorem for finite integrals and (9),

$$\int_{r_k}^{r_{k+1}}\frac{e^{-p\phi(r)}}{(\phi'(r))^{p+t}}dr=\frac{e^{-kp}(e^{-p}-1)}{-p(\phi'(y_k))^{p+t+1}}$$

for some $r_k < y_k < r_{k+1}$. Hence, we get

$$R \lesssim \frac{e^{-p}-1}{-p}\sum_{k=0}^{\infty} M_q^p(r_{k+1},\mathcal{R}f)\frac{r_{k+1}^{2n-1}}{r_k^p}\frac{e^{-kp}}{(\phi'(y_k))^{p+t+1}}.$$

By Lemma 2.4, we obtain

$$R \lesssim \sum_{k=0}^{\infty} M_q^p(r_{k+2},f)\left(\frac{r_{k+1}}{r_{k+2}-r_{k+1}}\right)^p \frac{r_{k+1}^{2n-1}}{r_k^p}\frac{e^{-kp}}{(\phi'(y_k))^{p+t+1}}.$$

Moreover, both the definition of a function $\phi(r)$ and the mean value theorem yield

$$\frac{1}{r_{k+1}-r_k} = \frac{\phi(r_{k+1})-\phi(r_k)}{r_{k+1}-r_k} = \phi'(z_k) \tag{10}$$

for some $r_k < z_k < r_{k+1}$. Therefore, it follows

$$R \lesssim \sum_{k=0}^{\infty} M_q^p(r_{k+2},f)(r_{k+1}\phi'(z_{k+1}))^p \frac{r_{k+1}^{2n-1}}{r_k^p}\frac{e^{-kp}}{(\phi'(y_k))^{p+t+1}}.$$

Note that by Lemma 2.2, there exists $C > 0$ such that

$$C^{-1} \le \frac{r_{k+1}\phi'(z_{k+1})}{r_k\phi'(y_k)} \le C$$

for $r_k < y_k < r_{k+1} < z_{k+1} < r_{k+2}$. Again, by Lemma 2.2, we have

$$R \lesssim \sum_{k=0}^{\infty} M_q^p(r_{k+2},f)\frac{r_{k+2}^{2n-1}e^{-(k+2)p}}{(\phi'(r_{k+2}))^{t+1}}.$$

Thus, we obtain

$$M_q^p(r_0,f) + R \lesssim L.$$

Now, we assume that

$$R = \int_0^{\infty} \frac{M_q^p(r,\mathcal{R}f)}{(1+r\phi'(r))^p}\frac{r^{2n-1}}{(1+\phi'(r))^t}dr < \infty.$$

Since $M_q^p(r,\mathcal{R}f)$ and r_k^{2n-1} are monotonically increasing, we have

$$R \gtrsim \sum_{k=0}^{\infty} M_q^p(r_k,\mathcal{R}f)\frac{r_k^{2n-1}}{r_{k+1}^p}\int_{r_k}^{r_{k+1}} \frac{e^{-p\phi(r)}}{(\phi'(r))^{p+t}}dr.$$

Then the mean value theorem for definite integrals and (9) give

$$R \gtrsim \sum_{k=0}^{\infty} M_q^p(r_k,\mathcal{R}f)\frac{r_k^{2n-1}}{r_{k+1}^p}\frac{e^{-kp}}{(\phi'(y_k))^{p+t+1}}$$

for some $r_k < y_k < r_{k+1}$. Lemma 2.2 yields

$$R \gtrsim \sum_{k=0}^{\infty} M_q^p(r_k,\mathcal{R}f)\frac{r_k^{2n-p-1}e^{-kp}}{(\phi'(r_k))^{p+t+1}}.$$

Furthermore, the estimate by the mean value theorem for definite integrals as in (8) and the monotonicity of $M_q^p(r,f)$ and r_k^{2n-1} yield

$$L \lesssim \sum_{k=0}^{\infty} M_q^p(r_{k+1},f) r_{k+1}^{2n-1} \frac{1}{-p(\phi'(x_k))^{t+1}} \int_{r_k}^{r_{k+1}} (-p\phi'(r)) e^{-p\phi(r)} dr$$

for some $r_k < x_k < r_{k+1}$. By applying (9) and Lemma 2.2(b), we obtain

$$L \lesssim \sum_{k=1}^{\infty} M_q^p(r_k,f) \frac{r_k^{2n-1} e^{-kp}}{(\phi'(r_k))^{t+1}}.$$

To apply Lemma 2.5, we consider two cases: $0 < q \le 1$ and $q > 1$. First, let $0 < q \le 1$, $\gamma = p/q$, and $A_k = M_q^q(r_k,f)$. Then Lemma 2.6 yields

$$L \lesssim M_q^p(r_0,f) + \sum_{k=1}^{\infty} \left[M_q^q(r_{k+1},f) - M_q^q(r_k,f) \right]^{\frac{p}{q}} \frac{r_k^{2n-1} e^{-kp}}{(\phi'(r_k))^{t+1}}.$$

By Lemma 2.5, we have

$$L \lesssim M_q^p(r_0,f) + \sum_{k=1}^{\infty} \left[\left(\frac{r_{k+1}-r_k}{r_k} \right)^q M_q^q(r_{k+1},\mathcal{R}f) \right]^{\frac{p}{q}} \frac{r_k^{2n-1} e^{-kp}}{(\phi'(r_k))^{t+1}}.$$

From (10), we obtain

$$L \lesssim M_q^p(r_0,f) + \sum_{k=1}^{\infty} (\phi'(z_k))^{-p} M_q^p(r_{k+1},\mathcal{R}f) \frac{r_k^{2n-p-1} e^{-kp}}{(\phi'(r_k))^{t+1}}$$

for some $r_k < z_k < r_{k+1}$. It follows from Lemma 2.2 that

$$L \lesssim M_q^p(r_0,f) + R.$$

Let $q > 1$, $\gamma = p$, and $A_k = M_q(r_k,f)$. Then the proof is similar to the case where $0 < q \le 1$; thus, we complete the proof. □

By applying the open mapping theorem as Remark 1 in [1], we prove Theorem 1.3. When $p = q$, Theorem 1.3 is represented by Corollary 3.2.

Corollary 3.2. *Let* $0 < p < \infty$ *and* $t \in \mathbb{R}$. *If* ϕ *satisfies Condition* $\mathcal{L}$, *then*

$$\int_{\mathbb{C}^n} |f(z)|^p dG_{p,\phi,t}(z) \approx |f(0)|^p + \int_{\mathbb{C}^n} \frac{|\mathcal{R}f(z)|^p}{(1+|z|\phi'(|z|))^p} dG_{p,\phi,t}(z)$$

for $f \in H(\mathbb{C}^n)$.

We denote a multi-index $M = (m_1, \ldots, m_n)$, which is an n-tuple of nonnegative integers, we use notation $|M| = m_1 + \cdots + m_n$ and $\partial^M = \partial_1^{m_1} \cdots \partial_n^{m_n}$, where ∂_j denotes partial differentiation with respect to the j-th component.

Lemma 3.3. *Let* $0 < p < \infty$, $t \in \mathbb{R}$, *and* $\delta > 0$. *Given a multi-index* M, *there exists a positive constant* C *such that*

$$\frac{|\partial^M g(z)|^p e^{-p\phi(|z|)}}{(1+\phi'(|z|))^{p|M|+2n+t}} \le C \int_{E_\delta(z)} |g(w)|^p \frac{e^{-p\phi(|w|)} \, dV(w)}{(1+\phi'(|w|))^t}, \quad z \in \mathbb{C}^n,$$

for $g \in H(\mathbb{C}^n)$.

Proof. Let $0 < p < \infty$, $g \in H(\mathbb{C}^n)$, and $z \in \mathbb{C}^n$. By subharmonicity, we have

$$\begin{aligned} |g(z)|^p &\le \frac{1}{Vol[E_\delta(z)]} \int_{E_\delta(z)} |g(w)|^p \, dV(w) \\ &= \frac{(1+\phi'(|z|))^{2n}}{\omega_n \delta^{2n}} \int_{E_\delta(z)} |g(w)|^p \, dV(w), \end{aligned}$$

where ω_n is the volume of the unit ball of $\mathbb{C}^n$. Hence, the Cauchy estimate over a ball $E_{\delta/2}(z)$ implies that

$$|\partial^M g(z)|^p \le C(1+\phi'(|z|))^{p|M|+2n} \int_{E_\delta(z)} |g(w)|^p \, dV(w)$$

for some $C > 0$. We apply Lemma 2.3. This completes the proof. □

Lemma 3.4. *Fock type space* $F^{p,q}_{\phi,\,t}(\mathbb{C}^n)$ *contains constant functions.*

Proof. It is sufficient to prove that there exists $r_0 > 0$ such that

$$\int_{r_0}^{\infty} \frac{s^{2n-1} e^{-p\phi(s)}}{(\phi'(s))^t} ds < \infty \tag{11}$$

for a weight function ϕ under Condition $\mathcal{L}$. To prove (11), we note that for $r \ge 1$,

$$\begin{aligned} \int_{r_0}^{r} \frac{s^{2n-1} e^{-p\phi(s)}}{(\phi'(s))^t} \, ds &= \left[\frac{s^{2n-1} e^{-p\phi(s)}}{-p(\phi'(s))^{t+1}} \right]_{r_0}^{r} + \int_{r_0}^{r} \left(\frac{s^{2n-1}}{p(\phi'(s))^{t+1}} \right)' e^{-p\phi(s)} \, ds \\ &\le \frac{r_0^{2n-1} e^{-p\phi(r_0)}}{p(\phi'(r_0))^{t+1}} + \beta \int_{r_0}^{r} \frac{s^{2n-1} e^{-p\phi(s)}}{(\phi'(s))^t} \, ds \end{aligned} \tag{12}$$

for some $-1 < \beta < 1$. For the above inequality, we claim that

$$\begin{aligned} \left(\frac{s^{2n-1}}{p(\phi'(s))^{t+1}} \right)' &= \frac{s^{2n-1}}{(\phi'(s))^t} \left[\frac{2n-1}{p} \frac{1}{s\phi'(s)} - \frac{t+1}{p} \frac{\phi''(s)}{(\phi'(s))^2} \right] \\ &\le \beta \frac{s^{2n-1}}{(\phi'(s))^t} \end{aligned}$$

for some $-1 < \beta < 1$ as follows: By Lemma 2.1(a), we consider that for any sufficiently small $\epsilon_1 > 0$, there exists r_0 such that

$$-\epsilon_1 < \frac{1}{s\phi'(s)} < \epsilon_1 \tag{13}$$

for all $s > r_0$. Moreover, we have

$$\frac{1}{s} \left(\frac{s}{\phi'(s)} \right)' = \frac{1}{s\phi'(s)} - \frac{\phi''(s)}{(\phi'(s))^2}.$$

Hence, (2) for Condition $\mathcal{L}$ implies that for any sufficiently small $\epsilon_2 > 0$, there exists r_0 such that

$$\frac{1}{s\phi'(s)} - \epsilon_2 < \frac{\phi''(s)}{(\phi'(s))^2} < \frac{1}{s\phi'(s)} + \epsilon_2 \tag{14}$$

for all $s > r_0$. Together with (13) and (14), we consider ϵ_1 and ϵ_2 that satisfy the following inequalities:

$$\begin{aligned}\left|\frac{2n-1}{p}\frac{1}{s\phi'(s)} - \frac{t+1}{p}\frac{\phi''(s)}{(\phi'(s))^2}\right| &\le \frac{2n-1}{p}\frac{1}{s\phi'(s)} + \frac{|t+1|}{p}\frac{\phi''(s)}{(\phi'(s))^2}\\ &< \frac{2n-1}{p}\frac{1}{s\phi'(s)} + \frac{|t+1|}{p}\left(\frac{1}{s\phi'(s)} + \epsilon_2\right)\\ &< \frac{2n-1+|t+1|}{p}\epsilon_1 + \frac{|t+1|}{p}\epsilon_2\\ &< 1.\end{aligned}$$

Therefore, we prove that there exists r_0 such that

$$\left|\frac{2n-1}{p}\frac{1}{s\phi'(s)} - \frac{t+1}{p}\frac{\phi''(s)}{(\phi'(s))^2}\right| < 1$$

for any $s > r_0$. Hence, the inequality in (12) holds.

As r approaches infinity in (12), we know

$$\int_{r_0}^{\infty} \frac{s^{2n-1}}{(\phi'(s))^t} e^{-p\phi(s)} ds \le \frac{1}{1-\beta}\frac{r_0^{2n-1}e^{-p\phi(r_0)}}{p(\phi'(r_0))^{t+1}}$$

for some $-1 < \beta < 1$. Hence, it follows that

$$\int_0^{\infty} \frac{r^{2n-1}e^{-p\phi(r)}}{(1+\phi'(r))^t} dr < \infty.$$

Thus, the proof is complete. □

Corollary 3.5. *Let $0 < p < \infty$ and $t \in \mathbb{R}$. Then the following norms*

$$\|f - f(0)\|_{F^{p,p}_{\phi,\,t}}, \quad \left\|\frac{\mathcal{R}f(z)}{1+|z|\phi'(|z|)}\right\|_{L^p(dG_{p,\,\phi,\,t})}, \quad \left\|\frac{\nabla f(z)}{1+\phi'(|z|)}\right\|_{L^p(dG_{p,\,\phi,\,t})}$$

are comparable to one another for $f \in H(\mathbb{C}^n)$.

Proof. First, we claim that the $F^{p,p}_{\phi,\,t}(\mathbb{C}^n)$-norm of $f - f(0)$ is equivalent to the norm of the product of its radial derivative and $\frac{1}{1+|z|\phi'(|z|)}$. By Lemma 3.4, we have that constant functions are contained in $F^{p,p}_{\phi,\,t}(\mathbb{C}^n)$. By applying Theorem 1.3 to $f - f(0)$, we obtain

$$\int_{\mathbb{C}^n} |f(z) - f(0)|^p \frac{e^{-p\phi(|z|)}dV(z)}{(1+\phi'(|z|))^t} \approx \int_{\mathbb{C}^n} \frac{|\mathcal{R}f(z)|^p}{(1+|z|\phi'(|z|))^p}\frac{e^{-p\phi(|z|)}dV(z)}{(1+\phi'(|z|))^t}.$$

Second, we prove that $F^{p,p}_{\phi,\,t}(\mathbb{C}^n)$-norm of $f-f(0)$ is equivalent to the norm of the product of its complex gradient ∇f and $\frac{1}{1+\phi'(|z|)}$. Note that $|\mathcal{R}f(z)| \le |z||\nabla f(z)|$. Hence, we have

$$\|f-f(0)\|_{F^{p,p}_{\phi,\,t}} \lesssim \left\| \frac{\nabla f(z)}{1+\phi'(|z|)} \right\|_{L^p(dG_{p,\,\phi,\,t})}.$$

Furthermore, Lemma 3.3 yields

$$\frac{|\nabla f(z)|^p}{(1+\phi'(|z|))^p} \lesssim (1+\phi'(|z|))^{2n} \int_{E_\delta(z)} |f(w)-f(0)|^p \, dV(w).$$

Let $\delta > 0$ and $\delta_1 := 2C_1\delta$, where C_1 is the same constant as in Lemma 2.3. Then it follows $E_\delta(z) \subset E_{\delta_1}(w)$ for $w \in E_\delta(z)$ since we have

$$\frac{\delta}{1+\phi'(|z|)} < \frac{C_1\delta}{1+\phi'(|w|)}. \tag{15}$$

By integrating both sides of the above inequality against $dG_{p,\,\phi,\,t}$ and applying Fubini's theorem, we have

$$\begin{aligned} &\left\| \frac{\nabla f(z)}{1+\phi'(|z|)} \right\|^p_{L^p(dG_{p,\,\phi,\,t})} \\ &\lesssim \int_{\mathbb{C}^n} \int_{E_{\delta_1}(w)} |f(w)-f(0)|^p \, (1+\phi'(|z|))^{2n} \, \frac{e^{-p\phi(|z|)} dV(z)}{(1+\phi'(|z|))^t} \, dV(w). \end{aligned}$$

Lemma 2.3 yields

$$\left\| \frac{\nabla f(z)}{1+\phi'(|z|)} \right\|^p_{L^p(dG_{p,\,\phi,\,t})} \lesssim \|f-f(0)\|^p_{F^{p,p}_{\phi,\,t}}.$$

This proves the desired result. □

4. Lipschitz type characterization

Proof of Theorem 1.5. First, we assume that (b) holds. By fixing z and taking limits $w \to z$ along the directions parallel to the coordinate axes, we have

$$|\partial_j f(z)| \lesssim (1+\phi'(|z|))^{1+s} g(z)$$

for each j. Thus, we get

$$\frac{|\nabla f(z)|}{1+\phi'(|z|)} \lesssim (1+\phi'(|z|))^s g(z), \quad z \in \mathbb{C}^n,$$

and hence, we obtain

$$\int_{\mathbb{C}^n} |\nabla f(z)|^p \frac{e^{-p\phi(|z|)} dV(z)}{(1+\phi'(|z|))^{p+t}} \lesssim \int_{\mathbb{C}^n} |g(z)|^p \, \frac{e^{-p\phi(|z|)} dV(z)}{(1+\phi'(|z|))^{-sp+t}}.$$

By Corollary 3.5, we have $g(z) \in L^p(G_{p,\,\phi,\,-sp+t})$, and thus $f \in F^{p,p}_{\phi,\,t}(\mathbb{C}^n)$.

Second, we assume that (a) holds. Fix any $\delta > 0$. We set

$$\Omega_\delta := \{(z,w) : |w-z|(1+\phi'(|z|)+\phi'(|w|)) < \delta\}.$$

Let $(z,w)\in\Omega_\delta$. Then we have $w\in E_\delta(z)$ and

$$1+\phi'(|z|)+\phi'(|w|)\approx 1+\phi'(|z|),$$

by Lemma 2.3. From the fundamental theorem of calculus, it follows

$$|f(z)-f(w)|\le|z-w|\int_0^1|\nabla f(\rho z+(1-\rho)w)|\,d\rho.$$

A line $\rho z+(1-\rho)w$ exists in $E_\delta(z)$ and it yields

$$|f(z)-f(w)|\le|z-w|\sup_{\zeta\in E_\delta(z)}|\nabla f(\zeta)|. \tag{16}$$

We note that

$$|\nabla f(\zeta)|\approx(1+\phi'(|z|)+\phi'(|w|))^{1+s}\frac{|\nabla f(\zeta)|}{(1+\phi'(|\zeta|))^{1+s}} \tag{17}$$

for $\zeta\in E_\delta(z)$. Let

$$h_s(z):=\sup_{\zeta\in E_\delta(z)}\frac{|\nabla f(\zeta)|}{(1+\phi'(|\zeta|))^{1+s}}.$$

By using (16) and (17), we obtain

$$|f(z)-f(w)|\lesssim|z-w|(1+\phi'(|z|)+\phi'(|w|))^{1+s}(h_s(z)+h_s(w))$$

for $(z,w)\in\Omega_\delta$.

Next, let $(z,w)\notin\Omega_\delta$. Then $|z-w|(1+\phi'(|z|)+\phi'(|w|))\ge\delta$. Hence, it follows

$$|f(z)-f(w)|\lesssim\frac{|z-w|(1+\phi'(|z|)+\phi'(|w|))^{1+s}}{\delta}\left(\frac{|f(z)|}{(1+\phi'(|z|))^s}+\frac{|f(w)|}{(1+\phi'(|w|))^s}\right)$$

for $s\in\mathbb{R}$. Let

$$g(z):=h_s(z)+\frac{|f(z)|}{\delta(1+\phi'(|z|))^s}$$

for $z\in\mathbb{C}^n$. Then we have

$$|f(z)-f(w)|\lesssim|z-w|(1+\phi'(|z|)+\phi'(|w|))^{1+s}(g(z)+g(w))$$

for each $z,w\in\mathbb{C}^n$ with $z\ne w$. Note that a function $g(z)$ is continuous on $\mathbb{C}^n$. However, it remains to prove that the function $g(z)$ is in $L^p(G_{p,\phi,-sp+t})$. Since $\frac{|f(z)|}{\delta(1+\phi'(|z|))^s}$ belongs to $L^p(G_{p,\phi,-sp+t})$ for $f\in F^{p,p}_{\phi,t}(\mathbb{C}^n)$, we claim that h_s exists in $L^p(G_{p,\phi,-sp+t})$. For $\zeta\in E_\delta(z)$, we have $E_{\delta_1}(\zeta)\subset E_{\delta_2}(z)$ by (15), where $\delta_1=2C_1\delta$ and $\delta_2=2C_1\delta_1$. From Lemma 3.3 and (6), we have

$$\frac{|\nabla f(\zeta)|^p}{(1+\phi'(|\zeta|))^{(1+s)p}}\lesssim(1+\phi'(|z|))^{2n-sp}\int_{E_{\delta_2}(z)}|f(w)|^p\,dV(w).$$

By taking the supremum over $\zeta\in E_\delta(z)$, we get

$$|h_s(z)|^p\lesssim(1+\phi'(|z|))^{2n-sp}\int_{E_{\delta_2}(z)}|f(w)|^p\,dV(w)$$

for all $z \in \mathbb{C}^n$. Let $\delta_3 = 2C_1\delta_2$. By integrating both sides of the above inequality against the measure $dG_{p,\phi,-sp+t}(z)$ and applying Fubini's theorem, we obtain

$$\|h_s\|^p_{L^p(G_{p,\phi,-sp+t})} \lesssim \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}(1+\phi'(|z|))^{2n}|f(w)|^p\chi_{E_{\delta_2}(z)}(w)\,\frac{e^{-p\phi(|z|)}dV(z)}{(1+\phi'(|z|))^t}\,dV(w),$$

where χ denotes the characteristic function in its subscripted set. For $z \in E_{\delta_3}(w)$, we apply Lemma 2.3, and it yields

$$\begin{aligned}\|h_s\|^p_{L^p(G_{p,\phi,-sp+t})} &\lesssim \int_{\mathbb{C}^n}|f(w)|^p(1+\phi'(|w|))^{2n}\frac{\omega_n\delta_3^{2n}}{(1+\phi'(|w|))^{2n}}\,\frac{e^{-p\phi(|w|)}dV(w)}{(1+\phi'(|w|))^t}\\ &\lesssim \|f\|^p_{F^{p,p}_{\phi,t}}.\end{aligned}$$

Thus, the proof is completed. □

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Hong Rae Cho
Department of Mathematics
Pusan National University
Pusan 46241, Korea
Email address: chohr@pusan.ac.kr

Jeong Min Ha
Department of Mathematics
Pusan National University
Pusan 46241, Korea
Email address: jm.ha@pusan.ac.kr