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THE H^1-UNIFORM ATTRACTOR FOR THE 2D NON-AUTONOMOUS TROPICAL CLIMATE MODEL ON SOME UNBOUNDED DOMAINS

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Abstract. In this paper, we study the uniform attractor of the 2D non-autonomous tropical climate model in an arbitrary unbounded domain on which the Poincaré inequality holds. We prove that the uniform attractor is compact not only in the L^2-spaces but also in the H^1-spaces. Our proof is based on the concept of asymptotical compactness. Finally, for the quasiperiodical external force case, the dimension estimates of such a uniform attractor are also obtained.

1. Introduction

In the present paper, we consider the following two-dimensional (2D) tropical climate model in an open bounded or unbounded set $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla p + \nabla\cdot(v\otimes v) = f^1, \\ \partial_t v + (u\cdot\nabla)v - \nu\Delta v + \nabla\theta + (v\cdot\nabla)u = f^2, \\ \partial_t \theta + (u\cdot\nabla)\theta - \eta\Delta\theta + \nabla\cdot v = f^3, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where $u = (u^1(x,t), u^2(x,t))$, $v = (v^1(x,t), v^2(x,t))$ are the barotropic mode and the first baroclinic mode of the velocity, respectively; $\theta = \theta(x,t)$ and $p = p(x,t)$ represent the scalar temperature and the scalar pressure. Here $v\otimes v$ is the standard tensor notation, i.e., $v\otimes v = (v^i v^j)_{1\le i,j\le 2}$. μ, ν, η are nonnegative constants where μ, ν are the viscosities and η is the thermal diffusivity. In the present paper, we consider $\mu=\nu=\eta=1$ and the (non-slip) boundary conditions $u|_{\partial\Omega}=0$, $v|_{\partial\Omega}=0$, $\theta|_{\partial\Omega}=0$.

The tropical climate model ($\mu=\nu=\eta=0$) was originally derived by Frierson-Majda-Pauluis [14]. The first baroclinic mode v of (1.1) was used

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in some studies of large-scale dynamics of precipitation fronts in the tropical atmosphere.

The tropical climate model is related to other equations in fluid mechanics. If we let θ be a constant function, the tropical climate model is similar to the magnetohydrodynamics (MHD) equations. If $v = 0$, the tropical climate model is analogous to the Boussinesq equations. If $v = 0$, $\theta = 0$, then (1.1) reduces to the classical incompressible Navier-Stokes equations. This kind of model is worth being studied and has attracted a lot of attentions recently. H. O. Bae and B. J. Jin in [4–8] considered the case $v = 0$, $\theta = 0$ in (1.1), and obtained many important temporal-spatial decay results in some classical domains. J. Li and E. Titi [25] established the global well-posedness of strong solutions with H^1 initial data for the Cauchy problem of (1.1) when $\mu > 0$, $\nu > 0$, $\eta = 0$. Under some smallness assumptions, R. Wan [34] proved the global well-posedness to (1.1) with $\mu = 0$, $\nu > 0$, $\eta = 0$. For tropical climate model with fractional dissipation, $\alpha, \mu, \nu, \eta > 0$, Z. Ye [36] studied the global regularity to (1.1) and obtained

$$\|(u, v, \theta)\|^2_{H^s(\mathbb{R}^2)} + \int_0^t [\|u\|^2_{H^{s+\alpha}(\mathbb{R}^2)} + \|v\|^2_{H^{s+1}(\mathbb{R}^2)} + \|\theta\|^2_{H^{s+1}(\mathbb{R}^2)}] d\tau \leq C.$$

If $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2), s > 2$, B. Dong, J. Wu, and Z. Ye [12] studied the global existence and regularity of weak solutions with fractional dissipation. Recently, H. Li and Y. Xiao [26] obtained for the tropical climate model (1.1) that

$$t^{\frac{s}{2}} \|(u, v, \theta)\|_{H^s(\mathbb{R}^2)} = 0 \quad \text{as } t \to +\infty,$$

when $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$. In [35], letting $(u_0, v_0, \theta_0) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, H. Xie and Z. Zhang mainly studied the rate of decay to n-dimensional ($n \geq 3$) problem of (1.1) with $\mu > 0$, $\nu > 0$, $\eta > 0$.

If the external force does not decay to zero (e.g. the forcing term is a quasiperiodic function), then the solutions may not decay to zero. However, the long time behavior of dynamic systems can be described in terms of attractors.

For autonomous fluid dynamic systems, the theory of global attractors of problems in bounded domains has been widely studied by many scholars (see [11, 24, 33]). However, some additional conditions need to be added when we study the global attractors of dynamic systems in unbounded domains because of the lack of compactness. If the forcing term lies in some weighted Sobolev spaces, F. Abergel [1] and A. V. Babin [3] obtained an existence result of the global attractor of the 2D Navier-Stokes equations. If the forcing term does not belong to any weighted Sobolev spaces but the Poincaré inequality is verified, R. Rosa [30] and N. Ju [22] showed the existence of the global L^2-attractor and the global H^1-attractor of the 2D Navier-Stokes equations, respectively. The dimension estimates of global attractors were studied in [30].

For non-autonomous dynamic systems, the concept of uniform attractors was firstly introduced by A. Haraux [18]. The systematic study of uniform attractors of the 2D Navier-Stokes equations in bounded domains was given by V. V. Chepyzhov and M. I. Vishik [10]. V. V. Chepyzhov and M. A. Efendiev [9] studied the finite dimensionality of the L^2-uniform attractor of a non-autonomous system in an unbounded strip domain. S. Lu, H. Wu, C. Zhong [28] and D. Gong, H. Song, C. Zhong [17] proved the existence of the H^1-uniform attractor of the 2D Navier-Stokes equations in a bounded domain. Recently, C. Ai, Z. Tan and J. Zhou [2] derived the existence of a uniform attractor of the 2D MHD equations in a smooth bounded domain.

Compared to the bounded domain case, the uniform attractor of the 2D Navier-Stokes equations in an unbounded domain is less well-understood. If the forcing term vanishes, I. Moise, R. Rosa and X. Wang [29] derived the existence of the L^2-uniform attractor of a noncompact system in an infinite strip domain. If the forcing term does not lie in any weighted Sobolev spaces but the Poincaré inequality holds, Y. Hou and K. Li [21] studied the existence of the L^2-uniform attractor of the 2D Navier-Stokes equations in some unbounded domains and established the estimates of the Hausdorff dimension of the uniform attractor for the quasiperiodic force case.

This paper is organized as follows. In Section 2, we shall introduce some function spaces and some operators. In Section 3, we will prove the existence and uniqueness for weak solutions and further define operators $\{U_f(t,\tau)\}$. In Section 4, we will recall the theory of semiprocesses. In Section 5, we shall study uniformly absorbing sets together with the H^1-uniformly asymptotical compactness, and obtain the H^1-uniform attractor of (1.1). The dimension estimates of the uniform attractor will be given in the last section.

2. Function spaces and weak formulation

Let Ω be an open subset of $\mathbb{R}^2$, either bounded or unbounded. The spaces we shall use in this paper are combinations of those used for the Navier-Stokes equations and the usual Sobolev spaces. For a Hilbert space $X(=L^2(\Omega)$ or $H^1(\Omega))$, we do not distinguish the inner products on X and on $[X]^2 := X \times X$, which will be denoted by $(\cdot,\cdot)_X$.

We always assume that the following Poincaré inequality holds on Ω:

$$\|\varphi\|_{L^2} \le \lambda_1^{-\frac{1}{2}} \|\nabla\varphi\|_{L^2} \quad \text{for all } \varphi \in H_0^1(\Omega), \tag{2.1}$$

where $\lambda_1 > 0$ is a positive constant.

Let

$$\begin{aligned} \mathscr{V} &= \{\varphi \in [C_c^\infty(\Omega)]^2 \mid \nabla\cdot\varphi = 0\}, \\ H &= \text{the closure of } \mathscr{V} \text{ in } [L^2(\Omega)]^2, \\ V &= \text{the closure of } \mathscr{V} \text{ in } [H_0^1(\Omega)]^2, \\ \widehat{V} &= \{u \in [H_0^1(\Omega)]^2 \mid \nabla\cdot u = 0\}. \end{aligned}$$

Obviously, for any domain Ω, the inclusion $V \subset \widehat{V}$ holds. However, V and $\widehat{V}$ can be different for some domains (see [15, 19]). In this paper, we assume that Ω satisfies

$$V = \widehat{V}. \tag{2.2}$$

We equip $V, H_0^1 (=H_0^1(\Omega)$ or $[H_0^1(\Omega)]^2)$ with the following inner products

$$(u_1, u_2)_V = \sum_{i=1}^{2} \left(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i} \right)_{L^2} \quad \text{for all } u_1, u_2 \in V,$$
$$(v_1, v_2)_{H_0^1} = \sum_{i=1}^{2} \left(\frac{\partial v_1}{\partial x_i}, \frac{\partial v_2}{\partial x_i} \right)_{L^2} \quad \text{for all } v_1, v_2 \in H_0^1.$$

Thanks to (2.1), equipped with such inner products, $V, H_0^1(\Omega), [H_0^1(\Omega)]^2$ are Hilbert spaces.

Now we introduce

$$\mathbb{H} = H \times [L^2(\Omega)]^2 \times L^2(\Omega),$$
$$\mathbb{V} = V \times [H_0^1(\Omega)]^2 \times H_0^1(\Omega),$$

and equip $\mathbb{H}$ and $\mathbb{V}$ with the following inner products, respectively,

$$(\varphi_1, \varphi_2)_{\mathbb{H}} = (u_1, u_2)_H + (v_1, v_2)_{L^2} + (\theta_1, \theta_2)_{L^2} \quad \text{for all } \varphi_i = (u_i, v_i, \theta_i) \in \mathbb{H},$$
$$(\varphi_1, \varphi_2)_{\mathbb{V}} = (u_1, u_2)_V + (v_1, v_2)_{H_0^1} + (\theta_1, \theta_2)_{H_0^1} \quad \text{for all } \varphi_i = (u_i, v_i, \theta_i) \in \mathbb{V}.$$

$\mathbb{H}$ is a Hilbert space with the norm $\|\varphi\|_{\mathbb{H}} = \sqrt{(\varphi, \varphi)_{\mathbb{H}}}$, and $\mathbb{V}$ is a Hilbert space with the norm $\|\varphi\|_{\mathbb{V}} = \sqrt{(\varphi, \varphi)_{\mathbb{V}}}$. If we identify $\mathbb{H}$ with its dual $\mathbb{H}'$, then

$$\mathbb{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathbb{V}',$$

where each space is dense and can be continuously embedded into the following one.

We define two linear bounded operators $A \in \mathcal{L}(V, V')$ and $\mathbb{A} \in \mathcal{L}(\mathbb{V}, \mathbb{V}')$ by setting

$$\langle Au_1, u_2 \rangle_{V', V} = (u_1, u_2)_V \quad \text{for all } u_1, u_2 \in V,$$
$$\langle \mathbb{A}\varphi_1, \varphi_2 \rangle_{\mathbb{V}', \mathbb{V}} = (\varphi_1, \varphi_2)_{\mathbb{V}} \quad \text{for all } \varphi_1, \varphi_2 \in \mathbb{V}.$$

Obviously,

$$\|Au\|_{V'} \leq \|u\|_V, \quad \|\mathbb{A}\varphi\|_{\mathbb{V}'} \leq \|\varphi\|_{\mathbb{V}}, \tag{2.3}$$

and

$$\|f\|_{\mathbb{V}'} \leq \lambda_1^{-\frac{1}{2}} \|f\|_{\mathbb{H}}. \tag{2.4}$$

From now on, we denote $\langle \cdot, \cdot \rangle_{\mathbb{V}', \mathbb{V}}$ by $\langle \cdot, \cdot \rangle$ for simplicity.

The boundary $\partial\Omega$ is said to be uniformly of class C^3 if we can choose suitable local cartesian coordinates (y_1, y_2) in a neighborhood $B(\eta, r)$ of each point $\eta \in \partial\Omega$, such that $\partial\Omega \cap B(\eta, r)$ can be represented by a function $y_2 = h(y_1; \eta)$ of class C^3 whose derivatives up to order 3 are bounded in $B(\eta, r)$ uniformly with respect to η, where $B(\eta, r)$ is a ball centered at η with radius r (independent of η).

If $\partial\Omega$ is uniformly of class C^3, then for all $u \in V$ satisfying $Au \in H$, we have $u \in [H^2(\Omega)]^2$ and

$$\|\nabla^2 u\|_{L^2} \le C_\partial(\|Au\|_H + \|\nabla u\|_{L^2}), \tag{2.5}$$

where C_∂ depends only on the C^3-regularity of $\partial\Omega$. Although the original proof (in [19, 20, 31]) of (2.5) is only for $n = 3$, it can be applied to $n = 2$ either. Combining (2.5) with similar estimates for elliptic operators (see [13, 16]), we obtain for all $\varphi \in \mathbb{V}$ satisfying $\mathbb{A}\varphi \in \mathbb{H}$ that

$$\|\nabla^2 \varphi\|_{L^2} \le C_\partial(\|\mathbb{A}\varphi\|_{\mathbb{H}} + \|\nabla \varphi\|_{L^2}). \tag{2.6}$$

Thus,

$$D(\mathbb{A}) = \{\varphi \in \mathbb{V} \mid \mathbb{A}\varphi \in \mathbb{H}\} = \mathbb{V} \cap ([H^2(\Omega)]^2 \times [H^2(\Omega)]^2 \times H^2(\Omega)).$$

In this paper, we always assume that Ω satisfies (2.1), (2.2), (2.6). For example, Ω is a smooth bounded domain or a straight strip.

For each $\varphi \in D(\mathbb{A})$, there holds $\mathbb{A}\varphi \in \mathbb{H}$ and

$$\|\varphi\|_{\mathbb{V}}^2 = (\varphi, \varphi)_{\mathbb{V}} = \langle \mathbb{A}\varphi, \varphi\rangle = (\mathbb{A}\varphi, \varphi)_{\mathbb{H}} \le \|\mathbb{A}\varphi\|_{\mathbb{H}}\|\varphi\|_{\mathbb{H}}.$$

This together with (2.1) gives

$$\|\varphi\|_{\mathbb{V}} \le \lambda_1^{-\frac{1}{2}}\|\mathbb{A}\varphi\|_{\mathbb{H}}. \tag{2.7}$$

Combining (2.1)-(2.7) yields

$$c_\Omega(\|\varphi\|_{L^2} + \|\nabla\varphi\|_{L^2} + \|\nabla^2\varphi\|_{L^2}) \le \|\mathbb{A}\varphi\|_{\mathbb{H}} \le C_\Omega\|\nabla^2\varphi\|_{L^2}, \tag{2.8}$$

where c_Ω and C_Ω depend on (both the regularity and the size of) Ω. Therefore, $D(\mathbb{A})$ is a closed linear subspace of $[H^2(\Omega)]^2 \times [H^2(\Omega)]^2 \times H^2(\Omega)$ with the equivalent norm $\|\varphi\|_{D(\mathbb{A})} = \|\mathbb{A}\varphi\|_{\mathbb{H}}$.

Now we define a trilinear form b_1 on $[H^1(\Omega)]^2 \times [H^1(\Omega)]^2 \times [H^1(\Omega)]^2$ by

$$b_1(v_1, v_2, v_3) = \sum_{j,k=1}^{2} \int_\Omega v_1^k \frac{\partial v_2^j}{\partial x_k} v_3^j dx, \quad v_i = (v_i^1, v_i^2),\ i = 1, 2, 3,$$

and a trilinear form b_2 on $[H^1(\Omega)]^2 \times H^1(\Omega) \times H^1(\Omega)$ by

$$b_2(v_1, \theta_2, \theta_3) = \sum_{i=1}^{2} \int_\Omega v_1^i \frac{\partial \theta_2}{\partial x_i} \theta_3 dx, \quad v_1 = (v_1^1, v_1^2).$$

We can check that

$$b_1(u_1, v_2, v_3) = -b_1(u_1, v_3, v_2) \quad \text{for all } u_1 \in V, v_2, v_3 \in [H^1(\Omega)]^2; \tag{2.9}$$

$$b_1(u, v, v) = 0 \qquad \text{for all } u \in V, v \in [H^1(\Omega)]^2. \tag{2.10}$$

Substituting the Gagliardo-Nirenberg inequality (see [15])

$$\|\varphi\|_{L^4} \le C\|\varphi\|_{L^2}^{\frac{1}{2}}\|\nabla\varphi\|_{L^2}^{\frac{1}{2}} \quad \text{for all } \varphi \in H_0^1(\Omega), \tag{2.11}$$

into

$$|b_1(v_1, v_2, v_3)| \le \|v_1\|_{L^4}\|\nabla v_2\|_{L^2}\|v_3\|_{L^4}, \tag{2.12}$$

one derives the following estimate for all $v_1, v_2, v_3 \in [H_0^1(\Omega)]^2$ (see [23]),

$$|b_1(v_1, v_2, v_3)| \le C\|v_1\|_{L^2}^{\frac{1}{2}}\|\nabla v_1\|_{L^2}^{\frac{1}{2}}\|\nabla v_2\|_{L^2}\|v_3\|_{L^2}^{\frac{1}{2}}\|\nabla v_3\|_{L^2}^{\frac{1}{2}}, \tag{2.13}$$

where C is independent of Ω. Moreover, by (2.8),

$$|b_1(v_1, v_2, v_3)| \le \begin{cases} C_\Omega\|v_1\|_{L^2}^{\frac{1}{2}}\|v_1\|_{H^2}^{\frac{1}{2}}\|v_2\|_{H^1}\|v_3\|_{L^2}, \\ C_\Omega\|v_1\|_{L^2}^{\frac{1}{2}}\|v_1\|_{H^1}^{\frac{1}{2}}\|v_2\|_{H^1}^{\frac{1}{2}}\|v_2\|_{H^2}^{\frac{1}{2}}\|v_3\|_{L^2}, \end{cases} \tag{2.14}$$

for all $v_1, v_2 \in [H^2(\Omega)]^2$, $v_3 \in [L^2(\Omega)]^2$, where C_Ω depends on (both the regularity and the size of) Ω. The trilinear form b_2 satisfies similar properties to (2.9)-(2.14), replacing v_2, v_3 with θ_2, θ_3.

Now we can define a continuous trilinear form b on $\mathbb{V} \times \mathbb{V} \times \mathbb{V}$,

$$\begin{aligned} b(\varphi_1, \varphi_2, \varphi_3) = b_1(u_1, u_2, u_3) - b_1(v_1, u_3, v_2) + b_1(u_1, v_2, v_3) \\ + b_1(v_1, u_2, v_3) + b_2(u_1, \theta_2, \theta_3) \end{aligned} \tag{2.15}$$

for all $\varphi_i = (u_i, v_i, \theta_i) \in \mathbb{V}$, $i = 1, 2, 3$. Obviously,

$$-b_1(v_1, u_3, v_2) + b_1(v_1, u_2, v_3) = b_1(v_1, u_2, v_3) - b_1(v_1, u_3, v_2).$$

Then, similar properties to (2.9) and (2.10) are valid, i.e.,

$$\begin{aligned} b(\varphi_1, \varphi_2, \varphi_3) &= -b(\varphi_1, \varphi_3, \varphi_2) \quad \text{for all } \varphi_1, \varphi_2, \varphi_3 \in \mathbb{V}, \\ b(\varphi, \psi, \psi) &= 0 \qquad\qquad\qquad \text{for all } \varphi, \psi \in \mathbb{V}. \end{aligned}$$

We note that, for $v_i = (v_i^1, v_i^2) \in [H_0^1(\Omega)]^2$, $i = 1, 2, 3$,

$$-b_1(v_1, v_3, v_2) = b_1(v_1, v_2, v_3) + \sum_{j,k=1}^{2} \int_\Omega v_2^j \frac{\partial v_1^k}{\partial x_k} v_3^j dx,$$

and the second term on the right-hand-side shares similar estimates to (2.13) and (2.14). So for all $\varphi_i \in \mathbb{V}$, $i = 1, 2, 3$,

$$|b(\varphi_1, \varphi_2, \varphi_3)| \le C(\|\varphi_1\|_{L^4}\|\varphi_2\|_{\mathbb{V}}\|\varphi_3\|_{L^4} + \|\varphi_1\|_{\mathbb{V}}\|\varphi_2\|_{L^4}\|\varphi_3\|_{L^4}), \tag{2.16}$$

and

$$\begin{aligned} |b(\varphi_1, \varphi_2, \varphi_3)| \le C(&\|\varphi_1\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}}\|\varphi_2\|_{\mathbb{V}}\|\varphi_3\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_3\|_{\mathbb{V}}^{\frac{1}{2}} \\ &+ \|\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}\|\varphi_3\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_3\|_{\mathbb{V}}^{\frac{1}{2}}). \end{aligned} \tag{2.17}$$

Moreover, for all $\varphi_1, \varphi_2 \in D(\mathbb{A})$, $\varphi_3 \in \mathbb{H}$,

$$\begin{aligned} |b(\varphi_1, \varphi_2, \varphi_3)| \le C_\partial[&\|\varphi_1\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}}\|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}}(\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}} + \|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}})\|\varphi_3\|_{\mathbb{H}} \\ &+ \|\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}}(\|\mathbb{A}\varphi_1\|_{\mathbb{H}}^{\frac{1}{2}} + \|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}})\|\varphi_3\|_{\mathbb{H}}], \end{aligned} \tag{2.18}$$

and

$$|b(\varphi_1, \varphi_2, \varphi_3)| + |b(\varphi_2, \varphi_1, \varphi_3)| \le C_\Omega(\|\varphi_1\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}}\|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}}\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_3\|_{\mathbb{H}} \tag{2.19}$$

$$+ \|\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}}\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}\|\varphi_3\|_{\mathbb{H}}),$$

where C_∂ depends on the regularity (but not the size) of $\partial\Omega$ and C_Ω depends on Ω.

Inequality (2.1) together with (2.17) implies that we can define a continuous bilinear operator $\mathbb{B} : \mathbb{V} \times \mathbb{V} \to \mathbb{V}'$ by setting

$$\langle \mathbb{B}(\varphi_1, \varphi_2), \varphi_3\rangle = b(\varphi_1, \varphi_2, \varphi_3) \quad \text{for all } \varphi_1, \varphi_2, \varphi_3 \in \mathbb{V}.$$

Inequality (2.18) also shows

$$\begin{aligned} \|\mathbb{B}(\varphi_1, \varphi_2)\|_{\mathbb{H}} \le C_\partial[&\|\varphi_1\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}}\|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}}(\|\mathbb{A}\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}} + \|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}}) \\ &+ \|\varphi_2\|_{\mathbb{H}}^{\frac{1}{2}}\|\varphi_2\|_{\mathbb{V}}^{\frac{1}{2}}\|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}}(\|\mathbb{A}\varphi_1\|_{\mathbb{H}}^{\frac{1}{2}} + \|\varphi_1\|_{\mathbb{V}}^{\frac{1}{2}})] \end{aligned} \tag{2.20}$$

for all $\varphi_1, \varphi_2 \in D(\mathbb{A})$.

We can also define a continuous linear operator $\mathbb{C} : \mathbb{V} \to \mathbb{H} \subset \mathbb{V}'$ by setting

$$\mathbb{C}\varphi = (0, \nabla\theta, \nabla \cdot v) \quad \text{for all } \varphi = (u, v, \theta) \in \mathbb{V}.$$

Especially,

$$\langle \mathbb{C}\varphi, \psi\rangle = (\mathbb{C}\varphi, \psi)_{\mathbb{H}}.$$

Direct calculation gives

$$(\mathbb{C}\varphi, \psi)_{\mathbb{H}} \le \|\varphi\|_{\mathbb{V}}\|\psi\|_{\mathbb{H}} \quad \text{for all } \varphi \in \mathbb{V}, \psi \in \mathbb{H}, \tag{2.21}$$

and

$$\langle \mathbb{C}\varphi, \psi\rangle \le \lambda_1^{-\frac{1}{2}}\|\varphi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}} \quad \text{for all } \varphi, \psi \in \mathbb{V}. \tag{2.22}$$

Moreover,

$$\langle \mathbb{C}\varphi, \varphi\rangle = (\mathbb{C}\varphi, \varphi)_{\mathbb{H}} = 0 \quad \text{for all } \varphi \in \mathbb{V}.$$

Definition. Let $f \in L^2(\tau, T; \mathbb{V}')$, $\Phi_\tau \in \mathbb{H}$. Then, $\Phi \in L^\infty(\tau, T; \mathbb{H}) \cap L^2(\tau, T; \mathbb{V})$ is called a weak solution to (1.1) with data Φ_τ at initial time τ, if

$$\frac{d}{dt}(\Phi, \psi)_{\mathbb{H}} + (\Phi, \psi)_{\mathbb{V}} + b(\Phi, \Phi, \psi) + (\mathbb{C}\Phi, \psi)_{\mathbb{H}} = \langle f, \psi\rangle, \quad \tau < t < T, \tag{2.23}$$

for all $\psi \in \mathbb{V}$, and

$$\Phi(t) \longrightarrow \Phi_\tau \text{ in } \mathbb{H} \text{ as } t \to \tau^+. \tag{2.24}$$

When $f \in L^2(\tau, T; \mathbb{H})$ and $\Phi_\tau \in \mathbb{V}$, if a weak solution Φ satisfies $\Phi \in L^\infty(\tau, T; \mathbb{V}) \cap L^2(\tau, T; D(\mathbb{A}))$, we call Φ a strong solution.

Remark 2.1. (i) Equation (2.23) is equivalent to

$$\partial_t\Phi + \mathbb{A}\Phi + \mathbb{B}(\Phi, \Phi) + \mathbb{C}\Phi = f \text{ in } \mathbb{V}'; \tag{2.25}$$

(ii) If Φ is a strong solution, then we deduce from (2.25) that $\partial_t\Phi \in L^2(0, T; \mathbb{H})$ and $\Phi \in C([0, T]; \mathbb{V})$.

3. Well-posedness

Theorem 3.1 (Uniqueness). *Let $f \in L^2(\tau, T; \mathbb{V}')$, $\Phi_\tau \in \mathbb{H}$, and Φ, Ψ be two weak solutions to the problem* (2.23) *with the same initial data Φ_τ. Then $\Phi \equiv \Psi \in C([\tau, T]; \mathbb{H})$,*

$$\frac{1}{2}\frac{d}{dt}\|\Phi(t)\|_{\mathbb{H}}^2 + \|\Phi(t)\|_{\mathbb{V}}^2 = \langle f(t), \Phi(t)\rangle, \tag{3.1}$$

and

$$\|\Phi(t)\|_{\mathbb{H}}^2 + 2\int_\tau^t \|\Phi(s)\|_{\mathbb{V}}^2 ds = \|\Phi_\tau\|_{\mathbb{H}}^2 + 2\int_\tau^t \langle f(s), \Phi(s)\rangle ds. \tag{3.2}$$

Proof. For all $\varphi \in \mathscr{V}$, combining (2.1) and (2.17) gives

$$b(\Phi, \Phi, \varphi) \le C\|\Phi\|_{\mathbb{H}}^{\frac{1}{2}}\|\Phi\|_{\mathbb{V}}^{\frac{3}{2}}\|\varphi\|_{\mathbb{V}}.$$

Because $\Phi \in L^\infty(\tau, T; \mathbb{H}) \cap L^2(\tau, T; \mathbb{V})$, we deduce from (2.25) that

$$\|\partial_t \Phi\|_{L^{\frac{4}{3}}(\tau, T; \mathbb{V}')} \le C, \tag{3.3}$$

which is not sufficient for us to obtain the uniqueness for weak solutions directly.

Notice that

$$\|\Phi\|_{L^4(\tau, T; L^4)} \le \|\Phi\|_{L^\infty(\tau, T; \mathbb{H})}^{\frac{1}{2}}\|\Phi\|_{L^2(\tau, T; \mathbb{V})}^{\frac{1}{2}} < \infty. \tag{3.4}$$

Thanks to (2.16) and (2.22), if we set

$$\Phi_1' = -\mathbb{A}\Phi - \mathbb{C}\Phi + f, \quad \Phi_2' = -\mathbb{B}(\Phi, \Phi),$$

then

$$\Phi_1' \in L^2(\tau, T; \mathbb{V}'), \quad \Phi_2' \in L^{\frac{4}{3}}(\tau, T; L^{\frac{4}{3}}),$$

and

$$\partial_t \Phi = \Phi_1' + \Phi_2' \in L^2(\tau, T; \mathbb{V}') + L^{\frac{4}{3}}(\tau, T; L^{\frac{4}{3}}).$$

By standard extension and mollification, we can derive

$$\frac{d}{dt}(\Phi, \Psi)_{\mathbb{H}} = \langle \Phi_1', \Psi\rangle_{\mathbb{V}', \mathbb{V}} + \langle \Phi_2', \Psi\rangle_{L^{\frac{4}{3}}, L^4} + \langle \Psi_1', \Phi\rangle_{\mathbb{V}', \mathbb{V}} + \langle \Psi_2', \Phi\rangle_{L^{\frac{4}{3}}, L^4}. \tag{3.5}$$

Then (3.1) follows from (3.5).

Noticing that $(\mathbb{C}\Psi, \Phi)_{\mathbb{H}} = -(\mathbb{C}\Phi, \Psi)_{\mathbb{H}}$, we integrate (3.5) on $[\tau+\varepsilon, t] \subset (\tau, T)$ and deduce

$$\begin{aligned} &(\Psi(t), \Phi(t))_{\mathbb{H}} - (\Psi(\tau+\varepsilon), \Phi(\tau+\varepsilon))_{\mathbb{H}} + 2\int_{\tau+\varepsilon}^t (\Psi, \Phi)_{\mathbb{V}} ds \\ = &-\int_{\tau+\varepsilon}^t [b(\Psi, \Psi, \Phi) + b(\Phi, \Phi, \Psi)] ds + \int_{\tau+\varepsilon}^t \langle f, \Phi + \Psi\rangle ds. \end{aligned}$$

Letting $\varepsilon \to 0^+$ and using (2.24), we obtain

$$(\Psi(t), \Phi(t))_{\mathbb{H}} - \|\Phi_\tau\|_{\mathbb{H}}^2 + 2\int_\tau^t (\Psi, \Phi)_{\mathbb{V}} ds \tag{3.6}$$

$$= -\int_\tau^t [b(\Psi,\Psi,\Phi)+b(\Phi,\Phi,\Psi)]ds + \int_\tau^t \langle f, \Phi+\Psi\rangle ds.$$

Then, (3.2) follows from (3.6).

Letting $Z = \Psi - \Phi$, then we deduce from (3.2) and (3.6) that

$$\begin{aligned}\|Z(t)\|_{\mathbb{H}}^2 &= \|\Psi(t)\|_{\mathbb{H}}^2 + \|\Phi(t)\|_{\mathbb{H}}^2 - 2(\Psi(t),\Phi(t))_{\mathbb{H}} \\ &= -\int_\tau^t \|Z(s)\|_{\mathbb{V}}^2 + 2\int_\tau^t [b(\Psi,\Psi,\Phi)+b(\Phi,\Phi,\Psi)]ds \\ &= -\int_\tau^t \|Z(s)\|_{\mathbb{V}}^2 + 2\int_\tau^t b(Z,Z,\Phi)ds.\end{aligned}$$

Here we use

$$b(\Psi,\Psi,\Phi) = b(Z,\Psi,\Phi) + b(\Phi,\Psi,\Phi) = b(Z,Z,\Phi) + b(\Phi,\Psi,\Phi),$$

and

$$b(\Phi,\Psi,\Phi) + b(\Phi,\Phi,\Psi) = b(\Phi,\Psi+\Phi,\Psi+\Phi) = 0.$$

By (2.11), (2.16) and (3.4), there holds

$$\begin{aligned}\int_\tau^t b(Z,Z,\Phi)ds &\le C\int_\tau^t \|Z\|_{\mathbb{H}}^{\frac{1}{2}}\|Z\|_{\mathbb{V}}^{\frac{3}{2}}\|\Phi\|_{L^4}ds \\ &\le \frac{1}{2}\int_\tau^t \|Z\|_{\mathbb{V}}^2 ds + C\int_\tau^t \|Z\|_{\mathbb{H}}^2\|\Phi\|_{L^4}^4 ds.\end{aligned}$$

Thus,

$$\|Z(t)\|_{\mathbb{H}}^2 \le C\int_\tau^t \|Z(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{L^4}^4 ds. \tag{3.7}$$

Finally, by the Gronwall inequality, we deduce $\|Z(t)\|_{\mathbb{H}}^2 \equiv 0$, i.e., $\Psi = \Phi$. □

Theorem 3.2 (Existence)**.** *For $f \in L^2(\tau,T;\mathbb{V}')$, $\Phi_\tau \in \mathbb{H}$, there exists a unique weak solution Φ to problem* (2.23) *with the initial data Φ_τ. Moreover, if $f \in L^2(\tau,T;\mathbb{H})$, $\Phi_\tau \in \mathbb{V}$, then Φ is a strong solution.*

Proof. We obtain from (3.2) that

$$\|\Phi(t)\|_{\mathbb{H}}^2 + \int_\tau^t \|\Phi(s)\|_{\mathbb{V}}^2 ds \le \|\Phi_\tau\|_{\mathbb{H}}^2 + \int_\tau^t \|f(s)\|_{\mathbb{V}'}^2 ds. \tag{3.8}$$

This gives

$$\sup_{t\in[\tau,T]} \|\Phi(t)\|_{\mathbb{H}}^2 \le \|\Phi_\tau\|_{\mathbb{H}}^2 + \int_\tau^T \|f\|_{\mathbb{V}'}^2 dt, \tag{3.9}$$

and

$$\int_\tau^T \|\Phi(t)\|_{\mathbb{V}}^2 dt \le \|\Phi_\tau\|_{\mathbb{H}}^2 + \int_\tau^T \|f\|_{\mathbb{V}'}^2 dt. \tag{3.10}$$

Thanks to (3.9) and (3.10), we can construct a weak solution to (2.23) by approximation of solutions on Ω_i, $i = 1, 2, \ldots$, where $\{\Omega_i\}_{i=1}^\infty$ is a sequence of smooth bounded regions satisfying $\Omega_1 \subset \Omega_2 \subset \cdots$, $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$. We omit the details here. Theorem 3.1 says that Φ is the unique weak solution.

Next, taking the $\mathbb{H}$-inner product of (2.25) with respect to $\mathbb{A}\Phi$, we obtain from (2.18) and (2.21) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Phi\|_{\mathbb{V}}^2 + \|\mathbb{A}\Phi\|_{\mathbb{H}}^2 &= -b(\Phi,\Phi,\mathbb{A}\Phi) - (\mathbb{C}\Phi,\mathbb{A}\Phi) + \langle f,\mathbb{A}\Phi\rangle \\ &\le \frac{1}{2}\|\mathbb{A}\Phi\|_{\mathbb{H}}^2 + C_\partial(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|\Phi\|_{\mathbb{V}}^2 + C\|f\|_{\mathbb{H}}^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}\|\Phi\|_{\mathbb{V}}^2 + \|\mathbb{A}\Phi\|_{\mathbb{H}}^2 \le C_\partial(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|\Phi\|_{\mathbb{V}}^2 + C\|f\|_{\mathbb{H}}^2. \tag{3.11}$$

From (3.9) and (3.10), we have

$$\int_\tau^T \|\Phi(t)\|_{\mathbb{H}}^2\|\Phi(t)\|_{\mathbb{V}}^2 dt \le C.$$

We deduce from the Gronwall inequality that

$$\sup_{t\in[\tau,T]} \|\Phi(t)\|_{\mathbb{V}}^2 \le C_\partial, \tag{3.12}$$

and

$$\int_\tau^T \|\mathbb{A}\Phi(t)\|_{\mathbb{H}}^2 dt \le C_\partial, \tag{3.13}$$

where $C_\partial = C_\partial(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{H})}, \|\Phi_\tau\|_{\mathbb{V}})$. Substituting (2.1), (2.20), (2.21) into (2.25), we derive from (3.9)–(3.13) that

$$\int_\tau^T \|\partial_t\Phi(t)\|_{\mathbb{H}}^2 dt \le C_\partial, \tag{3.14}$$

where $C_\partial = C_\partial(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{H})}, \|\Phi_\tau\|_{\mathbb{V}})$.

Because C_∂ in (3.9)–(3.14) depend on the regularity of $\partial\Omega$ (but not the size), we can let $\{\Omega_i\}_{i=1}^{\infty}$ be a sequence of bounded regions uniformly of class C^3 (see [19,20] for example). Then Φ is a strong solution because of (3.12) and (3.13). $\square$

By Theorem 3.2, for $f \in L^2_{\rm loc}(\mathbb{R}_+;\mathbb{V}') = L^2_{\rm loc}([0,+\infty);\mathbb{V}')$, we can define an operator from $\mathbb{H}$ into $\mathbb{V}$, denoted by $U_f(t,\tau) : \Phi_\tau \mapsto \Phi(t)$, where Φ is the unique weak solution to (2.23) with the initial data $\Phi_\tau \in \mathbb{H}$ and the external force f.

Theorem 3.3. *If $f \in L^2_{\rm loc}(\mathbb{R}_+;\mathbb{H})$, then $U_f(t,\tau) : \mathbb{H} \to \mathbb{V}$ is locally Lipschitz for $t > \tau$.*

Proof. For any $t > \tau$, we take $T > t$. Let Ψ, Φ be two weak solutions to (2.23) with $\Psi_\tau, \Phi_\tau \in \mathbb{H}$ and $Z = \Psi - \Phi$. Then $Z \in C([\tau,T];\mathbb{H}) \cap L^2(\tau,T;\mathbb{V})$ satisfies

$$\begin{cases} \frac{d}{dt}Z + \mathbb{A}Z + \mathbb{B}(\Psi,Z) + \mathbb{B}(Z,\Phi) + \mathbb{C}Z = 0, & \tau < t < T, \\ Z(\tau) = \Psi_\tau - \Phi_\tau. \end{cases} \tag{3.15}$$

By similar procedures to (3.7), we derive

$$\|Z(t)\|_{\mathbb{H}}^2 \le \|Z(\tau)\|_{\mathbb{H}}^2 + C\int_\tau^t \|Z(s)\|_{\mathbb{H}}^2 \|\Phi(s)\|_{L^4}^4 ds.$$

Estimates (3.9), (3.10) together with (3.4) show that

$$\int_\tau^T \|\Phi(s)\|_{L^4}^4 ds \le C.$$

Using the Gronwall inequality, we deduce that

$$\sup_{\tau\le t\le T} \|Z(t)\|_{\mathbb{H}}^2 \le C\|Z(\tau)\|_{\mathbb{H}}^2 = C\|\Psi_\tau - \Phi_\tau\|_{\mathbb{H}}^2, \tag{3.16}$$

$$\int_\tau^T \|Z(t)\|_{\mathbb{V}}^2 dt \le C\|Z(\tau)\|_{\mathbb{H}}^2 = C\|\Psi_\tau - \Phi_\tau\|_{\mathbb{H}}^2, \tag{3.17}$$

where $C = C(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{V}')}, \|\Psi_\tau\|_{\mathbb{H}}, \|\Phi_\tau\|_{\mathbb{H}})$.

Multiplying (3.11) by $t-\tau$, then

$$\begin{aligned}&\frac{d}{dt}[(t-\tau)\|\Phi\|_{\mathbb{V}}^2] + (t-\tau)\|\mathbb{A}\Phi\|_{\mathbb{H}}^2 \\ &\le C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)[(t-\tau)\|\Phi\|_{\mathbb{V}}^2] + \|\Phi\|_{\mathbb{V}}^2 + C\|f\|_{\mathbb{H}}^2.\end{aligned}$$

Using (3.9), (3.10) and the Gronwall inequality, we deduce that

$$\sup_{t\in[\tau,T]} [(t-\tau)\|\Phi(t)\|_{\mathbb{V}}^2] \le C, \tag{3.18}$$

where $C = C(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{H})}, \|\Phi_\tau\|_{\mathbb{H}})$.

Multiplying $(3.15)_1$ by $\mathbb{A}Z$ and using (2.19), (2.21), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z\|_{\mathbb{V}}^2 + \|\mathbb{A}Z\|_{\mathbb{H}}^2 &= -b(\Psi, Z, \mathbb{A}Z) - b(Z,\Phi,\mathbb{A}Z) - (\mathbb{C}Z, \mathbb{A}Z)_{\mathbb{H}} \\ &\le \frac{1}{2}\|\mathbb{A}Z\|_{\mathbb{H}}^2 + C(\|\Psi\|_{\mathbb{H}}^2\|\Psi\|_{\mathbb{V}}^2 + \|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|Z\|_{\mathbb{V}}^2 \\ &\quad + C(\|\Psi\|_{\mathbb{V}}^4 + \|\Phi\|_{\mathbb{V}}^4)\|Z\|_{\mathbb{H}}^2 + C\|Z\|_{\mathbb{V}}^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}[(t-\tau)\|Z\|_{\mathbb{V}}^2] + (t-\tau)\|\mathbb{A}Z\|_{\mathbb{H}}^2 \\ &\le C(\|\Psi\|_{\mathbb{H}}^2\|\Psi\|_{\mathbb{V}}^2 + \|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)[(t-\tau)\|Z\|_{\mathbb{V}}^2] \\ &\quad + C[(t-\tau)\|\Psi\|_{\mathbb{V}}^4 + (t-\tau)\|\Phi\|_{\mathbb{V}}^4]\|Z\|_{\mathbb{H}}^2 + C\|Z\|_{\mathbb{V}}^2.\end{aligned} \tag{3.19}$$

Estimates (3.9), (3.10) show that

$$\int_\tau^T (\|\Psi(t)\|_{\mathbb{H}}^2\|\Psi(t)\|_{\mathbb{V}}^2 + \|\Phi(t)\|_{\mathbb{H}}^2\|\Phi(t)\|_{\mathbb{V}}^2) dt \le C.$$

Combining (3.10), (3.16) and (3.18) yields that

$$\int_\tau^T [(t-\tau)\|\Psi(t)\|_{\mathbb{V}}^4 + (t-\tau)\|\Phi(t)\|_{\mathbb{V}}^4]\|Z(t)\|_{\mathbb{H}}^2 dt \le C\|\Psi_\tau - \Phi_\tau\|_{\mathbb{H}}^2.$$

Application of the Gronwall inequality to (3.19) gives

$$\sup_{t\in[\tau,T]}[(t-\tau)\|Z(t)\|_{\mathbb{V}}^2] \le C\|\Psi_\tau - \Phi_\tau\|_{\mathbb{H}}^2.$$

Therefore, for any fixed $t \in (\tau, T]$,

$$\|\Psi(t) - \Phi(t)\|_{\mathbb{V}}^2 = \|Z(t)\|_{\mathbb{V}}^2 \le \frac{C}{t-\tau}\|\Psi_\tau - \Phi_\tau\|_{\mathbb{H}}^2,$$

where $C = C(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{H})}, \|\Psi_\tau\|_{\mathbb{H}}, \|\Phi_\tau\|_{\mathbb{H}})$. □

Remark 3.4. Let $f \in L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{V}')$, $\mathcal{T}(s) : f(\cdot) \mapsto f(\cdot + s)$ be a translation, $s \ge 0$. Then:

(i) If $\Phi_\tau \in \mathbb{H}$, then $\Phi(\cdot) = U_f(\cdot,\tau)\Phi_\tau$ is a weak solution to (2.23);

(ii) $\begin{cases} U_f(t,\tau) = U_f(t,s)U_f(s,\tau) & \text{for all } t \ge s \ge \tau \ge 0, \\ U_f(\tau,\tau) = \mathrm{Id}_{\mathbb{H}} & \text{for all } \tau \ge 0; \end{cases}$

(iii) $U_{\mathcal{T}(s)f}(t,\tau) = U_f(t+s,\tau+s)$ for all $s \ge 0$, $t \ge \tau \ge 0$.

Now we study the linearized problem to (2.23). Let $f \in L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{V}')$, $\Phi_\tau \in \mathbb{H}$, $\Phi(t) = U_f(t,\tau)\Phi_\tau$, define $F'_{(\Phi_\tau,f)}(t,\tau) : D(\mathbb{A}) \to \mathbb{H}$,

$$F'_{(\Phi_\tau,f)}(t,\tau) : w \mapsto -\mathbb{A}w - \mathbb{B}(w,\Phi(t)) - \mathbb{B}(\Phi(t),w) - \mathbb{C}w.$$

Theorem 3.5. *If $f \in L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{V}')$, $\Phi_\tau \in \mathbb{H}$, then there exists a family of linear operators $\{L_{(\Phi_\tau,f)}(t,\tau) \mid t \ge \tau \ge 0\}$ such that*

(i) *For any $W_\tau \in \mathbb{H}$, let $\Phi(t) = U_f(t,\tau)\Phi_\tau$ and $W(t) = L_{(\Phi_\tau,f)}(t,\tau)W_\tau$, then W is the unique weak solution to*

$$\begin{cases} \frac{d}{dt}W + \mathbb{A}W + \mathbb{B}(W,\Phi) + \mathbb{B}(\Phi,W) + \mathbb{C}W = 0, \quad t > \tau, \\ W(\tau) = W_\tau; \end{cases} \tag{3.20}$$

(ii) $\begin{cases} L_{(\Phi_\tau,f)}(t,\tau) = L_{(\Phi(s),f)}(t,s)L_{(\Phi_\tau,f)}(s,\tau) & \textit{for all } t \ge s \ge \tau \ge 0, \\ L_{(\Phi_\tau,f)}(\tau,\tau) = \mathrm{Id}_{\mathbb{H}} & \textit{for all } \tau \ge 0; \end{cases}$

(iii) *If $f \in L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{H})$, then $L_{(\Phi_\tau,f)}(t,\tau) \in \mathcal{L}(\mathbb{H},\mathbb{V})$ for all $t > \tau$.*

Proof. The proof is similar to those of Theorems 3.1–3.3. Multiplying $(3.20)_1$ by W yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|W\|_{\mathbb{H}}^2 + \|W\|_{\mathbb{V}}^2 &= -b(W,\Phi,W) = b(W,W,\Phi) \\ &\le \frac{1}{2}\|W\|_{\mathbb{V}}^2 + C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2\|W\|_{\mathbb{H}}^2. \end{aligned}$$

Thus,

$$\frac{d}{dt}\|W\|_{\mathbb{H}}^2 + \|W\|_{\mathbb{V}}^2 \le C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2\|W\|_{\mathbb{H}}^2.$$

By (3.9) and (3.10),

$$\int_\tau^T \|\Phi(t)\|_{\mathbb{H}}^2\|\Phi(t)\|_{\mathbb{V}}^2 dt \le C.$$

Using the Gronwall inequality, we deduce

$$\sup_{t\in[\tau,T]}\|W(t)\|_{\mathbb{H}}^2 \le C\|W_\tau\|_{\mathbb{H}}^2, \tag{3.21}$$

and

$$\int_\tau^T \|W(t)\|_{\mathbb{V}}^2 dt \le C\|W_\tau\|_{\mathbb{H}}^2, \tag{3.22}$$

where $C = C(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{V}')}, \|\Phi_\tau\|_{\mathbb{H}})$.

Taking the $\mathbb{H}$-inner product of $(3.20)_1$ with respect to $\mathbb{A}W$ and using (2.19)–(2.21), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|W\|_{\mathbb{V}}^2 + \|\mathbb{A}W\|_{\mathbb{H}}^2 &= -b(W,\Phi,\mathbb{A}W) - b(\Phi,W,\mathbb{A}W) - (\mathbb{C}W,\mathbb{A}W)\\ &\le \frac{1}{2}\|\mathbb{A}W\|_{\mathbb{H}}^2 + C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|W\|_{\mathbb{V}}^2 + C\|\Phi\|_{\mathbb{V}}^4\|W\|_{\mathbb{H}}^2.\end{aligned}$$

Then,

$$\frac{d}{dt}\|W\|_{\mathbb{V}}^2 + \|\mathbb{A}W\|_{\mathbb{H}}^2 \le C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)\|W\|_{\mathbb{V}}^2 + C\|\Phi\|_{\mathbb{V}}^4\|W\|_{\mathbb{H}}^2. \tag{3.23}$$

Combining (3.9), (3.10), (3.12), (3.21) gives

$$\int_\tau^T \|\Phi(t)\|_{\mathbb{H}}^2\|\Phi(t)\|_{\mathbb{V}}^2 dt + \int_\tau^T \|\Phi(t)\|_{\mathbb{V}}^4\|W(t)\|_{\mathbb{H}}^2 dt \le C.$$

Application of the Gronwall inequality to (3.23) gives

$$\sup_{t\in[\tau,T]}\|W(t)\|_{\mathbb{V}}^2 + \int_\tau^T \|\mathbb{A}W(t)\|_{\mathbb{H}}^2 dt \le C, \tag{3.24}$$

where $C = C(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{H})}, \|\Phi_\tau\|_{\mathbb{V}}, \|W_\tau\|_{\mathbb{V}})$.

Substituting (3.24) into (2.20), we derive

$$\int_\tau^T \|\mathbb{B}(\Phi(t),W(t))\|_{\mathbb{H}}^2 dt + \int_\tau^T \|\mathbb{B}(W(t),\Phi(t))\|_{\mathbb{H}}^2 dt \le C. \tag{3.25}$$

Combining (3.20), (3.24), (3.25), we derive

$$\int_\tau^T \|\partial_t W(t)\|_{\mathbb{H}}^2 dt \le C, \tag{3.26}$$

where $C = C(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{H})}, \|\Phi_\tau\|_{\mathbb{V}}, \|\Psi_\tau\|_{\mathbb{V}})$. Estimates (3.24) and (3.26) allow us to obtain a unique strong solution to (3.20) because (3.20) is a linear problem. Let $L_{(\Phi_\tau,f)}(t,\tau) : W_\tau \mapsto W(t)$, then $L_{(\Phi_\tau,f)}(t,\tau) : \mathbb{H} \to \mathbb{V}$ for all $t > \tau$.

Multiplying (3.23) by $t-\tau$, we have

$$\begin{aligned}\frac{d}{dt}[(t-\tau)\|W\|_{\mathbb{V}}^2] + (t-\tau)\|\mathbb{A}W\|_{\mathbb{H}}^2 &\le C(1+\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2)[(t-\tau)\|W\|_{\mathbb{V}}^2]\\ &\quad + C(t-\tau)\|\Phi\|_{\mathbb{V}}^4\|W\|_{\mathbb{H}}^2 + \|W\|_{\mathbb{V}}^2.\end{aligned} \tag{3.27}$$

Estimates (3.9), (3.10), (3.18) together with (3.21) give

$$\int_\tau^T \|\Phi(t)\|_{\mathbb{H}}^2 \|\Phi(t)\|_{\mathbb{V}}^2 dt \le C,$$

and

$$\int_\tau^T (t-\tau)\|\Phi(t)\|_{\mathbb{V}}^4 \|W(t)\|_{\mathbb{H}}^2 dt \le C\|W_\tau\|_{\mathbb{H}}^2.$$

Using these two inequalities and (3.27), we obtain from the Gronwall inequality that

$$\sup_{\tau\le t\le T}[(t-\tau)\|W(t)\|_{\mathbb{V}}^2] \le C\|W_\tau\|_{\mathbb{H}}^2.$$

Therefore, for any fixed $t \in (\tau, T]$,

$$\|W(t)\|_{\mathbb{V}}^2 \le \frac{C}{t-\tau}\|W_\tau\|_{\mathbb{H}}^2, \tag{3.28}$$

where $C = C(T-\tau, \|f\|_{L^2(\tau,T;\mathbb{H})}, \|\Phi_\tau\|_{\mathbb{H}})$. Estimate (3.28) tells us that, the linear operator $L_{(\Phi_\tau,f)}(t,\tau) \in \mathcal{L}(\mathbb{H};\mathbb{V}) \subset \mathcal{L}(\mathbb{H})$. □

Remark 3.6. We shall prove in Section 6 that $L_{(\Phi_\tau,f)}(t,\tau)$ is the Fréchet differential of $U_f(t,\tau)$ at $\Phi_\tau \in \mathbb{H}$.

4. Semiprocesses

Definition. Let X be a Banach space.

(i) A two-parameter family of operators $\{U(t,\tau)\} = \{U(t,\tau) : X \to X \mid t \ge \tau \ge 0\}$ is called a semiprocess in X if

$$\begin{cases} U(t,\tau) = U(t,s)U(s,\tau) & \text{for all } t \ge s \ge \tau \ge 0, \\ U(\tau,\tau) = \mathrm{Id}_X & \text{for all } \tau \ge 0; \end{cases}$$

(ii) For a family of semiprocesses $\{U_f(t,\tau)\}$ depending on $f \in \mathcal{F}$, the parameter f is called the symbol of the semiprocess $\{U_f(t,\tau)\}$ and $\mathcal{F}$ is called the symbol space.

If $\mathcal{F} \subset L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}')$, then $\{U_f(t,\tau)\}$ defined in Theorem 3.3 is a family of semiprocesses with the symbol space $\mathcal{F}$.

Definition. Let $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, be a family of semiprocesses.

(i) A set $\mathcal{B}$ is said to be uniformly (with respect to $f \in \mathcal{F}$) absorbing for $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, if for any $\tau \ge 0$ and any bounded set $K \subset X$, there exists $t_0(\tau, K) \ge \tau$ such that

$$\bigcup_{f\in\mathcal{F}} U_f(t,\tau)K \subset \mathcal{B} \quad \text{for all } t \ge t_0(\tau,K);$$

(ii) A set $\mathcal{B}$ is said to be uniformly (with respect to $f \in \mathcal{F}$) attracting for $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, if for any $\tau \geq 0$ and any bounded set $K \subset X$,

$$\sup_{f\in\mathcal{F}} d_X(U_f(t,\tau)K, \mathcal{B}) \longrightarrow 0 \quad \text{as } t \to +\infty,$$

where $d_X(B_1, B_2)$ is the Hausdorff semidistance between two sets $B_1, B_2 \subset X$,

$$d_X(B_1, B_2) = \sup_{\varphi\in B_1} \inf_{\psi\in B_2} \|\varphi - \psi\|_X;$$

(iii) A closed uniform (with respect to $f \in \mathcal{F}$) attracting set $\mathcal{A}_{\mathcal{F}}$ is said to be the uniform (with respect to $f \in \mathcal{F}$) attractor of $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, if it is contained in any closed uniformly (with respect to $f \in \mathcal{F}$) attracting set $\mathcal{A}'$, i.e., $\mathcal{A}_{\mathcal{F}} \subset \mathcal{A}'$.

The uniform attractor is always constructed from a bounded uniformly absorbing set. The following asymptotical compactness is useful.

Definition. A family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, is said to be uniformly (with respect to $f \in \mathcal{F}$) asymptotically compact in X, if $\{U_{f_n}(t_n,\tau)\varphi_n\}_n$ is precompact in X whenever $\{\varphi_n\}_n$ is bounded in X, $\{f_n\}_n \subset \mathcal{F}$, and $t_n \to +\infty$.

Definition. For a family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, and an arbitrary bounded set $K \subset X$, the uniform (with respect to $f \in \mathcal{F}$) ω-limit set $\omega_{\tau,\mathcal{F}}(K)$ (with origin τ) is defined by

$$\omega_{\tau,\mathcal{F}}(K) = \bigcap_{t\geq\tau} \overline{\bigcup_{f\in\mathcal{F}} \bigcup_{s\geq t} U_f(s,\tau)K},$$

where the closures are taken in X.

Moreover, $\varphi \in \omega_{\tau,\mathcal{F}}(K)$ if and only if there exist $\{\varphi_n\}_n \subset K$, $\{f_n\}_n \subset \mathcal{F}$, and $t_n \to +\infty$, such that

$$U_{f_n}(t_n,\tau)\varphi_n \longrightarrow \varphi \text{ in } X \text{ as } n \to +\infty.$$

The existence of the uniform attractor is given by the following Theorem 4.1 (see [10, 21]).

Theorem 4.1. *Let $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, be a family of semiprocesses satisfying*:

(i) *There exists a semigroup $\{\mathcal{T}(s)\}$ on $\mathcal{F}$ such that $\mathcal{T}(s)\mathcal{F} \subset \mathcal{F}$ for all $s \geq 0$;*

(ii) *The following translation identity is valid*

$$U_{\mathcal{T}(s)f}(t,\tau) = U_f(t+s, \tau+s) \quad \textit{for all } s \geq 0,\ t \geq \tau \geq 0,\ f \in \mathcal{F};$$

(iii) *$\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, has a bounded uniformly (with respect to $f \in \mathcal{F}$) absorbing set $\mathcal{B}$;*

(iv) *$\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, is uniformly (with respect to $f \in \mathcal{F}$) asymptotically compact.*

Then there exists a unique nonempty compact uniform (with respect to $f \in \mathcal{F}$) attractor $\mathcal{A}_{\mathcal{F}}$ given by

$$\mathcal{A}_{\mathcal{F}} = \omega_{0,\mathcal{F}}(\mathcal{B}).$$

5. Existence of the compact uniform attractor

Now we consider the L^2-uniform attractor for system (1.1). Let $\mathcal{T}(\cdot)$ be the translation defined in Remark 3.4 and suppose

$$\begin{cases} \mathcal{F} \subset \{f \in L^2_b(\mathbb{R}_+;\mathbb{H}) \mid \|f\|_{L^2_b} \leq R_{\mathcal{F}}\}, \\ \mathcal{T}(s)\mathcal{F} \subset \mathcal{F}, \ \ \forall s > 0, \\ \mathcal{F} \text{ is compact in } L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{H}), \end{cases} \tag{5.1}$$

where $R_{\mathcal{F}}$ is a nonnegative constant and

$$\|f\|^2_{L^2_b} = \sup_{t\geq 0} \int_t^{t+1} \|f(s)\|^2_{\mathbb{H}} ds.$$

Lemma 5.1. *Let $\mathcal{F}$ satisfy (5.1), $\Phi_\tau \in \mathbb{H}$, $f \in \mathcal{F}$. Then $\Phi(t) = U_f(t,\tau)\Phi_\tau$ satisfies*

$$\|\Phi(t)\|^2_{\mathbb{H}} \leq \|\Phi_\tau\|^2_{\mathbb{H}} e^{-\lambda_1(t-\tau)} + \lambda_1^{-1}(1+\lambda_1^{-1})\|f\|^2_{L^2_b},$$

and

$$\frac{1}{t-\tau}\int_\tau^t \|\Phi(s)\|^2_{\mathbb{V}} ds \leq \frac{1}{t-\tau}\|\Phi_\tau\|^2_{\mathbb{H}} + \lambda_1^{-1}\frac{\lceil t-\tau\rceil}{t-\tau}\|f\|^2_{L^2_b},$$

where $\lceil\cdot\rceil$ is the ceiling function, i.e., $\lceil x\rceil$ is the smallest integer not less than x.

Proof. Substituting (2.1) and (2.4) into (3.1) gives

$$\frac{d}{dt}\|\Phi\|^2_{\mathbb{H}} + \lambda_1\|\Phi\|^2_{\mathbb{H}} \leq \lambda_1^{-1}\|f\|^2_{\mathbb{H}}.$$

By the Gronwall inequality, we deduce

$$\|\Phi(t)\|^2_{\mathbb{H}} \leq \|\Phi_\tau\|^2_{\mathbb{H}} e^{-\lambda_1(t-\tau)} + \lambda_1^{-1}\int_\tau^t e^{-\lambda_1(t-s)}\|f(s)\|^2_{\mathbb{H}} ds. \tag{5.2}$$

Set $f(t) = 0$ when $t < 0$. We can estimate the integral on the right-hand side of (5.2) as

$$\begin{aligned} \int_\tau^t e^{-\lambda_1(t-s)}\|f(s)\|^2_{\mathbb{H}} ds &= \int_0^{t-\tau} e^{-\lambda_1 s}\|f(t-s)\|^2_{\mathbb{H}} ds \\ &\leq \int_0^{+\infty} e^{-\lambda_1 s}\|f(t-s)\|^2_{\mathbb{H}} ds \\ &= \sum_{i=0}^\infty \int_i^{i+1} e^{-\lambda_1 s}\|f(t-s)\|^2_{\mathbb{H}} ds \\ &\leq \sum_{i=0}^\infty e^{-\lambda_1 i}\int_i^{i+1} \|f(t-s)\|^2_{\mathbb{H}} ds \end{aligned}$$

$$\begin{aligned}&\le \sum_{i=0}^{\infty} e^{-\lambda_1 i}\|f\|_{L_b^2}^2\\&= (1-e^{-\lambda_1})^{-1}\|f\|_{L_b^2}^2\\&\le (1+\lambda_1^{-1})\|f\|_{L_b^2}^2.\end{aligned}$$

Therefore,

$$\|\Phi(t)\|_{\mathbb{H}}^2 \le \|\Phi_\tau\|_{\mathbb{H}}^2 e^{-\lambda_1(t-\tau)} + \lambda_1^{-1}(1+\lambda_1^{-1})\|f\|_{L_b^2}^2.$$

Substituting (2.4) into (3.8), we derive

$$\begin{aligned}\|\Phi(t)\|_{\mathbb{H}}^2 + \int_\tau^t \|\Phi(s)\|_{\mathbb{V}}^2 ds &\le \|\Phi_\tau\|_{\mathbb{H}}^2 + \lambda_1^{-1}\int_\tau^t \|f(s)\|_{\mathbb{H}}^2 ds\\&\le \|\Phi_\tau\|_{\mathbb{H}}^2 + \lambda_1^{-1}\int_\tau^{\tau+\lceil t-\tau\rceil} \|f(s)\|_{\mathbb{H}}^2 ds\\&\le \|\Phi_\tau\|_{\mathbb{H}}^2 + \lambda_1^{-1}\lceil t-\tau\rceil \|f(s)\|_{L_b^2}^2.\end{aligned}$$

This ends the proof. □

Lemma 5.2. *Suppose $\mathcal{F}$ satisfies* (5.1). *Then*

$$\mathcal{B}_0 = \big\{\varphi \in \mathbb{H} \;\big|\; \|\varphi\|_{\mathbb{H}} \le R_0 := \sqrt{2\lambda_1^{-1}(1+\lambda_1^{-1})}R_{\mathcal{F}}\big\},$$

is uniformly (*with respect to $f \in \mathcal{F}$*) *absorbing in $\mathbb{H}$, and*

$$\mathcal{B}_1 = \{\varphi \in \mathbb{V} \mid \|\varphi\|_{\mathbb{V}} \le R_1 := CR_{\mathcal{F}}e^{CR_{\mathcal{F}}^4}\},$$

is uniformly (*with respect to $f \in \mathcal{F}$*) *absorbing in $\mathbb{V}$, where C depends on Ω.*

Proof. For any $r > 0$, let $\Phi_\tau \in B_{\mathbb{H}}(r)$ and

$$t_0(\tau, B_{\mathbb{H}}(r)) = \tau + \max\left\{0, \frac{1}{\lambda_1}\ln\frac{r^2}{\lambda_1^{-1}(1+\lambda_1^{-1})R_{\mathcal{F}}^2}\right\}.$$

Then, for all $t \ge t_0$, we deduce from Lemma 5.1 that

$$\|\Phi(t)\|_{\mathbb{H}}^2 \le 2\lambda_1^{-1}(1+\lambda_1^{-1})R_{\mathcal{F}}^2.$$

Inequality (3.11) gives

$$\frac{d}{dt}\|\Phi\|_{\mathbb{V}}^2 + \|\mathbb{A}\Phi\|_{\mathbb{H}}^2 \le C\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^4 + C\|\Phi\|_{\mathbb{V}}^2 + C\|f\|_{\mathbb{H}}^2.$$

Lemma 5.1 together with (5.1) shows that

$$\int_t^{t+1} \|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^2 ds \le CR_{\mathcal{F}}^4,$$

$$\int_t^{t+1} (\|\Phi(s)\|_{\mathbb{V}}^2 + \|f(s)\|_{\mathbb{H}}^2) ds \le CR_{\mathcal{F}}^2.$$

Using the uniform Gronwall inequality, we deduce

$$\|\Phi(t)\|_{\mathbb{V}}^2 \le CR_{\mathcal{F}}^2 e^{CR_{\mathcal{F}}^4} \quad \text{for all } t \ge t_0 + 1.$$ □

In order to obtain the asymptotical compactness, we need the following weak continuity of $\{U_f(t,\tau)\}$.

Lemma 5.3. *Let $\mathcal{F}$ satisfy* (5.1)*, $f_n, f \in \mathcal{F}$, $\varphi_n, \varphi \in \mathbb{H}$, and*

$$f_n \longrightarrow f \text{ in } L^2_{\text{loc,weak}}(\mathbb{R}_+;\mathbb{H}),$$
$$\varphi_n \longrightarrow \varphi \text{ weakly in } \mathbb{H},$$

as $n \to +\infty$. Then,

$$U_{f_n}(t,\tau)\varphi_n \longrightarrow U_f(t,\tau)\varphi \text{ weakly in } \mathbb{H},$$
$$U_{f_n}(\cdot,\tau)\varphi_n \longrightarrow U_f(\cdot,\tau)\varphi \text{ weakly in } L^2(\tau,T;\mathbb{V}),$$

as $n \to +\infty$ for all $T \geq t \geq \tau$.

Proof. Because of Remark 3.4 and (5.1), we only need to prove it for $\tau = 0$.

Let $\Phi_n(t) = U_{f_n}(t,0)\varphi_n$ and $\Phi(t) = U_f(t,0)\varphi$ for $t \geq 0$. From (3.3), (3.9) and (3.10), we find that $\{\Phi_n\}_n$ is bounded in $L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})$, and $\{\partial_t \Phi_n\}_n$ is bounded in $L^{\frac{4}{3}}(0,T;\mathbb{V}')$ for all $T > 0$. Therefore,

$$\Phi_{n'} \longrightarrow \widetilde{\Phi} \text{ weakly}-* \text{ in } L^\infty(0,T;\mathbb{H}),$$
$$\Phi_{n'} \longrightarrow \widetilde{\Phi} \text{ weakly in } L^2(0,T;\mathbb{V}),$$

for some $\widetilde{\Phi} \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})$.

Let $\Omega_r = \Omega \cap \{x \in \mathbb{R}^2 \mid |x| < r\}$. Now consider a smooth truncation function $\chi(s) = 1$ for $s \in [0,1]$, and $\chi(s) = 0$ for $s \in [2,\infty)$. For each $r > 0$, define $\chi_r(x) = \chi(|x|/r)$ and $\Psi_{n,r} = \chi_r \Phi_n$. Then, $\{\Psi_{n,r}\}_n$ is bounded in $L^\infty(0,T;L^2(\Omega_{2r})) \cap L^2(0,T;H_0^1(\Omega_{2r}))$ uniformly with respect to $r \geq 1$ and n. Meanwhile, for all $0 \leq t \leq t + a \leq T$,

$$\begin{aligned}
\|\Phi_n(t+a) - \Phi_n(t)\|_{\mathbb{H}}^2 &= \int_t^{t+a} \langle \partial_s \Phi_n(s), \Phi_n(t+a) - \Phi_n(t) \rangle ds \\
&\leq a^{\frac{1}{4}} \|\partial_t \Phi_n\|_{L^{\frac{4}{3}}(0,T;\mathbb{V}')} \|\Phi_n(t+a) - \Phi_n(t)\|_{\mathbb{V}} \\
&\leq C_T a^{\frac{1}{4}} \|\Phi_n(t+a) - \Phi_n(t)\|_{\mathbb{V}}.
\end{aligned}$$

Hence,

$$\int_0^{T-a} \|\Phi_n(t+a) - \Phi_n(t)\|_{\mathbb{H}}^2 dt \leq C_T a^{\frac{1}{4}} \int_0^{T-a} \|\Phi_n(t+a) - \Phi_n(t)\|_{\mathbb{V}} dt \leq C_T a^{\frac{1}{4}}.$$

Therefore,

$$\lim_{a \to 0^+} \sup_n \int_0^{T-a} \|\Psi_n(t+a) - \Psi_n(t)\|_{L^2(\Omega_{2r})}^2 dt = 0.$$

By Theorem 13.3 in [32], we can take a subsequence $\{\Psi_{n'}\}_{n'}$ such that

$$\Psi_{n'} \longrightarrow \chi_r \widetilde{\Phi} \text{ strongly in } L^2(0,T;L^2(\Omega_{2r})).$$

Thus,

$$\Phi_{n'} \longrightarrow \widetilde{\Phi} \text{ strongly in } L^2(0,T;L^2(\Omega_r)).$$

By a diagonal process,

$$\Phi_{n'} \longrightarrow \widetilde{\Phi} \text{ strongly in } L^2(0,T;L^2(\Omega_r)) \quad \text{for all } r \geq 1.$$

Noticing that

$$f_n \longrightarrow f \text{ weakly in } L^2(0,T;\mathbb{H}).$$

Passing the equations for $\Phi_{n'}$ to the limit shows that $\widetilde{\Phi}$ is a weak solution to (2.23). By Theorem 3.1, we must have $\widetilde{\Phi} = \Phi$. Then by a contradiction argument, the whole sequence $\{\Phi_n\}_n$ converges to Φ in the above senses.

For all $\psi \in \mathscr{V}$, the locally strong convergence for $\{\Phi_n\}_n$ gives

$$(\Phi_n(t),\psi)_{\mathbb{H}} \longrightarrow (\Phi(t),\psi)_{\mathbb{H}} \quad \text{a.e. in } [0,T].$$

Moreover, because $\{\partial_t \Phi_n\}_n$ is bounded in $L^{\frac{4}{3}}(0,T;\mathbb{V}')$, we see that

$$\{(\Phi_n(\cdot),\psi)_{\mathbb{H}}\}_n$$

is equibounded and equicontinuous on $[0,T]$. Therefore,

$$(\Phi_n(t),\psi) \longrightarrow (\Phi(t),\psi) \text{ in } C([0,T]).$$

Noticing that $\mathscr{V}$ is dense in $\mathbb{H}$, we deduce

$$\Phi_n(t) \longrightarrow \Phi(t) \text{ weakly in } \mathbb{H}. \qquad \square$$

Now we start to prove the L^2-asymptotical compactness. First, define $[[\cdot,\cdot]]$: $\mathbb{V}\times\mathbb{V}\to\mathbb{R}$ by

$$[[\Phi,\Psi]] = (\Phi,\Psi)_{\mathbb{V}} - \frac{\lambda_1}{2}(\Phi,\Psi)_{\mathbb{H}}.$$

Clearly, $[[\cdot,\cdot]]$ is bilinear and symmetric. Moreover,

$$[[\Phi]]^2 \equiv [[\Phi,\Phi]] = \|\Phi\|_{\mathbb{V}}^2 - \frac{\lambda_1}{2}\|\Phi\|_{\mathbb{H}}^2 \geq \|\Phi\|_{\mathbb{V}}^2 - \frac{1}{2}\|\Phi\|_{\mathbb{V}}^2 = \frac{1}{2}\|\Phi\|_{\mathbb{V}}^2.$$

Hence,

$$\frac{1}{2}\|\Phi\|_{\mathbb{V}}^2 \leq [[\Phi]]^2 \leq \|\Phi\|_{\mathbb{V}}^2.$$

Thus, $[[\cdot,\cdot]]$ defines an inner product in $\mathbb{V}$, equivalent to $(\cdot,\cdot)_{\mathbb{V}}$.

Lemma 5.4. *Suppose $\mathcal{F}$ satisfies* (5.1). *Then $\{U_f(t,\tau)\}$, $f\in\mathcal{F}$, is uniformly* (*with respect to $f\in\mathcal{F}$*) *asymptotically compact in $\mathbb{H}$.*

Proof. For any $\Phi(t) = U_f(t,\tau)\Phi_\tau$, $\Phi_\tau \in \mathbb{H}$, we rewrite (3.1) as

$$\frac{d}{dt}\|\Phi\|_{\mathbb{H}}^2 + \lambda_1\|\Phi\|_{\mathbb{H}}^2 + 2[[\Phi]]^2 = 2\langle f,\Phi\rangle.$$

Thus,

$$\|\Phi\|_{\mathbb{H}}^2 = \|\Phi_\tau\|_{\mathbb{H}}^2 e^{-\lambda_1(t-\tau)} + 2\int_\tau^t e^{-\lambda_1(t-s)}(\langle f(s),\Phi(s)\rangle - [[\Phi(s)]]^2)ds,$$

i.e.,

$$\|U_f(t,\tau)\Phi_\tau\|_{\mathbb{H}}^2 = \|\Phi_\tau\|_{\mathbb{H}}^2 e^{-\lambda_1(t-\tau)} + 2\int_\tau^t e^{-\lambda_1(t-s)}\langle f(s),U_f(s,\tau)\Phi_\tau\rangle ds \tag{5.3}$$

$$- 2\int_\tau^t e^{-\lambda_1(t-s)}[[U_f(s,\tau)\Phi_\tau]]^2 ds.$$

Let $K \subset \mathbb{H}$ be bounded and consider $\{\varphi_n\}_n \subset K$, $\{f_n\}_n \subset \mathcal{F}$, and $t_n \to +\infty$. Lemma 5.2 shows that, for all $t \geq t_0(\tau, K) + 1$,

$$U_f(t,\tau)K \subset \mathcal{B}_0 \cap \mathcal{B}_1.$$

Therefore, for $t_n \geq t_0(\tau, K) + 1$,

$$U_{f_n}(t_n,\tau)\varphi_n \in \mathcal{B}_0 \cap \mathcal{B}_1.$$

Thus, $\{U_{f_n}(t_n,\tau)\varphi_n\}_n$ is weakly precompact in $\mathbb{H}$,

$$U_{f_{n'}}(t_{n'},\tau)\varphi_{n'} \longrightarrow \psi \text{ weakly in } \mathbb{H} \tag{5.4}$$

for some subsequence n' and $\psi \in \mathcal{B}_0 \cap \mathcal{B}_1$. Similarly, for each $T > 0$, assume $t_{n'} \geq t_0(\tau, K) + 1 + T$, we also have

$$U_{f_{n'}}(t_{n'} - T,\tau)\varphi_{n'} \in \mathcal{B}_0 \cap \mathcal{B}_1.$$

By a diagonal process, we can assume that

$$\varphi_{T,n'} := U_{f_{n'}}(t_{n'} - T,\tau)\varphi_{n'} \longrightarrow \psi_T \text{ weakly in } \mathbb{H}, \tag{5.5}$$

with $\psi_T \in \mathcal{B}_0 \cap \mathcal{B}_1$ for all $T > 0$. By Remark 3.4, we know

$$\begin{aligned} U_{f_{n'}}(t_{n'},\tau) &= U_{f_{n'}}(t_{n'}, t_{n'} - T)U_{f'_n}(t_{n'} - T,\tau) \\ &= U_{\mathcal{T}(t_{n'}-T)f_{n'}}(T,0)U_{f_{n'}}(t_{n'} - T,\tau). \end{aligned}$$

Taking $g_{T,n'} = \mathcal{T}(t_{n'} - T)f_{n'} \in \mathcal{F}$, we derive

$$U_{f_{n'}}(t_{n'},\tau)\varphi_{n'} = U_{g_{T,n'}}(T,0)U_{f_{n'}}(t_{n'} - T,\tau)\varphi_{n'} = U_{g_{T,n'}}(T,0)\varphi_{T,n'}. \tag{5.6}$$

Since $\{g_{T,n'}\}_{n'} \subset \mathcal{F}$, taking subsequence, we derive

$$g_{T,n'} \longrightarrow g_T \text{ in } L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{H}) \tag{5.7}$$

for some $g_T \in \mathcal{F}$. Then using (5.5)–(5.7) and Lemma 5.3, we obtain

$$U_{f_{n'}}(t_{n'},\tau)\varphi_{n'} = U_{g_{T,n'}}(T,0)\varphi_{T,n'} \longrightarrow U_{g_T}(T,0)\psi_T \text{ weakly in } \mathbb{H}.$$

Comparing this with (5.4), we derive

$$\psi = U_{g_T}(T,0)\psi_T. \tag{5.8}$$

Now,

$$\|\psi\|_{\mathbb{H}} \leq \liminf_{n'\to+\infty} \|U_{g_{T,n'}}(T,0)\varphi_{T,n'}\|_{\mathbb{H}} = \liminf_{n'\to+\infty} \|U_{f_{n'}}(t_{n'},\tau)\varphi_{n'}\|_{\mathbb{H}},$$

and we shall show that

$$\limsup_{n'\to+\infty} \|U_{f_{n'}}(t_{n'},\tau)\varphi_{n'}\|_{\mathbb{H}} \leq \|\psi\|_{\mathbb{H}}.$$

Equality (5.3) says that

$$\begin{aligned} \|U_{g_{T,n'}}(T,0)\varphi_{T,n'}\|^2_{\mathbb{H}} &= \|\varphi_{T,n'}\|^2_{\mathbb{H}} e^{-\lambda_1 T} \\ &\quad + 2\int_0^T e^{-\lambda_1(T-s)}\langle g_{T,n'}(s), U_{g_{T,n'}}(s,0)\varphi_{T,n'}\rangle ds \end{aligned} \tag{5.9}$$

$$- 2\int_0^T e^{-\lambda_1(T-s)}[[U_{g_{T,n'}}(s,0)\varphi_{T,n'}]]^2 ds.$$

Because (5.5)–(5.7) and Lemma 5.3 again,

$$U_{g_{T,n'}}(\cdot,0)\varphi_{T,n'} \longrightarrow U_{g_T}(\cdot,0)\psi_T \text{ weakly in } L^2(0,T;\mathbb{V}).$$

Therefore,

$$\begin{aligned}
&\lim_{n'\to+\infty}\int_0^T e^{-\lambda_1(T-s)}\langle g_{T,n'}(s), U_{g_{T,n'}}(s,0)\varphi_{T,n'}\rangle ds \\
=\ &\lim_{n'\to+\infty}\int_0^T \langle e^{-\lambda_1(T-s)} g_{T,n'}(s), U_{g_{T,n'}}(s,0)\varphi_{T,n'}\rangle ds \\
=\ &\int_0^T \langle e^{-\lambda_1(T-s)} g_T(s), U_{g_T}(s,0)\psi_T\rangle ds \\
=\ &\int_0^T e^{-\lambda_1(T-s)}\langle g_T(s), U_{g_T}(s,0)\psi_T\rangle ds,
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
&\limsup_{n'\to+\infty}\left(-2\int_0^T e^{-\lambda_1(T-s)}[[U_{g_{T,n'}}(s,0)\varphi_{T,n'}]]^2 ds\right) \\
=\ &-2\liminf_{n'\to+\infty}\int_0^T e^{-\lambda_1(T-s)}[[U_{g_{T,n'}}(s,0)\varphi_{T,n'}]]^2 ds \\
=\ &-2\liminf_{n'\to+\infty}\int_0^T [[e^{-\frac{\lambda_1}{2}(T-s)}U_{g_{T,n'}}(s,0)\varphi_{T,n'}]]^2 ds \\
\le\ &-2\int_0^T [[e^{-\frac{\lambda_1}{2}(T-s)}U_{g_T}(s,0)\psi_T]]^2 ds \\
=\ &-2\int_0^T e^{-\lambda_1(T-s)}[[U_{g_T}(s,0)\psi_T]]^2 ds.
\end{aligned} \tag{5.11}$$

Thus, we substituting (5.10) and (5.11) into (5.9) and derive that

$$\begin{aligned}
&\limsup_{n'\to+\infty}\|U_{g_{T,n'}}(T,0)\varphi_{T,n'}\|_{\mathbb{H}}^2 \\
\le\ &\limsup_{n'\to+\infty}\|\varphi_{T,n'}\|_{\mathbb{H}}^2 e^{-\lambda_1 T} \\
&+ 2\int_0^T e^{-\lambda_1(T-s)}\langle g_T(s), U_{g_T}(s,0)\varphi_T\rangle ds \\
&- 2\int_0^T e^{-\lambda_1(T-s)}[[U_{g_T}(s,0)\psi_T]]^2 ds.
\end{aligned} \tag{5.12}$$

On the other hand, because of (5.3) and (5.8),

$$\|\psi\|_{\mathbb{H}}^2 = \|\psi_T\|_{\mathbb{H}}^2 e^{-\lambda_1 T} \tag{5.13}$$

$$+ 2\int_0^T e^{-\lambda_1(T-s)}\langle g_T(s), U_{g_T}(s,0)\psi_T\rangle ds$$
$$- 2\int_0^T e^{-\lambda_1(T-s)}[[U_{g_T}(s,0)\psi_T]]^2 ds.$$

Substituting (5.13) into (5.12), we derive from $\varphi_{T,n'} \in \mathcal{B}_0 \cap \mathcal{B}_1$ that

$$\begin{aligned}\limsup_{n'\to+\infty} \|U_{f_{n'}}(t_{n'},\tau)\varphi_{n'}\|_{\mathbb{H}}^2 &= \limsup_{n'\to+\infty} \|U_{g_{T,n'}}(T,0)\varphi_{T,n'}\|_{\mathbb{H}}^2 \\ &\le \|\psi\|_{\mathbb{H}}^2 + (R_0^2 - \|\psi_T\|_{\mathbb{H}}^2)e^{-\lambda_1 T} \\ &\le \|\psi\|_{\mathbb{H}}^2 + R_0^2 e^{-\lambda_1 T}.\end{aligned}$$

Letting $T \to +\infty$, then

$$\limsup_{n'\to+\infty} \|U_{f_{n'}}(t_{n'},\tau)\varphi_{n'}\|_{\mathbb{H}}^2 \le \|\psi\|_{\mathbb{H}}^2. \tag{5.14}$$

Combining (5.4) and (5.14), the lemma follows. ☐

From Theorem 4.1 and Lemmas 5.2, 5.4, we deduce the following theorem.

Theorem 5.5. *Suppose $\mathcal{F}$ satisfies* (5.1)*. Then $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, possesses a unique compact uniform* (*with respect to $f \in \mathcal{F}$*) *attractor $\mathcal{A}_{\mathcal{F}}$ in $\mathbb{H}$.*

Now we begin to study the uniform attractor in $\mathbb{V}$. We begin with a continuity property similar to Lemma 5.3.

Lemma 5.6. *Let $\mathcal{F}$ satisfy* (5.1)*, $f_n, f \in \mathcal{F}$, $\varphi_n, \varphi \in \mathbb{H}$, and*

$$f_n \longrightarrow f \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{H}),$$
$$\varphi_n \longrightarrow \varphi \text{ strongly in } \mathbb{H},$$

as $n \to +\infty$. Then,

$$U_{f_n}(\cdot,\tau)\varphi_n \longrightarrow U_f(\cdot,\tau)\varphi \text{ strongly in } C([\tau,T];\mathbb{H}) \cap L^2(\tau,T;\mathbb{V})$$

for all $T \ge \tau$. Furthermore, if additionally

$$\varphi_n \longrightarrow \varphi \text{ weakly in } \mathbb{V},$$

then

$$U_{f_n}(\cdot,\tau)\varphi_n \longrightarrow U_f(\cdot,\tau)\varphi \text{ weakly in } L^2(\tau,T;D(\mathbb{A}))$$

as $n \to +\infty$.

Proof. The proof of Lemma 5.6 is similar to that of Lemma 5.3. Letting $\Phi_n(t) = U_{f_n}(t,\tau)\varphi_n$, $\Phi(t) = U_f(t,\tau)\varphi$. It is not difficult to obtain

$$\frac{d}{dt}\|\Phi_n - \Phi\|_{\mathbb{H}}^2 + \|\Phi_n - \Phi\|_{\mathbb{V}}^2 \le C\|\Phi\|_{\mathbb{V}}^2\|\Phi_n - \Phi\|_{\mathbb{H}}^2 + C\|f_n - f\|_{\mathbb{H}}^2.$$

Application of the Gronwall inequality gives the first part of Lemma 5.6. The second part follows from (3.13). ☐

The L^2-asymptotical compactness gives the existence of a strong convergent (in $\mathbb{H}$) sequence which in turn satisfies Lemma 5.6. This indicates that we may deduce the H^1-asymptotical compactness. We define $\{\{\cdot,\cdot\}\} : D(\mathbb{A}) \times D(\mathbb{A}) \to \mathbb{R}$ as

$$\{\{\Phi, \Psi\}\} := (\mathbb{A}\Phi, \mathbb{A}\Psi) - \frac{\lambda_1}{2}(\Phi, \Psi)_{\mathbb{V}}.$$

Then,

$$\{\{\Phi\}\}^2 = \{\{\Phi, \Phi\}\} \geq \frac{1}{2}\|\mathbb{A}\Phi\|_{\mathbb{H}}^2.$$

Thus, $\{\{\cdot\}\}$ is an equivalent norm on $D(\mathbb{A})$.

Lemma 5.7. *Suppose $\mathcal{F}$ satisfies* (5.1). *Then $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, is uniformly* (*with respect to $f \in \mathcal{F}$*) *asymptotically compact in $\mathbb{V}$.*

Proof. For any $\Phi(t) = U_f(t,\tau)\Phi_\tau$, $\Phi_\tau \in \mathbb{H}$, taking the $\mathbb{H}$-inner product of (2.25) with respect to $\mathbb{A}\Phi$, we obtain

$$\frac{d}{dt}\|\Phi\|_{\mathbb{V}}^2 + \lambda_1\|\Phi\|_{\mathbb{V}}^2 = -2b(\Phi, \Phi, \mathbb{A}\Phi) - 2(\mathbb{C}\Phi, \mathbb{A}\Phi) + 2\langle f, \mathbb{A}\Phi\rangle - 2\{\{\Phi\}\}^2.$$

Therefore,

$$\begin{aligned} \|U_f(t,\tau)\Phi_\tau\|_{\mathbb{V}}^2 = {}& \|\Phi_\tau\|_{\mathbb{V}}^2 e^{-\lambda_1(t-\tau)} \\ & - 2\int_\tau^t e^{-\lambda_1(t-s)} b(\Phi(s), \Phi(s), \mathbb{A}\Phi(s))ds \\ & - 2\int_\tau^t e^{-\lambda_1(t-s)} (\mathbb{C}\Phi(s), \mathbb{A}\Phi(s))_{\mathbb{H}} ds \\ & + 2\int_\tau^t e^{-\lambda_1(t-s)} (f(s), \mathbb{A}\Phi(s))_{\mathbb{H}} ds \\ & - 2\int_\tau^t e^{-\lambda_1(t-s)} \{\{\Phi(s)\}\}^2 ds. \end{aligned} \tag{5.15}$$

For a bounded set $K \subset \mathbb{H}$, taking $t_n \geq t_0(\tau, K) + 1$, then $U_{f_n}(t_n,\tau)\varphi_n \in \mathcal{B}_0 \cap \mathcal{B}_1$. Lemma 5.4 allows us to select a subsequence such that $\{U_{f_{n'}}(t_{n'},\tau)\varphi_{n'}\}_{n'}$ converges to ψ in $\mathbb{H}$. Moreover, $U_{f_{n'}}(t_{n'},\tau)\varphi_{n'} \longrightarrow \psi$ weakly in $\mathbb{V}$ either. For any $T > 0$, we may assume $t_{n'} \geq t_0 + 1 + T$, then

$$U_{f_{n'}}(t_{n'} - T, \tau)\varphi_{n'} \in \mathcal{B}_0 \cap \mathcal{B}_1.$$

By a diagonal process and Lemma 5.4,

$$\varphi_{T,n'} := U_{f_{n'}}(t_{n'} - T, \tau)\varphi_{n'} \longrightarrow \psi_T \text{ strongly in } \mathbb{H}, \tag{5.16}$$

$$\varphi_{T,n'} = U_{f_{n'}}(t_{n'} - T, \tau)\varphi_{n'} \longrightarrow \psi_T \text{ weakly in } \mathbb{V}, \tag{5.17}$$

with $\psi_T \in \mathcal{B}_0 \cap \mathcal{B}_1$ for all $T > 0$. By Remark 3.4, we know

$$\begin{aligned} U_{f_{n'}}(t_{n'},\tau) &= U_{f_{n'}}(t_{n'}, t_{n'} - T)U_{f'_n}(t_{n'} - T, \tau) \\ &= U_{\mathcal{T}(t_{n'}-T)f_{n'}}(T, 0)U_{f_{n'}}(t_{n'} - T, \tau). \end{aligned}$$

Taking $g_{T,n'} = \mathcal{T}(t_{n'} - T)f_{n'} \in \mathcal{F}$, we derive

$$U_{f_{n'}}(t_{n'}, \tau)\varphi_{n'} = U_{g_{T,n'}}(T, 0)U_{f_{n'}}(t_{n'} - T, \tau)\varphi_{n'} = U_{g_{T,n'}}(T, 0)\varphi_{T,n'}. \tag{5.18}$$

Since $\{g_{T,n'}\}_{n'} \subset \mathcal{F}$, taking subsequence, we derive from (5.1) that

$$g_{T,n'} \longrightarrow g_T \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{H}) \tag{5.19}$$

for some $g_T \in \mathcal{F}$. Then, using (5.16)–(5.19) and Lemma 5.6, we obtain

$$U_{f_{n'}}(t_{n'}, \tau)\varphi_{n'} = U_{g_{T,n'}}(T, 0)\varphi_{T,n'} \longrightarrow U_{g_T}(T, 0)\psi_T \text{ strongly in } \mathbb{H},$$
$$U_{f_{n'}}(t_{n'}, \tau)\varphi_{n'} = U_{g_{T,n'}}(T, 0)\varphi_{T,n'} \longrightarrow U_{g_T}(T, 0)\psi_T \text{ weakly in } \mathbb{V}.$$

Thus,

$$\psi = U_{g_T}(T, 0)\psi_T, \tag{5.20}$$

and

$$\|\psi\|_{\mathbb{V}} \le \liminf_{n' \to +\infty} \|U_{g_{T,n'}}(T, 0)\varphi_{T,n'}\|_{\mathbb{V}} = \liminf_{n' \to +\infty} \|U_{f_{n'}}(t_{n'}, \tau)\varphi_{n'}\|_{\mathbb{V}}.$$

Letting $\Phi_{T,n'} = U_{g_{T,n'}}(\cdot, 0)\varphi_{T,n'}$, $\Phi_T = U_{g_T}(\cdot, 0)\psi_T$, then (5.15) gives that

$$\begin{aligned} &\|U_{g_{T,n'}}(T, 0)\varphi_{T,n'}\|^2_{\mathbb{V}} \\ &= \|\varphi_{T,n'}\|^2_{\mathbb{V}} e^{-\lambda_1 T} \\ &\quad - 2\int_0^T e^{-\lambda_1(T-s)} b(\Phi_{T,n'}(s), \Phi_{T,n'}(s), \mathbb{A}\Phi_{T,n'}(s))ds \\ &\quad - 2\int_0^T e^{-\lambda_1(T-s)} (\mathbb{C}\Phi_{T,n'}(s), \mathbb{A}\Phi_{T,n'}(s))_{\mathbb{H}} ds \\ &\quad + 2\int_0^T e^{-\lambda_1(T-s)} (g_{T,n'}(s), \mathbb{A}\Phi_{T,n'}(s))_{\mathbb{H}} ds \\ &\quad - 2\int_0^T e^{-\lambda_1(T-s)} \{\{\Phi_{T,n'}(s)\}\}^2 ds. \end{aligned} \tag{5.21}$$

By Lemma 5.6,

$$\Phi_{T,n'} \longrightarrow \Phi_T \text{ strongly in } C([0, T]; \mathbb{H}),$$
$$\Phi_{T,n'} \longrightarrow \Phi_T \text{ strongly in } L^2(0, T; \mathbb{V}),$$
$$\Phi_{T,n'} \longrightarrow \Phi_T \text{ weakly in } L^2(0, T; D(\mathbb{A})).$$

Combining these two strong convergences with the Sobolev embedding, the interpolation, and (3.12), (3.13), we derive

$$\Phi_{T,n'} \longrightarrow \Phi_T \text{ strongly in } C([0, T]; L^4),$$
$$\nabla\Phi_{T,n'} \longrightarrow \nabla\Phi_T \text{ strongly in } L^2(0, T; L^4).$$

Then,

$$\mathbb{B}(\Phi_{T,n'}, \Phi_{T,n'}) \longrightarrow \mathbb{B}(\Phi_T, \Phi_T) \text{ strongly in } L^2(0, T; \mathbb{H}).$$

Passing (5.21) to the limit, we derive

$$
\begin{aligned}
(5.22)\qquad & \limsup_{n'\to+\infty} \|U_{g_{T,n'}}(T,0)\varphi_{T,n'}\|_{\mathbb{V}}^2 \\
&\le R_1^2 e^{-\lambda_1 T} \\
&\quad - 2\int_0^T e^{-\lambda_1(T-s)} b(\Phi_T(s),\Phi_T(s),\mathbb{A}\Phi_T(s))ds \\
&\quad - 2\int_0^T e^{-\lambda_1(T-s)} (\mathbb{C}\Phi_T(s),\mathbb{A}\Phi_T(s))_{\mathbb{H}} ds \\
&\quad + 2\int_0^T e^{-\lambda_1(T-s)} (g_T(s,0),\mathbb{A}\Phi_T(s))_{\mathbb{H}} ds \\
&\quad - 2\int_0^T e^{-\lambda_1(T-s)} \{\{\Phi_T(s)\}\}^2 ds.
\end{aligned}
$$

Meanwhile, from (5.15) and (5.20), we have

$$
\begin{aligned}
(5.23)\qquad \|\psi\|_{\mathbb{V}}^2 &= \|\psi_T\|_{\mathbb{V}}^2 e^{-\lambda_1 T} \\
&\quad - 2\int_0^T e^{-\lambda_1(T-s)} b(\Phi_T(s),\Phi_T(s),\mathbb{A}\Phi_T(s))ds \\
&\quad - 2\int_0^T e^{-\lambda_1(T-s)} (\mathbb{C}\Phi_T(s),\mathbb{A}\Phi_T(s))_{\mathbb{H}} ds \\
&\quad + 2\int_0^T e^{-\lambda_1(T-s)} (g_T(s,0),\mathbb{A}\Phi_T(s))_{\mathbb{H}} ds \\
&\quad - 2\int_0^T e^{-\lambda_1(T-s)} \{\{\Phi_T(s)\}\}^2 ds.
\end{aligned}
$$

Putting (5.23) into (5.22), we derive

$$
\begin{aligned}
\limsup_{n'\to+\infty} \|U_{f_{n'}}(t_{n'},\tau)\varphi_{n'}\|_{\mathbb{V}}^2 &= \limsup_{n'\to+\infty} \|U_{g_{T,n'}}(T,0)\varphi_{T,n'}\|_{\mathbb{V}}^2 \\
&\le \|\psi\|_{\mathbb{V}}^2 + (R_1^2 - \|\psi_T\|_{\mathbb{V}}^2) e^{-\lambda_1 T} \\
&\le \|\psi\|_{\mathbb{V}}^2 + R_1^2 e^{-\lambda_1 T}.
\end{aligned}
$$

Letting $T \to +\infty$, then we obtain the desired strong convergence. $\square$

Theorem 5.8. *Suppose $\mathcal{F}$ satisfies* (5.1). *Then $\mathcal{A}_{\mathcal{F}}$ obtained in Theorem 5.5 is the unique compact uniform* (*with respect to $f \in \mathcal{F}$*) *attractor for $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, in $\mathbb{V}$.*

Proof. We only need to prove $\omega_{0,\mathcal{F}}(\mathcal{B}_1;\mathbb{H}) \subset \omega_{0,\mathcal{F}}(\mathcal{B}_1;\mathbb{V})$. Let $\varphi \in \omega_{0,\mathcal{F}}(\mathcal{B}_1;\mathbb{H})$, then there exist $\{\varphi_n\}_n \subset \mathcal{B}_1$, $\{f_n\}_n \subset \mathcal{F}$, $t_n \to +\infty$, such that

$$
U_{f_n}(t_n,\tau)\varphi_n \longrightarrow \varphi \text{ in } \mathbb{H}.
$$

As $\{U_f(t,\tau)\}$ is $\mathbb{V}$-asymptotically compact, then there exists a subsequence n', such that

$$U_{f_{n'}}(t_{n'},\tau)\varphi_{n'} \longrightarrow \varphi_* \text{ in } \mathbb{V}.$$

The uniqueness gives that $\varphi = \varphi_* \in \omega_{0,\mathcal{F}}(\mathcal{B}_1;\mathbb{V})$. □

6. Dimension estimates

We first prove that $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$ is so-called uniformly quasidifferentiable.

Lemma 6.1. *Let $\mathcal{A}_{\mathcal{F}}$ be the uniform attractor obtained in Theorem 5.5. Then $\{U_f(t,\tau)\}$, $f \in \mathcal{F}$, is uniformly quasidifferentiable in $\mathcal{A}_{\mathcal{F}} \subset \mathbb{H}$. That is,*

$$\|U_f(t,\tau)\psi - U_f(t,\tau)\phi - L_{(\phi,f)}(t,\tau)(\psi-\phi)\|_{\mathbb{H}} \le \gamma(t-\tau, \|\psi-\phi\|_{\mathbb{H}})\|\psi-\phi\|_{\mathbb{H}},$$

where $\psi,\phi \in \mathcal{A}_{\mathcal{F}}$, $\lim_{\xi\to 0^+}\gamma(s,\xi) = 0$ for all $s \ge 0$.

Proof. For $\phi,\psi \in \mathbb{H}$, set $\Phi(s) = U_f(t,\tau)\phi$, $\Psi(s) = U_f(t,\tau)\psi$, and $Z = \Psi - \Phi$, then Z satisfies

$$\left\{\begin{array}{l} \frac{d}{ds}Z + \mathbb{A}Z + \mathbb{B}(\Psi,Z) + \mathbb{C}Z = -\mathbb{B}(Z,\Phi), \quad \tau < s < t, \\ Z(\tau) = \psi - \phi. \end{array}\right. \tag{6.1}$$

Combining (3.16) and (3.17) gives

$$\sup_{\tau\le s\le t} \|Z(s)\|_{\mathbb{H}}^2 \le C\|\psi-\phi\|_{\mathbb{H}}^2, \tag{6.2}$$

$$\int_\tau^t \|Z(s)\|_{\mathbb{V}}^2 ds \le C\|\psi-\phi\|_{\mathbb{H}}^2, \tag{6.3}$$

where $C = C(t-\tau, \|f\|_{L^2(\tau,t;\mathbb{V}')}, \|\phi\|_{\mathbb{H}})$.

We write (3.20) as

$$\left\{\begin{array}{l} \frac{d}{ds}W + \mathbb{A}W + \mathbb{B}(\Psi,W) + \mathbb{C}W = -\mathbb{B}(W,\Phi) + \mathbb{B}(Z,W), \quad \tau < s < t, \\ W(\tau) = W_\tau, \end{array}\right. \tag{6.4}$$

and set

$$R = \Psi - \Phi - L_{(\phi,f)}(t,\tau)(\psi-\phi) = Z - W,$$

with $W_\tau = \psi - \phi$ and $W(s) = L_{(\phi,f)}(t,\tau)W_\tau$. Taking the difference of (6.1) and (6.4), we deduce

$$\left\{\begin{array}{l} \frac{d}{ds}R + \mathbb{A}R + \mathbb{B}(\Psi,R) + \mathbb{C}R = -\mathbb{B}(R,\Phi) - \mathbb{B}(Z,W), \quad \tau < s < t, \\ R(\tau) = 0. \end{array}\right. \tag{6.5}$$

Multiplying $(6.5)_1$ by R, we deduce from (2.1) and (2.17) that

$$\begin{aligned} \frac{d}{ds}\|R\|_{\mathbb{H}}^2 + \|R\|_{\mathbb{V}}^2 &\le C(\|\Phi\|_{\mathbb{H}}^2\|\Phi\|_{\mathbb{V}}^2 + \|W\|_{\mathbb{V}}^2 + \|Z\|_{\mathbb{V}}^2)\|R\|_{\mathbb{H}}^2 \\ &\quad + C(\|Z\|_{\mathbb{H}} + \|W\|_{\mathbb{H}})(\|Z\|_{\mathbb{V}}^2 + \|W\|_{\mathbb{V}}^2). \end{aligned}$$

Combining (3.9), (3.10), (3.21), (3.22), (6.2), (6.3) yields

$$\int_\tau^t (\|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^2 + \|W(s)\|_{\mathbb{V}}^2 + \|Z(s)\|_{\mathbb{V}}^2)ds \le C,$$

and

$$\int_\tau^t (\|Z(s)\|_{\mathbb{H}} + \|W(s)\|_{\mathbb{H}})(\|Z(s)\|_{\mathbb{V}}^2 + \|W(s)\|_{\mathbb{V}}^2)ds \le C\|\psi - \phi\|_{\mathbb{H}}^3,$$

where $C = C(t-\tau, \|f\|_{L^2(\tau,t;\mathbb{H})}, \|\phi\|_{\mathbb{H}}, \|\psi\|_{\mathbb{H}})$. Using the Gronwall inequality, we obtain

$$\sup_{\tau\le s\le t} \|R(s)\|_{\mathbb{H}}^2 \le C\|\psi - \phi\|_{\mathbb{H}}^3.$$

This tells us that

$$\frac{\|\Psi(s) - \Phi(s) - L_{(\phi,f)}(t,\tau)(\psi-\phi)\|_{\mathbb{H}}^2}{\|\psi-\phi\|_{\mathbb{H}}^2} = \frac{\|R(s)\|_{\mathbb{H}}^2}{\|\psi-\phi\|_{\mathbb{H}}^2} \le C\|\psi-\phi\|_{\mathbb{H}}, \tag{6.6}$$

where $C = C(t-\tau, \|f\|_{L^2(\tau,t;\mathbb{H})}, \|\phi\|_{\mathbb{H}}, \|\psi\|_{\mathbb{H}})$. Inequality (6.6) shows that $L_{(\phi,f)}(t,\tau)$ is the Fréchet differential of $U_f(t,\tau)$ at $\phi \in \mathbb{H}$. Moreover, if $\phi, \psi \in \mathcal{A}_{\mathcal{F}}$, then $\|\phi\|_{\mathbb{H}}, \|\psi\|_{\mathbb{H}}$ are bounded according to Lemma 5.2. Thus the constant C in (6.6) is independent of ϕ and $\{U_f(t,\tau)\}$ is uniformly quasidifferentiable. □

Let $T = 1$, then (3.21) says that

$$\sup_{t\in[0,1]} \sup_{f\in\mathcal{F}} \sup_{\varphi\in\mathcal{A}_{\mathcal{F}}} \|L_{(\varphi,f)}(t,\lfloor t\rfloor)\|_{\mathcal{L}(\mathbb{H})} \le C,$$

where $C = C(R_{\mathcal{F}})$. We set

$$f_0(t) = g(\alpha_1 t, \alpha_2 t, \ldots, \alpha_k t),$$

where $g(\omega_1,\ldots,\omega_k)$ is a 2π-periodic function in each argument ω_i, $i = 1,\ldots,k$; $\alpha_1,\ldots,\alpha_k$ are rationally independent numbers; g is a Lipschitz continuous function on $\mathbb{T}^k$ with values in $\mathbb{H}$, i.e.,

$$\|g(\omega) - g(\omega')\|_{\mathbb{H}} \le L|\omega - \omega'|_{\mathbb{T}^k} \quad \text{for all } \omega, \omega' \in \mathbb{T}^k,$$

for some positive constants $L > 0$. We take

$$\mathcal{F} = \{f_\omega \mid f_\omega(t) = g(\omega + \alpha t),\ \omega \in \mathbb{T}^k\},$$

where $\alpha = (\alpha_1,\ldots,\alpha_k)$. Then (5.1) is satisfied, and $\mathcal{A}_{\mathcal{F}}$ is obtained by Theorem 5.5.

We define

$$\widetilde{q}_j = \limsup_{t\to+\infty} \sup_{\substack{f_\omega\in\mathcal{F}\\ \varphi\in\mathcal{A}_{\mathcal{F}}}} \sup_{\substack{\psi_i\in\mathbb{H}\\ \|\psi_i\|_{\mathbb{H}}\le 1\\ i=1,\ldots,j}} \frac{1}{t}\int_0^t \mathrm{Tr}\{F'_{(\varphi,f_\omega)}(s,0)\circ \mathbb{Q}_j(s)\}ds,$$

where $\mathbb{Q}_j(s) = \mathbb{Q}_j(s,\varphi;\psi_1,\ldots,\psi_j)$ is the projection from $\mathbb{H}$ onto the space spanned by $L_{(\varphi,f_\omega)}(s,0)\psi_1,\ldots,L_{(\varphi,f_\omega)}(s,0)\psi_j$. The number $\mathrm{Tr}\{F'_{(\varphi,f_\omega)}(s,0)\circ \mathbb{Q}_j(s)\}$ is the trace of the linear operator (of finite rank) $F'_{(\varphi,f_\omega)}(s,0)\circ\mathbb{Q}_j(s)$ (F' is defined in Theorem 3.5). The Hausdorff and the fractal dimension of $\mathcal{A}_{\mathcal{F}}$ relies on the negativeness of $\widetilde{q}_j$. The following theorem is from [10].

Theorem 6.2. *Under the above assumptions in this section, if*

$$\widetilde{q}_j \le q_j, \quad j = 1, 2, \ldots,$$

for some concave function q_j with respect to j, and

$$q_m \ge 0 > q_{m+1}$$

for some integer m, then the Hausdorff dimension and the fractal dimension can be estimated by

$$\dim_H(\mathcal{A}_{\mathcal{F}}) \le \dim_F(\mathcal{A}_{\mathcal{F}}) \le m + \frac{q_m}{q_m - q_{m+1}} + k.$$

Moreover, if $m = 0$, then $\dim_H(\mathcal{A}_{\mathcal{F}}) = \dim_F(\mathcal{A}_{\mathcal{F}}) = 0$.

The following lemma is important when we study dynamic systems on unbounded domains (see [10, 27, 30]).

Lemma 6.3 (Lieb-Thirring Inequality)**.** *Let $e_1, \ldots, e_j \in [H^1(\mathbb{R}^2)]^m$ be an orthonormal family of vectors in $[L^2(\mathbb{R}^2)]^m$. Then*

$$\Big\| \sum_{i=1}^{j} |e_i|^2 \Big\|_{L^2}^2 \le C \sum_{i=1}^{j} \|\nabla e_i\|_{L^2}^2,$$

where C depends on m only.

Application of Theorem 6.2 gives:

Theorem 6.4. *Under the assumptions of Theorem 6.2, the uniform attractor $\mathcal{A}_{\mathcal{F}}$ obtained in Theorem 5.5 satisfies*

$$\dim_H(\mathcal{A}_{\mathcal{F}}) \le \dim_F(\mathcal{A}_{\mathcal{F}}) \le C_0 \lambda_1^{-3}(1 + \lambda_1^{-1}) R_{\mathcal{F}}^4 + k,$$

where $R_{\mathcal{F}}^2 = \sup_{\omega \in \mathbb{T}^k} \int_0^1 \|f_\omega(s)\|_{\mathbb{H}}^2 ds$, C_0 is an absolute constant independent of Ω or k.

Moreover, if $C_0 \lambda_1^{-3}(1 + \lambda_1^{-1}) R_{\mathcal{F}}^4 < 1$, then $\dim_H(\mathcal{A}_{\mathcal{F}}) = \dim_F(\mathcal{A}_{\mathcal{F}}) = k$.

Proof. Let $\Phi(t) = U_{f_\omega}(t, 0)\varphi$, then

$$\begin{aligned} \operatorname{Tr}\{F'_{(\varphi, f_\omega)}(s, 0) \circ \mathbb{Q}_j(s)\} &= \sum_{i=1}^{j} \langle F'_{(\varphi, f_\omega)}(s, 0) e_i(s), e_i(s) \rangle \\ &= -\sum_{i=1}^{j} [\|e_i(s)\|_{\mathbb{V}}^2 + b(e_i(s), \Phi(s), e_i(s))] \\ &= -\sum_{i=1}^{j} [\|e_i(s)\|_{\mathbb{V}}^2 - b(e_i(s), e_i(s), \Phi(s))]. \end{aligned}$$

For the last term, using (2.15), Hölder inequality, Young's inequality and Lemma 6.3, we deduce

$$\Big| \sum_{i=1}^{j} b(e_i(s), e_i(s), \Phi(s)) \Big| \le C \sum_{i=1}^{j} \int_\Omega |\Phi(s)| |e_i(s)| |\nabla e_i(s)| dx$$

$$
\begin{aligned}
&= C\int_\Omega |\Phi(s)|\sum_{i=1}^{j}|e_i(s)||\nabla e_i(s)|dx\\
&\le C\int_\Omega |\Phi(s)|\Big(\sum_{i=1}^{j}|e_i(s)|^2\Big)^{\frac12}\Big(\sum_{i=1}^{j}|\nabla e_i(s)|^2\Big)^{\frac12}dx\\
&\le C\|\Phi(s)\|_{L^4}\Big\|\sum_{i=1}^{j}|e_i(s)|^2\Big\|_{L^2}^{\frac12}\Big\|\sum_{i=1}^{j}|\nabla e_i(s)|^2\Big\|_{L^1}^{\frac12}\\
&\le C\|\Phi(s)\|_{L^2}^{\frac12}\|\nabla\Phi(s)\|_{L^2}^{\frac12}\Big(\sum_{i=1}^{j}\|e_i(s)\|_{\mathbb{V}}^2\Big)^{\frac34}\\
&\le \frac12\sum_{i=1}^{j}\|e_i(s)\|_{\mathbb{V}}^2 + C\|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^2.
\end{aligned}
$$

Here we use

$$
\Big\|\sum_{i=1}^{j}|\nabla e_i(s)|^2\Big\|_{L^1}^{\frac12} = \Big(\int_\Omega\sum_{i=1}^{j}|\nabla e_i(s)|^2dx\Big)^{\frac12} = \Big(\sum_{i=1}^{j}\int_\Omega|\nabla e_i(s)|^2dx\Big)^{\frac12}.
$$

Hence,

$$
\begin{aligned}
\mathrm{Tr}\{F'_{(\varphi,f_\omega)}(s,0)\circ\mathbb{Q}_j(s)\} &\le -\frac12\sum_{i=1}^{j}\|e_i(s)\|_{\mathbb{V}}^2 + C\|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^2\\
&\le -\frac{\lambda_1}{2}\sum_{i=1}^{j}\|e_i(s)\|_{\mathbb{H}}^2 + C\|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^2\\
&\le -\frac{\lambda_1}{2}j + C\|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^2.
\end{aligned}
$$

Thus by Lemma 5.1 and Lemma 5.2, we obtain

$$
\begin{aligned}
\widetilde{q}_j &\le -\frac{\lambda_1}{2}j + C\limsup_{t\to+\infty}\frac1t\int_0^t\|\Phi(s)\|_{\mathbb{H}}^2\|\Phi(s)\|_{\mathbb{V}}^2ds\\
&\le -\frac{\lambda_1}{2}j + C\lambda_1^{-1}(1+\lambda_1^{-1})R_{\mathcal{F}}^2\limsup_{t\to+\infty}\frac1t\int_0^t\|\Phi(s)\|_{\mathbb{V}}^2ds\\
&\le -\frac{\lambda_1}{2}j + C\lambda_1^{-2}(1+\lambda_1^{-1})R_{\mathcal{F}}^4\\
&= -\frac{\lambda_1}{2}[j - C_0\lambda_1^{-3}(1+\lambda_1^{-1})R_{\mathcal{F}}^4] \triangleq q_j.
\end{aligned}
$$

Take

$$
m = \lfloor C_0\lambda_1^{-3}(1+\lambda_1^{-1})R_{\mathcal{F}}^4\rfloor,
$$

where $\lfloor x\rfloor$ is the smallest integer not greater than x. Then $q_m \ge 0 > q_{m+1}$. Finally, by Theorem 6.2, we deduce

$$
\dim_H(\mathcal{A}_{\mathcal{F}}) \le \dim_F(\mathcal{A}_{\mathcal{F}}) \le C_0\lambda_1^{-3}(1+\lambda_1^{-1})R_{\mathcal{F}}^4 + k. \qquad \square
$$

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