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BIHARMONIC HYPERSURFACES WITH RECURRENT OPERATORS IN THE EUCLIDEAN SPACE

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Abstract. We show how some of well-known recurrent operators such as recurrent curvature operator, recurrent Ricci operator, recurrent Jacobi operator, recurrent shape and Weyl operators have the significant role for biharmonic hypersurfaces to be minimal in the Euclidean space.

1. Introduction

The phrase harmonic map $f : (M, g) \to (N, h)$ between two Riemannian manifolds is which refers to the critical points of the energy functional $E(f) = \frac{1}{2}\int_M |df|^2 \star 1$. The studying K-harmonic maps, correspondingly, k-harmonic submanifolds began with J. Eells and L. Lemair. It was proposed to investigate K-harmonic maps as critical points of the functional

$$E : C^\infty(M, N) \to R, \quad E_K(f) = \int_m \|d + d^\star\|^k f \star 1,$$

where d and $d^\star$ are the exterior differentiation and codifferentiation on the vector bundle on M, respectively (see [7, 8]). The idea was supported in case $K = 2$, which is called biharmonic maps and deal with $E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 dv$, where $\tau(f) = \mathsf{trace}\nabla df$ is the tension field of f [13,14]. Furthermore, the Euler-Lagrange equation associated to E_2 is given by vanishing of the bitension field written as:

$$\tau_2(f) = -\Delta\tau(f) - \mathsf{trace}R^N(df, \tau(f))df = 0.$$

The interesting is in the non harmonic biharmonic maps which are called proper biharmonic. The first ambient spaces to investigate the proper biharmonic submanifolds are spaces of the constant sectional curvature. In this case, the biharmonic concept of submanifold in the Euclidean space with the harmonic mean curvature vector was established by B. Y. Chen. Indeed the well known conjecture was posted: any biharmonic submanifold in Euclidean space is harmonic see [4]. By following the Chen's conjecture, hypersurfaces are the first class

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of submanifolds to be studied such that up to now, the following classification results reached.

- Biharmonic hypersurfaces in $E^n, n = 3, 4, 5$, are minimal [5, 11, 12];
- Biharmonic hypersurfaces in 4-dimensional space form H^4 are minimal [2];
- The biharmonic submanifold with the constant mean curvature and biharmonic hypersurfaces with at most two distinct principal curvatures in the Euclidean space are minimal [6];
- Biharmonic hypersurfaces with three distinct principal curvatures in R^n and S^n are minimal [9, 10];

Furthermore, a result of K. Akutagawa and Maeta [1] states that the biharmonic complete submanifolds in the Euclidean space are minimal too. Motivated by the results, authors in [15, 16] deal with the biharmonic Hopf hypersurfaces in the complex Euclidean spaces and in the odd dimensional spheres and showed they are minimal. Specifically, they proved the nonexistence result of the proper biharmonic Ricci Soliton hypersurfaces in the Euclidean space E^{n+1}, if the potential vector field is a principal direction.

In this survey we shall focus on the biharmonic hypersurfaces in the Euclidean space E^{n+1} with an important object, so-called the recurrent operator, attached to them. The key observation throughout is that the recurrent operators can be a property of the biharmonic hypersurfaces, which can not be proper one. Indeed, we show that the biharmonic hypersurfaces with some recurrent operators in the Euclidean space are minimal. Clearly, the results are given in Section 3, following the works in [6, 9] .

2. Preliminaries

Let $x : M^n \to E^{n+1}$ be an isometric immersion of an n-dimensional hypersurface (M^n, g) into the Euclidean space E^{n+1}. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections on M^n and E^{n+1}, respectively. Let N be a local unit normal vector field to M^n in E^{n+1} and $\overrightarrow{H} = HN$ be the mean curvature vector field. One of the considerable equations in differential geometry is $\triangle x = -n\overrightarrow{H}$, where $\triangle$ the Laplacian-Beltrami operator is defined $\triangle =$ - trace ∇^2. It is well-known that the tension field and bitension field are satisfied

$$\tau(x) = n\overrightarrow{H}, \quad \tau_2(x) = -n\Delta\overrightarrow{H}.$$

So, the immersion x is biharmonic if and only if $\triangle\overrightarrow{H} = 0$, where

$$0 = \triangle\overrightarrow{H} = 2A(\mathsf{grad}H) + nH\mathsf{grad}H + (\triangle^{\perp}H + H\mathsf{trace}A^2).$$

In particular, by considering the normal component and tangential component in the above equation, one can obtain one of main tools to study the proper biharmonic hypersurfaces in the Euclidean spaces.

Theorem 2.1 ([3]). *Let $x : M^n \to E^{n+1}$ be an isometric immersion of an n-dimensional hypersurface (M^n, g) into the Euclidean space E^{n+1}. Then M^n*

is a biharmonic hypersurface if and only if

$$\begin{cases} \triangle^{\perp} H + H\,\mathsf{trace}A^2 = 0; \\ 2A(\mathsf{grad}H) + nH\mathsf{grad}H = 0, \end{cases}$$

where A denotes the Weingarten operator and $\Delta^{\perp}$ the Laplacian in the normal bundle of M^n in E^{n+1}.

In the rest of the content, we deal with an orthonormal frame field $\{e_i\}_{i=1}^n$ on biharmonic hypersurface M^n in such a way that e_i are the principal directions and $e_1 = \frac{\mathsf{grad}H}{|\mathsf{grad}H|}$ and we call it is an *appropriate frame field.*

Lemma 2.2. *Let M^n be a biharminic hypersurface in the Euclidean space E^{n+1}. Suppose that the mean curvature of M^n is not constant. Then for the appropriate frame field $\{e_i\}_{i=1}^n$*

$$\nabla_{e_1} e_i = \sum_{k=1}^{n} \omega_{1i}^k e_k = 0 \quad \textit{for} \quad i = 1, \ldots, n, \quad \nabla_{e_i} e_1 = -\omega_{ii}^1 e_i \quad \textit{for} \quad i \neq 1,$$

where ω_{ij}^k are called connection forms for any $i, j, k = 1, \ldots, n$.

Proof. Let $x : M^n \to E^{n+1}$ be an isometric immersion of the biharmonic hypersurface M^n with the non constant mean curvature. So, there exists a point $p \in M^n$, where $\mathsf{grad}H \neq 0$ at p then there is an open subset U of M^n such that $\mathsf{grad}H \neq 0$ on U. By Theorem 2.1 we have $\mathsf{grad}H$ is a principal direction corresponding to the unique principal curvature $\frac{-n}{2}H$. Suppose that the Weingarten operator A takes the form $Ae_i = \lambda_i e_i$, i.e., e_i is an eigenvector of A with eigenvalue λ_i. We choose e_1 such that e_1 is parallel to $\mathsf{grad}H$ where it expresses $\mathsf{grad}H = \sum_{i=1}^n (e_i H) e_i$, this shows that $(e_1 H) \neq 0$ and $(e_i H) = 0$ for any $i = 1, \ldots, n$. For following our approach, we need to estimate the connection forms ω_{ij}^k which is given $\nabla_{e_i} e_j = \sum_{i=1}^n \omega_{ij}^k e_k$. By this we have

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0 \quad i \neq j, \quad i, j, k = 1, \ldots, n \tag{1}$$

since $\nabla_{e_k} \langle e_i, e_j \rangle = 0$. Moreover, by the above and the Codazzi equation we find

$$e_k(\lambda_j) e_i + (\lambda_i - \lambda_j) \omega_{ki}^j e_j = e_i(\lambda_k) e_k + (\lambda_k - \lambda_j) \omega_{ik}^j e_j$$

which yields

$$\begin{aligned} e_i(\lambda_j) &= (\lambda_i - \lambda_j) \omega_{ji}^j, \\ (\lambda_i - \lambda_j) \omega_{ki}^j &= (\lambda_k - \lambda_j) \omega_{ik}^j \end{aligned} \tag{2}$$

for distinct i, j and k where $i, j, k = 1, \ldots, n$. Now, we set $\lambda_1 = \frac{-n}{2} H$, this implies $(e_1 \lambda_1) \neq 0$ and $(e_i \lambda_1) = 0$ for any $i = 2, \ldots, n$. Then we have

$$0 = [e_i, e_j] \lambda_1 = (\omega_{ij}^1 - \omega_{ji}^1)(e_1 \lambda_1), \quad 2 \leq i, j \leq n, \quad i \neq j$$

which shows

$$\omega_{ij}^1 = \omega_{ji}^1, \quad 2 \leq i, j \leq n, \quad i \neq j. \tag{3}$$

Observe that for indices $j = 1$ and $2 \leq i, k \leq n$ the equation (2) follows

$$(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1,$$

where $\lambda_i \neq \lambda_k$. Then, because of uniqueness of λ_1 and (3) by the above we have

$$\omega_{ij}^1 = \omega_{ji}^1 = 0, \quad i \neq j, \quad 2 \leq i, j \leq n.$$

On the one hand, from (1) it follows $\omega_{k1}^1 = 0$ and $\omega_{k1}^j + \omega_{kj}^1 = 0$ for any $i, j, k = 1, \ldots, n$. Then, $\omega_{1i}^1 = \omega_{11}^i = 0$ where $i = 1, \ldots, n$. So,

$$\omega_{ij}^1 = \omega_{ji}^1 = 0, \quad i \neq j, \quad i, j = 1, \ldots, n.$$

Afterall, putting this all together, we obtain our claim. □

3. Biharmonic hypersurfaces in the Euclidean space E^{n+1} with the recurrent operators

Let T be a tensor on a Rimannian manifold M^n. We say T is *recurrent* if there exists a certain 1-form η on M^n such that for any X tangent to M^n satisfies $\nabla_X T = \eta(X)T$. So, the recurrent $(1, 1)$-tensor is an extension of the parallel one.

Theorem 3.1. *Let M^n be a biharmonic hypersurface with the recurrent Ricci operator in the Euclidean space E^{n+1}. Then M^n is a minimal hypersurface.*

Proof. Let $x : M^n \to E^{n+1}$ be an isometric biharmonic immersion. Consider the appropriate frame field $\{e_i\}_{i=1}^n$ on M^n consisting of eigenvectors of the Weingarten operator A. Then, by the Gauss equation we have $\mathsf{Ric}(e_i) = \alpha_i e_i$ for any $i = 1, \ldots, n$ where $\alpha_i = nH\lambda_i - \lambda_i^2$. Since the Ricci operator is recurrent, i.e., $\big(\nabla_X \mathsf{Ric}\big)Y = \eta(X)\mathsf{Ric}(Y)$ for X and Y tangent to M^n, we get

$$\begin{aligned}
\nabla_{e_i}\big(\mathsf{Ric}(e_i)\big) &= \big(\nabla_{e_i}\mathsf{Ric}\big)(e_i) + \mathsf{Ric}(\nabla_{e_i} e_i) \\
&= \eta(e_i)\alpha_i e_i + \sum_k \omega_{ii}^k \mathsf{Ric}(e_k), \\
&= \eta(e_i)\alpha_i e_i + \sum_k \omega_{ii}^k \alpha_k e_k, \quad i = 1, \ldots, n. \qquad (4)
\end{aligned}$$

On the one hand

$$\begin{aligned}
\nabla_{e_i}\big(\mathsf{Ric}(e_i)\big) &= \nabla_{e_i}(\alpha_i e_i) \\
&= (e_i\alpha_i)e_i + \alpha_i \sum_k \omega_{ii}^k e_k, \quad i = 1, \ldots, n. \qquad (5)
\end{aligned}$$

Hence, we get from (4) and (5) the following

$$\big(e_i\alpha_i - \eta(e_i)\alpha_i\big)e_i = \sum_k \big(\alpha_k - \alpha_i\big)\omega_{ii}^k e_k.$$

From linear independent, the above equation for $i \neq 1$ follows $(\alpha_1 - \alpha_i)\omega_{ii}^1 = 0$ in which by applying Lemma 2.2 we have $\omega_{ii}^1 \neq 0$. Now, the equation $\alpha_i =$

$nH\lambda_i - \lambda_i^2$ has at most two distinct roots. Then, we reach that M^n has at most two distinct principal curvatures at each point. Therefore, by following the studying in [6], we obtain the result. □

Theorem 3.2. *Let M^n be a biharmonic hypersurface in the Euclidean space E^{n+1} with the recurrent curvature operator. Then M^n is minimal.*

Proof. Let $x : M^n \to E^{n+1}$ be an isometric immersion of a biharmonic hypersurface M^n in the Euclidean space E^{n+1}. Choosing the appropriate frame field $\{e_1, \ldots, e_n\}$, the Guass equation yields $R(e_i, e_j)e_k = 0$ for distinct i, j and k. According to the assumption the curvature operator R is recurrent, i.e., $(\nabla_X R(Y, Z))W = \eta(X)R(Y, Z)W$ for all X, Y, Z and W tangent to M^n so, $(\nabla_{e_i} R(e_j, e_k))e_l = \eta(e_i)R(e_j, e_k)e_l = 0$. Then, take the Guass equation we have

$$\begin{aligned} 0 = \nabla_{e_i} R(e_j, e_k)e_l &= \big(\nabla_{e_i} R(e_j, e_k)\big)e_l + R(e_j, e_k)\nabla_{e_i} e_l \\ &= \eta(e_i)R(e_j, e_k)e_l + R(e_j, e_k)\sum_{t=1}^{n} \omega_{il}^s e_s \\ &= \omega_{il}^k \lambda_k \lambda_j e_j - \omega_{il}^j \lambda_j \lambda_k e_k, \end{aligned}$$

where $i, j, k, l \neq 1$ because by Lemma 2.2 $\omega_{ij}^1 = 0$ for $i \neq j$. Then, from the linear independence of $\{e_i\}$ follows that $\omega_{il}^k \lambda_k \lambda_j = 0$. Now, for all nonzero principal curvatures it follows $\omega_{il}^k = 0$ for distinct indices. Thus, all we need is to use the Codazzi equation (2) in which

$$0 = (\lambda_l - \lambda_k)\omega_{il}^k = (\lambda_i - \lambda_k)\omega_{li}^k,$$

this yields $\lambda_i = \lambda_k$ or $\omega_{li}^k = 0$ for $j \neq k$. In particular if $\omega_{li}^k = 0$, then the Codazzi equation implies $\lambda_l = \lambda_k$ too. Hence, by the above there exist at most two distinct principal curvatures at each point of M^n. Note that $\lambda_1 = \frac{-n}{2}H$ that is corresponding to the principal direction $e_1 = \frac{\mathrm{grad} H}{|\mathrm{grad} H|}$. Now, by following the studying in [6], we obtain the result. □

Now, directly by the above theorem we will have the following result.

Corollary 3.3. *The biharmonic locally symmetric hypersurfaces in the Euclidean space E^{n+1} are minimal.*

Theorem 3.4. *Let M^n be a biharmonic hypersurface in the Euclidean space E^{n+1}, with the recurrent Jacobi operator R_X for any $X \in \Gamma(T(M^n))$. Then M^n is minimal.*

Proof. Let $x : M^n \to E^{n+1}$ be an isometric immersion of a biharmonic hypersurface M^n in the Euclidean space E^{n+1}. Now we use the assumption that the Jacobi operator is recurrent, i.e., $(\nabla_Y R_X)(Z) = \eta(Y)R_X(Z)$ for all X, Y and Z tangent to M^n. Consider the appropriate frame field $\{e_i\}_{i=1}^n$ and the Guass equation then we see that the recurrent Jacobi operator expresses

$$\nabla_{e_i} R_{e_j}(e_k) = \big(\nabla_{e_i} R_{e_j}\big)(e_k) + R_{e_j}(\nabla_{e_i} e_k)$$

$$
\begin{aligned}
&= \eta(e_i)R_{e_j}(e_k) + R_{e_j}(\nabla_{e_i}e_k)\\
&= \eta(e_i)R(e_k, e_j)e_j + R(\nabla_{e_i}e_k, e_j)e_j\\
&= -\eta(e_i)\lambda_j\lambda_k e_k - \lambda_j \sum_{l=1, l\neq j}^{n} \omega_{ik}^l \lambda_l e_l.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\nabla_{e_i} R_{e_j}(e_k) &= \nabla_{e_i}\big(R(e_k, e_j)e_j\big)\\
&= -e_i(\lambda_j\lambda_k)e_k - \lambda_j\lambda_k \sum_{l=1}^{n} \omega_{ik}^l e_l.
\end{aligned}
$$

Comparing the components follows that $\lambda_j \sum_{l=1, l\neq j}^n \omega_{il}^l \lambda_l e_l = \lambda_j\lambda_k \sum_{l=1}^n \omega_{ik}^l e_l$. If $\lambda_j \neq 0$, then

$$
\sum_{l=2}^{n} (\lambda_l - \lambda_k)\omega_{ik}^l e_l - \lambda_j \omega_{ik}^j e_j = 0.
$$

One consequence of the above is that $\lambda_l = \lambda_k$ for $2 \leq l, k \leq n$. Then take $\lambda_1 = -\frac{n}{2}H$ and its uniqueness turns out that there are two distinct principal curvatures at each points of M^n. Furthermore, because $\lambda_j \neq 0$ so $\omega_{ik}^j = 0$ that the Codazzi equation follows $(\lambda_i - \lambda_j)\omega_{ki}^j = 0$, which yields $\lambda_i = \lambda_j$ for $i \neq j$. Similarly, we get the same result. Now, by following the work in [6] we obtain what was claimed. □

Theorem 3.5. *Let M^n be a biharmonic hypersurface with the recurrent Weyl operator $W_{X,Y}$ for any $X, Y \in \Gamma(T(M^n))$ in the Euclidean space E^{n+1}. Then M^n is minimal.*

Proof. Let $x : M^n \to E^{n+1}$ be an isometric immersion of a biharmonic hypersurface M^n in the Euclidean space E^{n+1}. In this case we see that with the appropriate frame field $\{e_i\}_{i=1}^n$ on M^n, the Weyl operator $W_{e_i,e_j}(e_k)$ vanishes for distinct indices, since

$$
\begin{aligned}
W_{e_i,e_j}(e_k) = R(e_i, e_j)e_k &- \frac{1}{n-2}\{\mathsf{Ricci}(e_j, e_k)e_i - \mathsf{Ricci}(e_i, e_k)e_j\\
&+ g(e_j, e_k)\mathsf{Ricci}(e_i) - g(e_i, e_k)\mathsf{Ricci}(e_j)\}\\
&+ \frac{s}{(n-1)(n-2)}\{g(e_j, e_k)e_i - g(e_i, e_k)e_j\},
\end{aligned}
$$

where R and s are the curvature tensor and the scalar curvature, respectively and all terms are zero. Note that, $W_{e_i,e_j}(e_j) = \alpha e_i$ where $\alpha = \lambda_i\lambda_j - (\lambda_i + \lambda_j)(H - \lambda_i - \lambda_j) + \frac{s}{n-2}$. Consider the assumption that the Weyl operator is recurrent, i.e., $(\nabla_V W_{X,Y})(Z) = \eta(V)W_{X,Y}(Z)$ for all X, Y, Z and V tangent to M^n. In particular, it shows

$$
0 = \nabla_{e_j} W_{e_i,e_j}(e_1) = W_{e_i,e_j}(\nabla_{e_j}e_1) = \omega_{j1}^j W_{e_i,e_j}(e_j) = \omega_{j1}^j \alpha e_i,
$$

where by Lemma 2.2 $\nabla_{e_j} e_1 = \omega_{jj}^1 e_j$ for $j \neq 1$. Thus, $\alpha = 0$, i.e.,

$$\lambda_i \lambda_j - (\lambda_i + \lambda_j)(H - \lambda_i - \lambda_j) = a, \quad i \neq j \tag{6}$$

in which $a = \frac{s}{2-n}$. Now, to reach the purpose we need to consider

$$\lambda_i \lambda_k - (\lambda_i + \lambda_k)(H - \lambda_i - \lambda_k) = a, \quad i \neq k. \tag{7}$$

Then from (6) and (7) it follows

$$3\lambda_i + \lambda_j + \lambda_k - H = 0, \quad 2 \leq i, j, k \leq n,$$

which leads to that all the principal curvatures are equal. Now, take the unique principal curvature $\lambda_1 = -\frac{nH}{2}$ corresponding to the principal direction $e_1 = \frac{\operatorname{grad} H}{|\operatorname{grad} H|}$. So, there exist two distinct principal curvatures at each point of M^n. Then, by following the studying in [6] we get the result. □

Theorem 3.6. *Let M^n be a biharmonic hypersurface with the recurrent shape operator in the Euclidean space E^{n+1}. Then M^n is minimal.*

Proof. Let $x : M^n \to E^{n+1}$ be an isometric immersion of a biharmonic hypersurface M^n in the Euclidean space E^{n+1}. We use the assumption that the shape operator is recurrent, i.e., $(\nabla_X A)Y = \eta(X)A(Y)$ for X and Y tangent to M^n such that for the appropriate frame field $\{e_i\}_{i=1}^n$ it satisfies

$$g((\nabla_{e_i} A)e_j, e_k) = \eta(e_i) g(\lambda_j e_j, e_k) = 0.$$

Then the Codazzi equation yields

$$\begin{aligned} 0 = g((\nabla_{e_i} A)e_j, e_k) &= (\lambda_j - \lambda_k) g(\nabla_{e_i} e_j, e_k) \\ &= (\lambda_j - \lambda_k)\omega_{ij}^k \end{aligned}$$

for $2 \leq i, j, k \leq n$ where by Lemma 2.2 $\nabla_{e_i} e_j = \sum_{l=2, l\neq j}^n \omega_{ij}^l e_l$. By the above, one consequence is $\lambda_j = \lambda_k$ for $2 \leq j, k \leq n$. Add the unique principal curvature $\frac{-nH}{2}$ corresponding with the principal direction $e_1 = \frac{\operatorname{grad} H}{|\operatorname{grad} H|}$ then it determines that there exist two distinct principal curvatures at each point of M^n. If $\lambda_j \neq \lambda_k$, then $\omega_{ij}^k = 0$ and in this case the Codazzi equation expresses

$$0 = (\lambda_j - \lambda_k)\omega_{ij}^k = (\lambda_i - \lambda_k)\omega_{ji}^k,$$

so, $\lambda_i = \lambda_k$ where $2 \leq i, k \leq n$. Similarly, take the $\lambda_1 = \frac{-nH}{2}$ it leads to there are two distinct principal curvatures at each point. Then we get the result by the work in [6]. □

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References

[1] K. Akutagawa and S. Maeta, *Biharmonic properly immersed submanifolds in Euclidean spaces*, Geom. Dedicata **164** (2013), 351–355. https://doi.org/10.1007/s10711-012-9778-1

[2] A. Balmuş, S. Montaldo, and C. Oniciuc, *Biharmonic hypersurfaces in 4-dimensional space forms*, Math. Nachr. **283** (2010), no. 12, 1696–1705. https://doi.org/10.1002/mana.200710176

[3] B.-Y. Chen, *Total mean curvature and submanifolds of finite type*, Series in Pure Mathematics, 1, World Scientific Publishing Co., Singapore, 1984. https://doi.org/10.1142/0065

[4] B.-Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. **17** (1991), no. 2, 169–188.

[5] B.-Y. Chen and S. Ishikawa, *Biharmonic surfaces in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ. Ser. A **45** (1991), no. 2, 323–347. https://doi.org/10.2206/kyushumfs.45.323

[6] I. Dimitrić, *Submanifolds of E^m with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica **20** (1992), no. 1, 53–65.

[7] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, 50, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1983. https://doi.org/10.1090/cbms/050

[8] J. Eells, Jr., and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. https://doi.org/10.2307/2373037

[9] Y. Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in spheres*, Math. Nachr. **288** (2015), no. 7, 763–774. https://doi.org/10.1002/mana.201400101

[10] Y. Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean space*, Tohoku Math. J. (2) **67** (2015), no. 3, 465–479. https://doi.org/10.2748/tmj/1446818561

[11] R. S. Gupta and A. Sharfuddin, *Biharmonic hypersurfaces in Euclidean space E^5*, J. Geom. **107** (2016), no. 3, 685–705. https://doi.org/10.1007/s00022-015-0310-2

[12] Th. Hasanis and Th. Vlachos, *Hypersurfaces in E^4 with harmonic mean curvature vector field*, Math. Nachr. **172** (1995), 145–169. https://doi.org/10.1002/mana.19951720112

[13] G. Y. Jiang, *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A **7** (1986), no. 2, 130–144.

[14] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402.

[15] N. Mosadegh and E. Abedi, *Biharmonic Hopf hypersurfaces of complex Euclidean space and odd dimensional sphere*, Zh. Mat. Fiz. Anal. Geom. **16** (2020), no. 2, 161–173. https://doi.org/10.15407/mag16.02.161

[16] N. Mosadegh, E. Abedi, and M. Ilmakchi, *Ricci soliton biharmonic hypersurfaces in the Euclidean space*, Ukraïn. Mat. Zh. **73** (2021), no. 7, 931–937. https://doi.org/10.37863/umzh.v73i7.495

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