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BLOW UP OF SOLUTIONS FOR A PETROVSKY TYPE EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper aims to investigate the initial boundary value problem of the nonlinear viscoelastic Petrovsky type equation with nonlinear damping and logarithmic source term. We derive the blow-up results by the combination of the perturbation energy method, concavity method, and differential-integral inequality technique.

1. Introduction

In this article, we are concerned with the following viscoelastic Petrovsky type equation with logarithmic nonlinearity,

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s) \triangle^2 u(s)\,ds + |u_t|^{m-2} u_t = |u|^{p-2} u \ln|u|, \tag{1.1}$$

where $(x,t) \in \Omega \times \mathbb{R}^+$ with initial data

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega \tag{1.2}$$

and boundary conditions

$$u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0, \tag{1.3}$$

being Ω a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, and v the unit outer normal to $\partial\Omega$.

The Petrovsky [17, 18] type of equation

$$u_{tt} + \Delta^2 u = f(x,t,u,u_t)$$

originated from the study of beams and plates, and it can also be used in many branches of physics such as optics, geophysics, nuclear physics, and ocean acoustics. Many authors gave big attention to this problem for quite a long time. They made a lot of progress, as reported in [1–3, 6, 8–10, 13–15, 21, 22] with references therein. Memory damping has its origin in the mathematical

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description of viscoelastic materials. The viscoelastic materials exhibit natural stabilization mechanism, which is due to the special property of these materials, to retain a memory of their past history. For nonlinearly damped viscoelastic equations with logarithmic source effect and time delay, see [16] with references therein.

Alabau-Boussouira et al. [1] discussed the initial-boundary value problem of a linear Petrovsky equation related to a plate model with memory in absence of source term,

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s) \triangle^2 u(s)\, ds = 0, \tag{1.4}$$

and showed that the solution decays exponentially or polynomially as $t \to \infty$ if the initial data is sufficient small.

An energy source that acts in opposition to damping can cause destabilization in the model. The Petrovsky type models without memory and source term were discussed by Messaoudi [15],

$$u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u \ln |u|, \tag{1.5}$$

where was established an existence result, and proved that the solution exists globally if $m \geq p$, however, if $m < p$ and the initial energy is negative, the solution blows up in finite time. Later, Chen and Zhou [6] proved that the solution of (1.5) blows up with positive initial energy. Moreover, they claimed that the solution blows up in finite time for vanishing initial energy under the condition $m = 2$.

Li and Gao [10] considered (1.4) with memory and source term,

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s) \triangle^2 u(s)\, ds + |u_t|^{m-2} u_t = |u|^{p-2} u. \tag{1.6}$$

They showed that the solution of the initial-boundary value problem of the model (1.6) blows up in finite time. In the case $m = 2$, they studied the blow-up of solution with a different method. In [13], Liu et al. studied the finite time blow-up for solutions of problem (1.6) with arbitrary high initial energy. Also, Liu et al. [14] obtained global nonexistence results of the problem (1.6) with a weak damping term for positive initial energy. Chen and Zhou [6] proved that the solution blows up with positive initial energy.

In recent years, a great deal of mathematical effort has been devoted to the study of nonlinear wave equations with logarithmic nonlinearity. The logarithmic source term has a wide range of applications of physics such as quantum field theory, nuclear physics, geophysics, optics [4, 5]. The hyperbolic type equations with logarithmic source terms drew mathematicians' attention. The global existence, blow up, local existence, and asymptotic behavior of solutions were studied by many authors, see for instance [3, 5, 7, 9, 11, 12, 19, 20, 23].

Gharabli et al. [3] considered the Petrovsky type equation with memory term and logarithmic nonlinearity given by

$$|u_t|^{\rho} u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^t g(t-s) \triangle^2 u(s)\, ds = ku \ln u. \tag{1.7}$$

They established the local existence and general decay rate of the solutions of the problem (1.7). Later, in [2] was treated (1.7) in the absence $\Delta^2 u_{tt}$ and in the presence of u for $\rho = 0$ and then, was proved the existence and general decay result under different suitable conditions.

As far as we know, in works about the Petrovsky type equation with viscoelastic and logarithmic nonlinearity, as seen from the above statements, there are not enough results on the blow-up problem (1.1) with positive initial energy. Hence, we intend to study finite time blow-up of the solution to the problem (1.1).

This work is written as follows. In Section 2, we prepare preliminaries, some notations. In the last section, we establish the blow-up results of the solution.

Throughout the present paper, we denote by $(\cdot,\cdot)$ the inner product of L^2 space. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms.

2. Preliminaries

Firstly, we present the Sobolev's embedding inequality: suppose that p is a constant such that $1 \le p \le \frac{2n}{n-4}$ if $n \ge 5$; $p \ge 1$ if $n \le 4$, then $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$ continuously, and for $u \in H_0^2(\Omega)$

$$\|u\|_p \le C \|\Delta u\|_2, \tag{2.1}$$

where C denotes the best embedding constant.

With regard to problem (1.1), we assume that the parameter p, m and the kernel function g satisfy the following assumptions.

$(A1)$ $2 < p < \infty$, if $n \le 4$; $2 < p < \frac{2(n-2)}{n-4}$, if $n \ge 5$.

$(A2)$ $2 \le m < \infty$, if $n \le 4$; $2 \le m \le \frac{2n}{n-4}$, if $n \ge 5$.

$(A3)$ $g : [0,+\infty) \to [0,+\infty)$ is a nonincreasing and differentiable function satisfying

$$1 - \int_0^{\infty} g(s)\, ds := l > 0. \tag{2.2}$$

In addition to (2.2), g satisfies, the inequalities

$$g(s) \ge 0,\ g'(s) \le 0,$$

$$\int_0^{\infty} g(s)\, ds < \frac{(p(1-b)/2) - 1}{(p(1-b)/2) - 1 + (1/2p(1-b))}. \tag{2.3}$$

We first state a local existence theorem. Using the Faedo-Galerkin method and the contraction mapping principle, the local existence result can also be proved in a similar way to those of [8].

Theorem 2.1. *Assume that* $(A1)$ *and* $(A3)$ *hold. Then, for the initial data* $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, *there exists* $T > 0$, *such that the problem* (1.1) *admits a unique local weak solution on* $[0;T]$.

Remark 2.2. Condition (2.2) is necessary to guarantee the hyperbolicity and well-posedness of system (1.1).

In this part, we will introduce some material necessary in the proof of our main result (Theorem 3.1). For this purpose, we give some lemmas which will be used throughout this work.

Let us begin with defining the following total energy functional

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|^2 + \frac{1}{2}(g \circ \Delta u)(t) + \frac{1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega u^p \ln|u|\,dx, \tag{2.4}$$

where

$$(g \circ \Delta u)(t) = \int_0^t g(t-s)\|\Delta u(s) - \Delta u(t)\|^2\,ds. \tag{2.5}$$

Lemma 2.3. *Suppose that* $(A1)$ *and* $(A2)$ *hold and let* $u(t)$ *be the solution of the problem* (1.1). *Then the energy functional* $E(t)$ *is decreasing with respect to* t *and*

$$E'(t) = \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}\int_0^t g(s)\,ds\,\|\Delta u(t)\|^2 - \|u_t\|_m^m \le 0, \tag{2.6}$$

where

$$(g' \circ \Delta u)(t) = \int_0^t g'(t-s)\int_\Omega |\Delta u(s) - \Delta u(t)|^2\,dxdt.$$

Proof. We multiply both sides of (1.1) by u_t and then integrating from 0 to t and using Green formula, we have

$$E'(t) = \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}\int_0^t g(s)\,ds\,\|\Delta u(t)\|^2 - \|u_t\|_m^m \le 0,$$

that is,

$$E(t) = \int_0^t \left[\frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}\int_0^t g(s)\,ds\,\|\Delta u(t)\|^2 - \|u_t\|_m^m\right] + E(0). \tag{2.7}$$

□

Lemma 2.4. *Suppose that* $(A1)$-$(A3)$ *hold. We also define*

$$G(\alpha) = \frac{1}{2}\alpha^2 - \frac{C_1^{p+1}}{p}\alpha^{p+1} \quad \textit{with } C/l^{1/2} = C_1,$$

and

$$\alpha(t) = \left(l\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\right)^{\frac{1}{2}}.$$

Then

$$E(t) \geq G(\alpha(t)). \tag{2.8}$$

Proof. We first note that, by (2.4), we obtain

$$\begin{aligned}
E(t) &\geq \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\Delta u\|^2+\frac{1}{2}(g\circ\Delta u)(t)+\frac{1}{p^2}\|u\|_p^p-\frac{1}{p}\int_\Omega u^p\ln|u|\,dx \\
&\geq \frac{1}{2}l\|\Delta u\|^2+\frac{1}{2}(g\circ\Delta u)(t)-\frac{1}{p}\|u\|_{p+1}^{p+1} \\
&\geq \frac{1}{2}l\|\Delta u\|^2+\frac{1}{2}(g\circ\Delta u)(t)-\frac{1}{p}C^{p+1}\|\Delta u\|_2^{p+1} \\
&\geq \frac{1}{2}\left(l\|\Delta u\|^2+(g\circ\Delta u)(t)\right)-\frac{1}{p}C_1^{p+1}\left(l\|\Delta u(t)\|^2+(g\circ\Delta u)(t)\right)^{\frac{p+1}{2}} \\
&= \frac{1}{2}\alpha^2-\frac{C_1^{p+1}}{p}\alpha^{p+1} \\
&= G(\alpha(t)).
\end{aligned}$$

□

Remark 2.5. We can easily see that G is increasing for $0<\alpha<\lambda$, decreasing for $\alpha>\lambda$, $\lim_{\alpha\to\infty}G(\alpha)\to-\infty$, G has a maximum at $\lambda=\left(\frac{p}{p+1}\right)^{\frac{1}{p-1}}C_1^{-\frac{p+1}{p-1}}$ and the maximum value is

$$G(\lambda)=\left(\frac{1}{2}\left(\frac{p}{p+1}\right)^{\frac{2}{p-1}}-\frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{p+1}{p-1}}\right)C_1^{-\frac{2(p+1)}{p-1}}=E_1. \tag{2.9}$$

Lemma 2.6. *Assume that* (*A*1)*-*(*A*3) *hold. Suppose that the solution* u *of problem* (1.1) *satisfies*

$$E(0)<E_1,\ \alpha(0)=l^{\frac{1}{2}}\|\Delta u_0\|^2>\lambda=\left(\frac{p}{p+1}\right)^{\frac{1}{p-1}}C_1^{-\frac{p+1}{p-1}}. \tag{2.10}$$

Then there is a constant $\lambda_1>\lambda$ *such that*

$$\alpha(t)=\left(l\|\Delta u(t)\|^2+(g\circ\Delta u)(t)\right)^{\frac{1}{2}}\geq\lambda_1$$

and

$$\|u(t)\|_{p+1}\geq C_1\lambda_1. \tag{2.11}$$

Proof. Since $E(0)<E_1$ and $G(\alpha)$ is a continuous function, there exist λ_1' and λ_1 with $\lambda_1'<\lambda<\lambda_1$ such that

$$G(\lambda_1')=G(\lambda_1)=E(0),$$

which combined with Lemma 2.4 gives

$$G(\alpha(0))<E(0)=G(\lambda_1). \tag{2.12}$$

By (2.9)-(2.12) and Remark 2.5, we deduce

$$\alpha(0)\geq\lambda_1. \tag{2.13}$$

Now we prove the first conclusion. If not, then there exists $t_0 > 0$ such that

$$\alpha^2(t_0) = \left(l \left\| \Delta u(t_0) \right\|^2 + (g \circ \Delta u)(t_0) \right) < \lambda_1^2.$$

Case1. If $\lambda_1' < \alpha(t_0) < \lambda_1$, according to Lemma 2.3 and Remark 2.5, we know $G(\alpha(t_0)) > E(0) \geq E(t_0)$, which contradicts to Lemma 2.4.

Case 2. If $\alpha(t_0) < \lambda_1'$, then $\alpha(t_0) < \lambda_1' < \lambda_1$. Set

$$h(t) = \alpha(t) - \frac{\lambda_1' + \lambda_1}{2}.$$

Clearly, $h(t)$ is a continuous function, $h(t_0) < 0$ and $h(0) > 0$ by applying (2.13). Hence, there exists $t_1 \in (0, t_0)$ such that $h(t_1) = 0$, that is $\alpha(t_1) = \frac{\lambda_1' + \lambda_1}{2}$ which implies

$$G(\alpha(t_1)) > E(0) \geq E(t_1).$$

This contradicts to Lemma 2.4.

To establish (2.11), by using the definition of energy function, we obtain

$$E(0) + \frac{1}{p} \int_\Omega u^p \ln|u| \, dx \geq \frac{1}{2} \left(1 - \int_0^t g(s)\, ds \right) \left\| \Delta u \right\|^2 + \frac{1}{2} (g \circ \Delta u)(t).$$

Consequently, we obtain

$$\begin{aligned} \frac{1}{p} \left\| u(t) \right\|_{p+1}^{p+1} \geq \frac{1}{p} \int_\Omega u^p \ln|u| \, dx &\geq \frac{1}{2} \left(1 - \int_0^t g(s)\, ds \right) \left\| \Delta u \right\|^2 \\ &\quad + \frac{1}{2} (g \circ \Delta u)(t) - E(0) \\ &\geq \frac{1}{2} \left(l \left\| \Delta u \right\|^2 + (g \circ \Delta u)(t) \right) - E(0) \\ &\geq \frac{1}{2} \lambda_1^2 - G(\lambda_1) = \frac{C_1^{p+1}}{p} \lambda_1^{p+1}. \end{aligned} \tag{2.14}$$

The proof is completed. □

We give some lemmas which will be used in our proof. For proofs of Lemmas 2.7-2.10, we refer the readers to Kafini and Messaoudi [9].

Lemma 2.7. *Suppose that (A1)-(A2) hold. There exists a positive constant depending on* Ω *only such that*

$$\left(\int_\Omega u^p \ln|u| \, dx \right)^{\frac{s}{p}} \leq C \left[\int_\Omega u^p \ln|u| \, dx + \left\| \Delta u \right\|_2^2 \right] \tag{2.15}$$

for any $u \in L^{p+1}(\Omega)$ *and* $2 \leq s \leq p$, *provided that* $\int_\Omega u^p \ln|u| \, dx \geq 0$.

Lemma 2.8. *Suppose that (A1)-(A2) hold. There exists a positive constant depending on* Ω *only such that*

$$\left\| u \right\|_p^p \leq C \left[\int_\Omega u^p \ln|u| \, dx + \left\| \Delta u \right\|_2^2 \right] \tag{2.16}$$

for any $u \in L^p(\Omega)$, *provided that* $\int_\Omega u^p \ln|u| \, dx \geq 0$.

Thus, the following estimate holds.

Corollary 2.9. *Let the assumptions of Lemma 2.7 and $m < p$ hold. Using the fact that* $\|u\|_m^m \le C\|u\|_p^m \le C\left(\|u\|_p^p\right)^{\frac{m}{p}}$. *Then we obtain the following*

$$\|u\|_m^m \le C\left[\left(\int_\Omega u^p \ln|u|\,dx\right)^{\frac{m}{p}} + \|\Delta u\|^{\frac{2m}{p}}\right]. \tag{2.17}$$

Lemma 2.10. *Suppose that (A1)-(A2) hold. The there exists a positive constant depending on Ω only such that*

$$\|u\|_p^s \le C\left[\|u\|_p^p + \|\Delta u\|_2^2\right] \tag{2.18}$$

for any $u \in L^p(\Omega)$ and $2 \le s \le p$.

We define

$$H(t) = E_1 - E(t), \tag{2.19}$$

and we will use throughout this paper. As a result of Lemma 2.10 and definition of $H(t)$ and $E(t)$, we have the following estimative:

Lemma 2.11. *Let u be a solution of (1.1). Suppose that (A1) and (A2) hold. Then we get*

$$\|u\|_p^s \le C\left(-H(t) - \|u_t\|^2 - (g\circ\Delta u)(t) + \|u\|_{p+1}^{p+1} + \|u\|_p^p\right) \tag{2.20}$$

for any $u \in L^p(\Omega)$ and $2 \le s \le p$.

Proof. Using (2.2) and definition of $E(t)$, we obtain

$$\begin{aligned}\frac{1}{2}l\|\Delta u\|^2 &\le \frac{1}{2}\left(1 - \int_0^\infty g(s)\,ds\right)\|\Delta u\|^2\\ &\le E(t) - \frac{1}{2}\left(\|u_t\|^2 + (g\circ\Delta u)(t) + \frac{2}{p^2}\|u\|_p^p\right) + \frac{1}{p}\int_\Omega u^p\ln|u|\,dx\\ &\le E_1 - H(t) - \frac{1}{2}\left(\|u_t\|^2 + (g\circ\Delta u)(t) + \frac{2}{p^2}\|u\|_p^p\right) + \frac{1}{p}\|u\|_{p+1}^{p+1}.\end{aligned}$$

Using Remark 2.5 and Lemma 2.6, we note that

$$\left(\frac{p+1}{p}\right)^{\frac{p+1}{p-1}}\|u\|_{p+1}^{p+1} \ge C_1^{-\frac{2(p+1)}{p-1}}$$

and

$$E_1 = \left(\frac{p}{p+1}\right)^{\frac{p+1}{p-1}}\left[\frac{1}{2}\frac{(p+1)}{p} - \frac{1}{p}\right].$$

Consequently, we obtain

$$E_1 \le \left[\frac{1}{2}\left(\frac{p+1}{p}\right) - \frac{1}{p}\right]\|u\|_{p+1}^{p+1}. \tag{2.21}$$

Then, a combination of Lemma 2.10 and (2.21) leads to

$$\begin{aligned}\|u\|_p^s &\leq \frac{p+1}{2p}\|u\|_{p+1}^{p+1} - H(t) - \frac{1}{2}\|u_t\|^2 - \frac{1}{2}(g\circ\Delta u)(t) - \frac{1}{p^2}\|u\|_p^p \\ &\leq C\left(-H(t) - \|u_t\|^2 - (g\circ\Delta u)(t) + \|u\|_{p+1}^{p+1} + \|u\|_p^p\right).\end{aligned}$$

Finally we give the desired result. □

3. Blow up result

In this part, we state and prove a blow up result for problem (1.1) in finite time with $E(0) < E_1$.

Theorem 3.1. *Assume that* $(A1)$, $(A2)$ *and* $m < p$ *hold. Assume further that* g *satisfies* (2.2), (2.3). *Then any solution of* (1.1) *with initial data blows up in finite time provided that* $E(0) < E_1$.

Proof. Using (2.4), Lemma 2.3, (2.19) and $E(0) < E_1$, we get

$$\begin{aligned}0 &< E_1 - E(0) = H(0) \\ &\leq H(t) = E_1 - \frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|^2 \\ &\qquad - \frac{1}{2}(g\circ\Delta u)(t) - \frac{1}{p^2}\|u\|_p^p + \frac{1}{p}\int_\Omega u^p \ln|u|\,dx.\end{aligned}$$

By (2.9) and Lemma 2.6, we note that

$$\begin{aligned}&E_1 - \frac{1}{2}\left[\|u_t\|^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\Delta u\|^2 + (g\circ\Delta u)(t) + \frac{2}{p^2}\|u\|_p^p\right] \\ &< E_1 - \frac{1}{2}\lambda^2 = -\frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{p+1}{p-1}} C_1^{-\frac{2(p+1)}{p-1}} < 0\end{aligned} \tag{3.1}$$

for $\forall t \geq 0$, which implies

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega u^p \ln|u|\,dx. \tag{3.2}$$

Let us define the following function

$$L(t) = H^{1-\beta}(t) + \varepsilon\int_\Omega u u_t dx, \tag{3.3}$$

where β is a suitable positive constant which will be determined later and for

$$\frac{2(p-m)}{(m-1)p^2} < \beta < \frac{p-m}{(m-1)p} < 1. \tag{3.4}$$

Now, differentiating $L(t)$ with respect to t and using equation (1.1) we have

$$\begin{aligned}L'(t) &= (1-\beta)H^{-\beta}(t)H'(t) + \varepsilon\int_\Omega |u_t|^2\,dx + \int_\Omega u u_{tt} dx \\ &\geq (1-\beta)H^{-\beta}(t)\left[\|u_t\|_m^m - \frac{1}{2}(g'\circ\Delta u)(t) + \frac{1}{2}\int_0^t g(s)\,ds\,\|\Delta u(t)\|^2\right] + \varepsilon\|u_t\|^2\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon \int_{\Omega} u\left(-\Delta^2 u+\int_0^t g(t-s) \triangle^2 u(s)\, ds-|u_t|^{m-2} u_t+|u|^{p-2} u \ln|u|\right) dx \\
\geq\ & (1-\beta) H^{-\beta}(t)\|u_t\|_m^m+\varepsilon\|u_t\|^2-\varepsilon\|\Delta u\|^2 \\
&+\varepsilon \int_0^t g(t-s) \int_{\Omega} \triangle u(t) \triangle u(s)\, dx ds-\varepsilon \int_{\Omega}|u_t|^{m-2} u_t u dx+\varepsilon \int_{\Omega} u^p \ln|u|\, dx.
\end{aligned}
$$

That is

$$
\begin{aligned}
L'(t) \geq\ & (1-\beta) H^{-\beta}(t)\|u_t\|_m^m+\varepsilon\|u_t\|^2-\varepsilon\|\Delta u\|^2 \\
&+\varepsilon \int_0^t g(t-s)\|\Delta u(t)\|^2 \\
&+\varepsilon \int_0^t g(t-s) \int_{\Omega} \triangle u(t)[\triangle u(s)-\triangle u(t)]\, dx ds \\
&-\varepsilon \int_{\Omega}|u_t|^{m-2} u_t u dx+\varepsilon \int_{\Omega} u^p \ln|u|\, dx.
\end{aligned} \tag{3.5}
$$

In view of Young's inequality, for any $\delta>0$ we obtain

$$
\begin{aligned}
&\left|\int_0^t g(t-s) \int_{\Omega} \triangle u(t)[\triangle u(s)-\triangle u(t)]\, dx ds\right| \\
&\leq \delta \int_0^t g(t-s)\|\triangle u(s)-\triangle u(t)\|^2 ds+\frac{1}{4 \delta} \int_0^t g(s)\, ds\|\triangle u(t)\|^2 \\
&=\delta(g \circ \Delta u)(t)+\frac{1}{4 \delta} \int_0^t g(s)\, ds\|\triangle u(t)\|^2 .
\end{aligned} \tag{3.6}
$$

Taking (3.6) into (3.5) yields that

$$
\begin{aligned}
L'(t) \geq\ & (1-\beta) H^{-\beta}(t)\|u_t\|_m^m+\varepsilon\|u_t\|^2-\varepsilon \int_{\Omega}|u_t|^{m-2} u_t u dx \\
&-\varepsilon\left(1-\int_0^t g(s)\, ds\right)\|\Delta u(t)\|^2+\varepsilon \int_{\Omega} u^p \ln|u|\, dx \\
&-\varepsilon \delta(g \circ \Delta u)(t)-\frac{\varepsilon}{4 \delta} \int_0^t g(s)\, ds\|\triangle u(t)\|^2 .
\end{aligned} \tag{3.7}
$$

We use the definition of $H(t)$ to substitute for $\int_{\Omega} u^p \ln|u|\, dx$ for $0<b<\frac{p-1}{p}$. Therefore (3.7) takes form

$$
\begin{aligned}
L'(t) \geq\ & (1-\beta) H^{-\beta}(t)\|u_t\|_m^m+\varepsilon\left(1+\frac{p(1-b)}{2}\right)\|u_t\|^2 \\
&+\varepsilon p(1-b) H(t)-\varepsilon p(1-b) E_1+\varepsilon \frac{(1-b)}{p}\|u\|_p^p \\
&+\varepsilon\left[\left(\frac{p(1-b)}{2}-1\right)-\left(\frac{p(1-b)}{2}-1+\frac{1}{4 \delta}\right) \int_0^{\infty} g(s)\, ds\right]\|\triangle u(t)\|^2 \\
&+\varepsilon\left(\frac{p(1-b)}{2}-\delta\right)(g \circ \Delta u)(t)+\varepsilon b \int_{\Omega} u^p \ln|u|\, dx-\varepsilon \int_{\Omega}|u_t|^{m-2} u_t u dx
\end{aligned}
$$

for a positive number δ, $0 < \delta < \frac{p(1-b)}{2}$. Recalling (2.3) and $p > 2$, the inequality above reduces to

$$L'(t) \geq (1-\beta) H^{-\beta}(t) \|u_t\|_m^m + \varepsilon \left(1 + \frac{p(1-b)}{2}\right) \|u_t\|^2 + \varepsilon p(1-b) H(t) - \varepsilon p(1-b) E_1 + \varepsilon \frac{(1-b)}{p} \|u\|_p^p + \varepsilon \mu_1 \|\triangle u(t)\|^2 + \varepsilon \mu_2 (g \circ \Delta u)(t) + \varepsilon b \int_\Omega u^p \ln|u|\, dx - \varepsilon \int_\Omega |u_t|^{m-2} u_t u dx. \tag{3.8}$$

At this point we choose $0 < b < \frac{p-1}{p}$. This implies that

$$\frac{p(1-b)}{2} - 1 + \frac{1}{4\delta} > \frac{p(1-b)}{2} - 1 + \frac{1}{2p(1-b)} = \frac{[p(1-b)-1]^2}{2p(1-b)} > 0.$$

Thus if we choose b small enough, then we get from (2.3)

$$\mu_1 = \left(\frac{p(1-b)}{2} - 1\right) - \left(\frac{p(1-b)}{2} - 1 + \frac{1}{4\delta}\right) \int_0^\infty g(s)\, ds > 0,$$

$$\mu_2 = \frac{p(1-b)}{2} - \delta > 0.$$

To estimate the ninth term of (3.8), we again use Young's inequality

$$AB \leq \frac{\delta^r}{r} A^r + \frac{\delta^{-q}}{q} A^q,\ A, B \geq 0 \ \text{ for all } \delta > 0,\ \frac{1}{r} + \frac{1}{q} = 1,$$

with $r = m$ and $q = \frac{m}{m-1}$. So we get

$$\int_\Omega |u_t|^{m-2} u_t u dx \leq \frac{\tau^m}{m} \|u\|_m^m + \frac{m-1}{m} \tau^{-\frac{m}{m-1}} \|u_t\|_m^m,$$

which yields, by substitution in (3.8),

$$L'(t) \geq \left[(1-\beta) H^{-\beta}(t) - \varepsilon \frac{m-1}{m} \tau^{-\frac{m}{m-1}}\right] \|u_t\|_m^m - \varepsilon \frac{\tau^m}{m} \|u\|_m^m - \varepsilon p(1-b) E_1 + \varepsilon \left(1 + \frac{p(1-b)}{2}\right) \|u_t\|^2 + \varepsilon \mu_1 \|\triangle u(t)\|^2 + \varepsilon \mu_2 (g \circ \Delta u)(t) + \varepsilon p(1-b) H(t) + \varepsilon \frac{(1-b)}{p} \|u\|_p^p + \varepsilon b \int_\Omega u^p \ln|u|\, dx. \tag{3.9}$$

Of course (3.9) holds even if τ is time dependent since the integral is taken over the x-variable. Therefore by choosing τ so that $\tau^{-\frac{m}{m-1}} = M_1 H^{-\beta}(t)$, for M_1 to be specified later, and substituting in (3.9), we get

$$L'(t) \geq \left[(1-\beta) - \varepsilon \frac{m-1}{m} M_1\right] H^{-\beta}(t) \|u_t\|_m^m \tag{3.10}$$

$$
\begin{aligned}
& - \varepsilon \frac{(M_1)^{1-m}}{m} H^{\beta(m-1)}(t) \|u\|_m^m - \varepsilon p (1-b) E_1 \\
& + \varepsilon \left(1 + \frac{p(1-b)}{2}\right) \|u_t\|^2 + \varepsilon \mu_1 \|\triangle u(t)\|^2 + \varepsilon \mu_2 (g \circ \Delta u)(t) \\
& + \varepsilon p (1-b) H(t) + \varepsilon \frac{(1-b)}{p} \|u\|_p^p + \varepsilon b \int_\Omega u^p \ln |u| \, dx.
\end{aligned}
$$

By exploiting (3.2), Corollary 2.9 and Young's inequality, we have

$$
\begin{aligned}
(3.11) \quad & H^{\beta(m-1)} \|u\|_m^m \\
\leq & \left(\frac{1}{p} \int_\Omega u^p \ln |u| \, dx\right)^{\beta(m-1)} \|u\|_m^m \\
\leq & \, C \left(\int_\Omega u^p \ln |u| \, dx\right)^{\beta(m-1)} \left[\left(\int_\Omega u^p \ln |u| \, dx\right)^{\frac{m}{p}} + \|\Delta u\|^{\frac{2m}{p}}\right] \\
\leq & \, C \left[\left(\int_\Omega u^p \ln |u| \, dx\right)^{\beta(m-1)+\frac{m}{p}} + \left(\int_\Omega u^p \ln |u| \, dx\right)^{\beta(m-1)} \|\Delta u\|^{\frac{2m}{p}}\right] \\
\leq & \, C \left[\left(\int_\Omega u^p \ln |u| \, dx\right)^{\beta(m-1)+\frac{m}{p}} + \left(\int_\Omega u^p \ln |u| \, dx\right)^{\beta(m-1)\frac{p}{p-m}} + \|\Delta u\|^2\right].
\end{aligned}
$$

From (3.4)

$$
2 < \beta p (m-1) + m \leq p \text{ and } 2 < \frac{\beta (m-1) p^2}{p-m} \leq p.
$$

Making using of Lemma 2.7, (3.11) yields that

$$
H^{\beta(m-1)} \|u\|_m^m \leq C \left[\int_\Omega u^p \ln |u| \, dx + \|\Delta u\|^2\right]. \tag{3.12}
$$

Combining (3.12) and (3.10) we arrive at

$$
\begin{aligned}
(3.13) \quad L'(t) \geq & \left[(1-\beta) - \varepsilon \frac{m-1}{m} M_1\right] H^{-\beta}(t) \|u_t\|_m^m \\
& + \varepsilon \left(1 + \frac{p(1-b)}{2}\right) \|u_t\|^2 \\
& - \varepsilon p (1-b) E_1 + \varepsilon \mu_2 (g \circ \Delta u)(t) \\
& + \varepsilon \left(\mu_1 - \frac{(M_1)^{1-m}}{m} C\right) \|\triangle u(t)\|^2 + \varepsilon p (1-b) H(t) \\
& + \varepsilon \frac{(1-b)}{p} \|u\|_p^p + \varepsilon \left(b - \frac{(M_1)^{1-m}}{m} C\right) \int_\Omega u^p \ln |u| \, dx.
\end{aligned}
$$

Noting that

$$
H(t) \geq E_1 - \frac{1}{2} \|u_t\|^2 - \frac{1}{2} \|\Delta u\|^2 - \frac{1}{2} (g \circ \Delta u)(t) \tag{3.14}
$$

$$-\frac{1}{p}\|u\|_p^p+\frac{1}{p}\int_\Omega u^p\ln|u|\,dx.$$

Inserting (3.14) into (3.13) and taking $p=p-2a+2a$, with

$$\frac{p}{2}>a>\max\left\{\frac{p(1-2\mu_1)}{2l},\frac{p(1-\mu_2)}{2}\right\}>1,$$

$$\begin{aligned}L'(t)\geq&\left[(1-\beta)-\varepsilon\frac{m-1}{m}M\right]H^{-\beta}(t)\|u_t\|_m^m\\&+\varepsilon(1+a(1-b))\|u_t\|^2\\&-2\varepsilon(1-b)a\left[\frac{p+1}{2p}-\frac{1}{p}\right]\|u\|_{p+1}^{p+1}\\&+\varepsilon\left(\mu_2-\frac{(p-2a)(1-b)}{2}\right)(g\circ\Delta u)(t)\\&+\varepsilon\left(\mu_1-\frac{(M_1)^{1-m}}{m}C-\frac{(p-2a)(1-b)l}{2}\right)\|\triangle u(t)\|^2\\&+\varepsilon(1-b)(p-2a)H(t)+\varepsilon\frac{(1-b)}{p}(1-p+2a)\|u\|_p^p\\&+\varepsilon\left(b-\frac{(M_1)^{1-m}}{m}C+\frac{(p-2a)(1-b)}{p}\right)\int_\Omega u^p\ln|u|\,dx.\end{aligned}\tag{3.15}$$

Then by $\ln|u|\geq 1$,

$$\int_\Omega u^p\ln|u|\,dx\geq\|u\|_p^p,$$

and (3.15) becomes

$$\begin{aligned}L'(t)\geq&\left[(1-\beta)-\varepsilon\frac{m-1}{m}M\right]H^{-\beta}(t)\|u_t\|_m^m\\&+\varepsilon(1+a(1-b))\|u_t\|^2\\&-2\varepsilon(1-b)a\left[\frac{p+1}{2p}-\frac{1}{p}\right]\|u\|_{p+1}^{p+1}\\&+\varepsilon\left(\mu_2-\frac{(p-2a)(1-b)}{2}\right)(g\circ\Delta u)(t)\\&+\varepsilon\left(\mu_1-\frac{(M_1)^{1-m}}{m}C-\frac{(p-2a)(1-b)l}{2}\right)\|\triangle u(t)\|^2\\&+\varepsilon(1-b)(p-2a)H(t)\\&+\varepsilon\left(b+\frac{1-b}{p}-\frac{(M_1)^{1-m}}{m}C\right)\|u\|_p^p.\end{aligned}$$

At this point, we take M_1 sufficiently large such that

$$b+\frac{1-b}{p}-\frac{(M_1)^{1-m}}{m}C>0,$$

and

$$\mu_1-\frac{(M_1)^{1-m}}{m}C-\frac{(p-2a)(1-b)l}{2}>0.$$

Since $H(0)=E_1-E(0)>0$, and M_1 and b are fixed, taking ε small enough yields

$$(1-\beta)-\varepsilon\frac{m-1}{m}M\geq 0,$$

and

$$L(0)=H^{1-\beta}(0)+\varepsilon\int_\Omega u_0u_1dx>0. \tag{3.16}$$

Therefore, (3.15) takes the form

$$L'(t)\geq\lambda\left[H(t)+\|u_t\|^2+\|\Delta u\|^2+(g\circ\Delta u)(t)+\|u\|_{p+1}^{p+1}+\|u\|_p^p\right], \tag{3.17}$$

where $\lambda>0$ is the minimum of the coefficients of $H(t)$, $\|u_t\|^2$, $(g\circ\Delta u)(t)$, $\|\Delta u\|^2$, $\|u\|_p^p$ and $\int_\Omega u^p\ln|u|\,dx$.

Consequently, we obtain

$$L(t)>L(0),\ t\geq 0. \tag{3.18}$$

Now we estimate

$$\left|\int_\Omega uu_tdx\right|\leq\|u\|\,\|u_t\|\leq C\,\|u\|_p\,\|u_t\|,$$

which implies

$$\left|\int_\Omega uu_tdx\right|^{\frac{1}{1-\alpha}}\leq C\,\|u\|_p^{\frac{1}{1-\alpha}}\,\|u_t\|^{\frac{1}{1-\alpha}}.$$

Applying Young's inequality we get

$$\left|\int_\Omega uu_tdx\right|^{1/(1-\alpha)}\leq C\left[C\,\|u\|_p^{\frac{\mu}{1-\alpha}}\,\|u_t\|^{\frac{\kappa}{1-\alpha}}\right]\ \text{for}\ \frac{1}{\mu}+\frac{1}{\kappa}=1. \tag{3.19}$$

To be able to use Lemma 2.11, we take $\kappa=2/(1-\alpha)$, to get

$$\mu=2(1-\alpha)/(1-2\alpha).$$

Therefore (3.19) has the form

$$\left|\int_\Omega uu_tdx\right|^{1/(1-\alpha)}\leq C\left[\|u_t\|^2+\|u\|_p^s\right],$$

where $s=2/(1-2\alpha)\leq p$. By using Lemma 2.11 we get

$$\left|\int_\Omega uu_tdx\right|^{1/(1-\alpha)}\leq C\left[H(t)+\|u_t\|^2+\|\Delta u\|^2+(g\circ\Delta u)(t)+\|u\|_{p+1}^{p+1}+\|u\|_p^p\right].$$

On the other hand by $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, we have

$$L(t)^{\frac{1}{1-\alpha}} = \left[H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx\right]^{\frac{1}{1-\alpha}} \tag{3.20}$$
$$\leq 2^{1/(1-\alpha)}\left[H(t) + \left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\alpha}}\right]$$
$$\leq C\Big[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + (g \circ \Delta u)(t) + \|u\|_{p+1}^{p+1} + \|u\|_p^p\Big].$$

By associating (3.17) and (3.20) we arrive at

$$L'(t) \geq \xi L^{\frac{1}{1-\alpha}}(t), \tag{3.21}$$

where ξ is a positive constant.

Integration of (3.21) over $(0,t)$ we reach

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\xi\alpha t}{1-\alpha}}.$$

□

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