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SOME PROPERTIES INVOLVING (p,q)-HERMITE POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study differential equations arising from the generating functions of (p,q)-Hermite polynomials. We use this differential equation to give explicit identities for (p,q)-Hermite polynomials.

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1. Introduction

Special functions are functions that are used specifically in mathematical physics or other fields of mathematics(see [1-16]). The ordinary Hermite numbers H_n and Hermite polynomials $H_n(x)$ are usually defined by the generating functions

$$e^{(2x-t)t}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}$$

and

$$e^{-t^2}=\sum_{n=0}^{\infty}H_n\frac{t^n}{n!}.$$

Clearly, $H_n=H_n(0)$.

We use the following notation:

$$[x]_{p,q}=\frac{p^x-q^x}{p-q},\quad 0<q<p\le 1,\quad (\text{see } [6,7,9]).$$

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Note that $p=1, \lim_{q\to 1}[x]_{p,q}=x$. We recall that the (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$ defined by the generating function

$$\sum_{n=0}^{\infty}\mathbf{H}_{n,p,q}(x)\frac{t^n}{n!}=e^{2[x]_{p,q}t-t^2}=\mathcal{F}(t,[x]_{p,q}). \tag{1}$$

Observe that if $p=1, q\to 1$, then $\mathbf{H}_{n,p,q}(x)\to H_n(x)$.

Mathematicians have studied the differential equations arising from the generating functions of special numbers and polynomials(see [1, 5, 8, 10-16]). Based on the results so far, in the present work, we can derive the differential equations generated from the generating function of (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$. By using the coefficients of this differential equation, we obtain explicit identities for the (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$. The rest of the paper is organized as follows. In Section 2, we derive the differential equations generated from the generating function of (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$. Using the coefficients of this differential equation, we have explicit identities for the (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$.

2. Basic properties for the (p,q)-Hermite polynomials

The generating function (1) is useful for deriving several properties of the (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$. The following basic properties of the (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$ are derived form (1). We, therefore, choose to omit the details involved.

Theorem 2.1. *For any positive integer n, we have*

(1) $$\mathbf{H}_{n,p,q}(x)=\sum_{k=0}^{n}\binom{n}{k}2^{n-k}[x]_{p,q}^{n-k}H_k.$$

(2) $$\mathbf{H}_{n,p,q}(x)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k 2^{n-2k}[x]_{p,q}^{n-2k}}{k!(n-2k)!}.$$

(3) $$\mathbf{H}_{n,p,q}(x)=\sum_{k=0}^{n}\binom{n}{k}\mathbf{H}_{k,p,q}(x)(-1)^k 4^{n-k}[x]_{p,q}^{n-k}.$$

where [ ] denotes taking the integer part.

Note that

$$\mathcal{F}(t,[x]_{p,q})=e^{2[x]_{p,q}t-t^2}$$

satisfies

$$\frac{\partial\mathcal{F}(t,[x]_{p,q})}{\partial t}-(2[x]_{p,q}-2t)\,\mathcal{F}(t,[x]_{p,q})=0. \tag{2}$$

Substitute the series in (2) for $\mathcal{F}(t,[x]_{p,q})$ to get

$$\mathbf{H}_{n+1,p,q}(x)-2[x]_{p,q}\mathbf{H}_{n,p,q}(x)+2n\mathbf{H}_{n-1,p,q}(x)=0, n=1,2,\ldots \tag{3}$$

This is the recurrence relation for (p, q)-Hermite polynomials. Another recurrence relation comes from

$$\left(\frac{d}{d[x]_{p,q}}\right)\mathcal{F}(t,[x]_{p,q}) - 2t\mathcal{F}(t,[x]_{p,q}) = 0.$$

This implies

$$\left(\frac{d}{d[x]_{q}}\right)\mathbf{H}_{n,q}(x) - 2n\mathbf{H}_{n-1,q}(x) = 0, n = 1, 2, \dots. \tag{4}$$

Eliminate $\mathbf{H}_{n-1,p,q}(x)$ from (3) and (4) to obtain

$$\mathbf{H}_{n+1,p,q}(x) - 2[x]_{p,q}\mathbf{H}_{n,p,q}(x) + \left(\frac{d}{d[x]_{p,q}}\right)\mathbf{H}_{n,p,q}(x) = 0.$$

Differentiate this equation and use (3) and (4) again to get

$$\left(\frac{d}{d[x]_{p,q}}\right)^2\mathbf{H}_{n,p,q}(x) - 2[x]_{p,q}\left(\frac{d}{d[x]_{p,q}}\right)\mathbf{H}_{n,p,q}(x) + 2n\mathbf{H}_{n,p,q}(x) = 0, n = 0, 1, 2, \dots.$$

Thus we have the following theorem.

Theorem 2.2. *The (p, q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$ in generating function (1) are the solution of equation*

$$\left(\left(\frac{d}{d[x]_{p,q}}\right)^2 - 2[x]_{p,q}\left(\frac{d}{d[x]_{p,q}}\right) + 2n\right)\mathbf{H}_{n,p,q}(x) = 0,$$

$$\mathbf{H}_{n,p,q}(0) = \begin{cases} (-1)^l\dfrac{(2l)!}{l!}, & \text{if } n = 2l, \\ 0, & \text{otherwise} \end{cases}$$

As another application of the differential equation for $\mathbf{H}_{n,p,q}(x)$ is as follows: Note that

$$\mathcal{F}(t,[x]_q) = e^{2[x]_{p,q}t - t^2}$$

satisfies

$$\frac{d\mathcal{F}(t,[x]_{p,q})}{dx} - \left(\frac{p^x\log p - q^x\log q}{p-q}\right)2t\mathcal{F}(t,[x]_{p,q}) = 0. \tag{5}$$

Substitute the series in (5) for $\mathcal{F}(t,[x]_{p,q})$ to get

$$\frac{d\mathbf{H}_{n,p,q}(x)}{dx} - \frac{2n\log p}{p-q}p^x\mathbf{H}_{n-1,p,q}(x) + \frac{2n\log q}{p-q}q^x\mathbf{H}_{n-1,p,q}(x) = 0, n = 1, 2, \dots \tag{6}$$

Differentiate this equation and use (3) and (6) again to derive

$$\begin{aligned} &2n\left(\frac{p^x\log p - q^x\log q}{p-q}\right)\mathbf{H}_{n,p,q}(x) \\ &-\left(\frac{2p^x - 2q^x}{p-q} + \frac{(p-q)(p^x(\log p)^2 - q^x(\log q)^2)}{(p^x\log p - q^x\log q)^2}\right)\frac{d\mathbf{H}_{n,p,q}(x)}{dx} \\ &+\left(\frac{p-q}{p^x\log p - q^x\log q}\right)\frac{d^2\mathbf{H}_{n,q}(x)}{dx^2} = 0. \end{aligned} \tag{7}$$

Hence we have the following theorem.

Theorem 2.3. *The (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$ in generating function (1) are the solution of equation*

$$\left(\left(\frac{p-q}{p^x\log p-q^x\log q}\right)\frac{d^2}{dx^2}-\left(\frac{2p^x-2q^x}{p-q}+\frac{(p-q)(p^x(\log p)^2-q^x(\log q)^2)}{(p^x\log p-q^x\log q)^2}\right)\frac{d}{dx}\right.$$
$$\left.+2n\left(\frac{p^x\log p-q^x\log q}{p-q}\right)\right)\mathbf{H}_{n,p,q}(x)=0,$$
$$\mathbf{H}_{n,p,q}(0)=\begin{cases}(-1)^l\dfrac{(2l)!}{l!}, & \textit{if } n=2l,\\ 0, & \textit{otherwise}\end{cases}$$

Recently, many mathematicians have studied differential equations that occur in the generating functions of special polynomials(see [1, 5, 8, 10-15]). The paper is organized as follows. We derive the differential equations generated from the generating function of (p,q)-Hermite polynomials:

$$\left(\frac{\partial}{\partial t}\right)^N\mathcal{F}(t,[x]_{p,q})-a_0(N,[x]_{p,q})\mathcal{F}(t,[x]_{p,q})-\cdots-a_N(N,[x]_{p,q})t^N\mathcal{F}(t,[x]_{p,q})=0.$$

By obtaining the coefficients of this differential equation, we get explicit identities for the (p,q)–Hermite polynomials in Sect. 3.

3. DIFFERENTIAL EQUATIONS ASSOCIATED WITH (p,q)-HERMITE POLYNOMIALS

In order to obtain explicit identities for special polynomials, differential equations arising from the generating functions of special polynomials are studied by many authors(see [1, 5, 8, 10-16]). In this section, we introduce differential equations arising from the generating functions of (p,q)-Hermite polynomials and use these differential equations to obtain the explicit identities for the (p,q)-Hermite polynomials.

Let

$$\mathcal{F}=\mathcal{F}(t,[x]_{p,q})=e^{2[x]_{p,q}t-t^2}=\sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{t^n}{n!},\quad x,t\in\mathbb{R}. \tag{8}$$

Then, by (8), we have

$$\begin{aligned}\mathcal{F}^{(1)}&=\frac{\partial}{\partial t}\mathcal{F}(t,[x]_{p,q})=\frac{\partial}{\partial t}\left(e^{2[x]_{p,q}t-t^2}\right)=e^{2[x]_{p,q}t-t^2}(2[x]_{p,q}-2t)\\&=(2[x]_{p,q}-2t)\mathcal{F}(t,[x]_{p,q})\\&=(2[x]_{p,q})\mathcal{F}(t,[x]_{p,q})\\&\quad+(-2)t\mathcal{F}(t,[x]_{p,q}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{(2)} &= \frac{\partial}{\partial t}\mathcal{F}^{(1)}(t,[x]_{p,q}) = -2\mathcal{F}(t,[x]_{p,q}) + (2[x]_{p,q}-2t)\mathcal{F}^{(1)}(t,[x]_{p,q})\\ &= (-2+4[x]_{p,q}^2)\mathcal{F}(t,[x]_{p,q})\\ &\quad + (-8[x]_{p,q})t\mathcal{F}(t,[x]_{p,q})\\ &\quad + (-2)^2t^2\mathcal{F}(t,[x]_{p,q}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^{(3)} &= \frac{\partial}{\partial t}\mathcal{F}^{(2)}(t,[x]_{p,q})\\ &= (-8[x]_{p,q}+8t)\mathcal{F}(t,[x]_{p,q}) + (-2+4[x]_{p,q}^2 - 8[x]_{p,q}t + 4t^2)\mathcal{F}^{(1)}(t,[x]_{p,q})\\ &= (-12[x]_{p,q}+8[x]_{p,q}^3)\mathcal{F}(t,[x]_{p,q}) + (12-24[x]_{p,q}^2)t\mathcal{F}(t,[x]_{p,q})\\ &\quad + (24[x]_q)t^2\mathcal{G}(t,[x]_q) + (-2)^3t^3\mathcal{F}(t,[x]_{p,q}).\end{aligned}$$

If we continue this process, we can guess as follows.

$$\mathcal{F}^{(N)} = \left(\frac{\partial}{\partial t}\right)^N \mathcal{F}(t,[x]_{p,q}) = \sum_{i=0}^{N} a_i(N,[x]_{p,q})t^i\mathcal{F}(t,[x]_{p,q}), (N=0,1,2,\ldots). \tag{9}$$

Differentiating (9) with respect to t, we have

$$\begin{aligned}\mathcal{F}^{(N+1)} &= \frac{\partial \mathcal{F}^{(N)}}{\partial t}\\ &= \sum_{i=0}^{N} a_i(N,[x]_{p,q})it^{i-1}\mathcal{F}(t,[x]_{p,q}) + \sum_{i=0}^{N} a_i(N,[x]_{p,q})t^i\mathcal{F}^{(1)}(t,[x]_{p,q})\\ &= \sum_{i=0}^{N} a_i(N,[x]_{p,q})it^{i-1}\mathcal{F}(t,[x]_{p,q}) + \sum_{i=0}^{N} a_i(N,[x]_{p,q})t^i(2[x]_{p,q}-2t)\mathcal{F}(t,[x]_{p,q})\\ &= \sum_{i=0}^{N} ia_i(N,[x]_{p,q})t^{i-1}\mathcal{F}(t,[x]_{p,q}) + \sum_{i=0}^{N}(2[x]_{p,q})a_i(N,[x]_{p,q})t^i\mathcal{F}(t,[x]_{p,q})\\ &\qquad + \sum_{i=0}^{N}(-2)a_i(N,[x]_{p,q})t^{i+1}\mathcal{F}(t,[x]_{p,q})\\ &= \sum_{i=0}^{N-1}(i+1)a_{i+1}(N,[x]_{p,q})t^i\mathcal{F}(t,[x]_{p,q}) + \sum_{i=0}^{N}(2[x]_{p,q})a_i(N,[x]_{p,q})t^i\mathcal{F}(t,[x]_{p,q})\\ &\qquad + \sum_{i=1}^{N+1}(-2)a_{i-1}(N,[x]_{p,q})t^i\mathcal{F}(t,[x]_{p,q}).\end{aligned} \tag{10}$$

Now replacing N by $N+1$ in (9), we find

$$\mathcal{F}^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,[x]_{p,q})t^i\mathcal{F}(t,[x]_{p,q}). \tag{11}$$

Comparing the coefficients on both sides of (10) and (11), we obtain

$$\begin{aligned}
&a_0(N+1,[x]_{p,q}) = a_1(N,[x]_{p,q}) + (2[x]_{p,q})a_0(N,[x]_{p,q}),\\
&a_N(N+1,[x]_{p,q}) = (2[x]_{p,q})a_N(N,[x]_{p,q}) + (-2)a_{N-1}(N,[x]_{p,q}),\\
&a_{N+1}(N+1,[x]_{p,q}) = (-2)a_N(N,[x]_{p,q}),
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
a_i(N+1,[x]_{p,q}) &= (i+1)a_{i+1}(N,[x]_{p,q})\\
&\quad + (2[x]_{p,q})a_i(N,[x]_{p,q}) + (-2)a_{i-1}(N,[x]_{p,q}), (1 \le i \le N-1).
\end{aligned} \tag{13}$$

In addition, by (9), we have

$$\mathcal{F}(t,[x]_{p,q}) = \mathcal{F}^{(0)}(t,[x]_{p,q}) = a_0(0,[x]_{p,q})\mathcal{F}(t,[x]_{p,q}), \tag{14}$$

which gives

$$a_0(0,[x]_{p,q}) = 1. \tag{15}$$

It is not difficult to show that

$$\begin{aligned}
&(2[x]_{p,q})\mathcal{F}(t,[x]_{p,q}) + (-2)t\mathcal{F}(t,[x]_{p,q})\\
&= \mathcal{F}^{(1)}(t,[x]_{p,q})\\
&= \sum_{i=0}^{1} a_i(1,[x]_{p,q})\mathcal{F}(t,[x]_{p,q})\\
&= a_0(1,[x]_{p,q})\mathcal{F}(t,[x]_{p,q}) + a_1(1,[x]_{p,q})t\mathcal{F}(t,[x]_{p,q}).
\end{aligned} \tag{16}$$

Thus, by (11), we also find

$$a_0(1,[x]_{p,q}) = 2[x]_{p,q}, \quad a_1(1,[x]_{p,q}) = -2. \tag{17}$$

From (12), we note that

$$\begin{aligned}
&a_0(N+1,[x]_{p,q}) = a_1(N,[x]_{p,q}) + (2[x]_{p,q})a_0(N,[x]_{p,q}),\\
&a_0(N,[x]_{p,q}) = a_1(N-1,[x]_{p,q}) + (2x)a_0(N-1,[x]_{p,q}), \dots\\
&a_0(N+1,[x]_{p,q}) = \sum_{i=0}^{N}(2[x]_{p,q})^i a_1(N-i,[x]_{p,q}) + (2[x]_{p,q})^{N+1},
\end{aligned} \tag{18}$$

$$\begin{aligned}
&a_N(N+1,[x]_{p,q}) = (2[x]_{p,q})a_N(N,[x]_{p,q}) + (-2)a_{N-1}(N,[x]_{p,q}),\\
&a_{N-1}(N,[x]_{p,q}) = (2[x]_{p,q})a_{N-1}(N-1,[x]_{p,q}) + (-2)a_{N-2}(N-1,[x]_{p,q}), \dots\\
&a_N(N+1,[x]_{p,q}) = (-2)^N(N+1)(2[x]_{p,q}),
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&a_{N+1}(N+1,[x]_{p,q}) = (-2)a_N(N,[x]_{p,q}),\\
&a_N(N,[x]_{p,q}) = (-2)a_{N-1}(N-1,[x]_{p,q}), \dots\\
&a_{N+1}(N+1,[x]_{p,q}) = (-2)^{N+1}.
\end{aligned} \tag{20}$$

For $i=1$ in (13), we have

$$\begin{aligned} &a_1(N+1,[x]_{p,q}) \\ &= 2\sum_{k=0}^{N}(2[x]_{p,q})^k a_2(N-k,[x]_{p,q}) + (-2)\sum_{k=0}^{N}(2[x]_{p,q})^k a_0(N-k,[x]_{p,q}), \end{aligned} \tag{21}$$

Continuing this process, we can deduce that, for $1 \le i \le N-1$,

$$\begin{aligned} a_i(N+1,[x]_{p,q}) = (i+1)&\sum_{k=0}^{N}(2[x]_{p,q})^k a_{i+1}(N-k,[x]_{p,q}) \\ &+ (-2)\sum_{k=0}^{N}(2[x]_{p,q})^k a_{i-1}(N-k,[x]_{p,q}). \end{aligned} \tag{22}$$

Note that, here the matrix $a_i(j,[x]_{p,q})_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & 2[x]_{p,q} & -2+4[x]_{p,q}^2 & -12[x]_{p,q}+8[x]_{p,q}^3 & \cdots & \cdot \\ 0 & (-2) & (-2)2(2[x]_{p,q}) & 12-24[x]_{p,q}^2 & \cdots & \cdot \\ 0 & 0 & (-2)^2 & (-2)^2 3(2[x]_{p,q}) & \cdots & \cdot \\ 0 & 0 & 0 & (-2)^3 & \ddots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & (-2)^N(N+1)(2[x]_{p,q}) \\ 0 & 0 & 0 & 0 & \cdots & (-2)^{N+1} \end{pmatrix}$$

Therefore, by (12)-(22), we obtain the following theorem.

Theorem 3.1. *For $N=0,1,2,\ldots,$ the differential equation*

$$\mathcal{F}^{(N)} = \left(\frac{\partial}{\partial t}\right)^N \mathcal{F}(t,[x]_{p,q}) = \left(\sum_{i=0}^{N} a_i(N,[x]_{p,q})t^i\right)\mathcal{F}(t,[x]_{p,q}) \tag{23}$$

has a solution

$$\mathcal{F} = \mathcal{F}(t,[x]_{p,q}) = e^{2[x]_{p,q}t-t^2},$$

where

$$\begin{aligned} a_0(N,[x]_{p,q}) &= \sum_{k=0}^{N-1}[x]_{p,q}^i a_1(N-1-k,[x]_{p,q}) + (2[x]_{p,q})^N,\\ a_{N-1}(N,[x]_{p,q}) &= (-2)^{N-1}N(2[x]_{p,q}),\\ a_N(N,[x]_{p,q}) &= (-2)^N,\\ a_i(N+1,[x]_{p,q}) &= (i+1)\sum_{k=0}^{N}2^k[x]_{p,q}^k a_{i+1}(N-k,[x]_{p,q})\\ &\quad + (-2)\sum_{k=0}^{N}2^k[x]_{p,q}^k a_{i-1}(N-k,[x]_{p,q}), (1\le i\le N-2). \end{aligned}$$

Making N-times derivative for (1) with respect to t, we have

$$\left(\frac{\partial}{\partial t}\right)^N \mathcal{F}(t,[x]_q) = \left(\frac{\partial}{\partial t}\right)^N e^{2[x]_{p,q}t-t^2} = \sum_{m=0}^{\infty}\mathbf{H}_{m+N,p,q}(x)\frac{t^m}{m!}. \tag{24}$$

By (23) and (24), we have

$$a_0(N,[x]_{p,q})\mathcal{F}(t,[x]_{p,q}) + \cdots + a_1(N,[x]_{p,q})t^N\mathcal{F}(t,[x]_{p,q}) = \sum_{m=0}^{\infty}\mathbf{H}_{m+N,p,q}(x)\frac{t^m}{m!}.$$

Hence we have the following theorem.

Theorem 3.2. *For $N=0,1,2,\ldots,$ we have*

$$\mathbf{H}_{m+N,p,q}(x) = \sum_{i=0}^{m}\frac{\mathbf{H}_{m-i,p,q}(x)a_i(N,[x]_{p,q})m!}{(m-i)!}, \tag{25}$$

where

$$\begin{aligned} a_0(N,[x]_{p,q}) &= \sum_{k=0}^{N-1}2^k[x]_{p,q}^k a_1(N-1-k,[x]_{p,q}) + (2[x]_{p,q})^N,\\ a_{N-1}(N,[x]_{p,q}) &= (-2)^{N-1}N(2[x]_{p,q}),\\ a_N(N,[x]_{p,q}) &= (-2)^N,\\ a_i(N+1,[x]_{p,q}) &= (i+1)\sum_{k=0}^{N}2^k[x]_{p,q}^k a_{i+1}(N-k,[x]_{p,q})\\ &\quad + (-2)\sum_{k=0}^{N}2^k[x]_{p,q}^k a_{i-1}(N-k,[x]_{p,q}), (1\le i\le N-2). \end{aligned}$$

If we take $m=0$ in (25), then we have the following corollary.

Corollary 3.3. *For $N=0,1,2,\ldots,$ we have*

$$\mathbf{H}_{N,p,q}(x) = a_0(N,[x]_{p,q}),$$

where,

$$a_0(N,[x]_{p,q}) = \sum_{k=0}^{N-1} 2^k [x]_{p,q}^k a_1(N-1-k,[x]_{p,q}) + (2[x]_{p,q})^N,$$

$$a_1(N,[x]_{p,q}) = 2\sum_{k=0}^{N-1} (2[x]_{p,q})^k a_2(N-k-1,[x]_{p,q}) + (-2)\sum_{k=0}^{N-1} (2[x]_{p,q})^k a_0(N-k-1,[x]_{p,q}).$$

For $N = 0,1,2,\ldots,$ the differential equation

$$\mathcal{F}^{(N)} = \left(\frac{\partial}{\partial t}\right)^N \mathcal{F}(t,[x]_q) = \left(\sum_{i=0}^{N} a_i(N,[x]_{p,q}) t^i\right) \mathcal{F}(t,[x]_{p,q})$$

has a solution

$$\mathcal{F} = \mathcal{F}(t,[x]_{p,q}) = e^{2[x]_{p,q}t - t^2}.$$

Here is a plot of the surface for this solution. In Figure 1(left), we choose

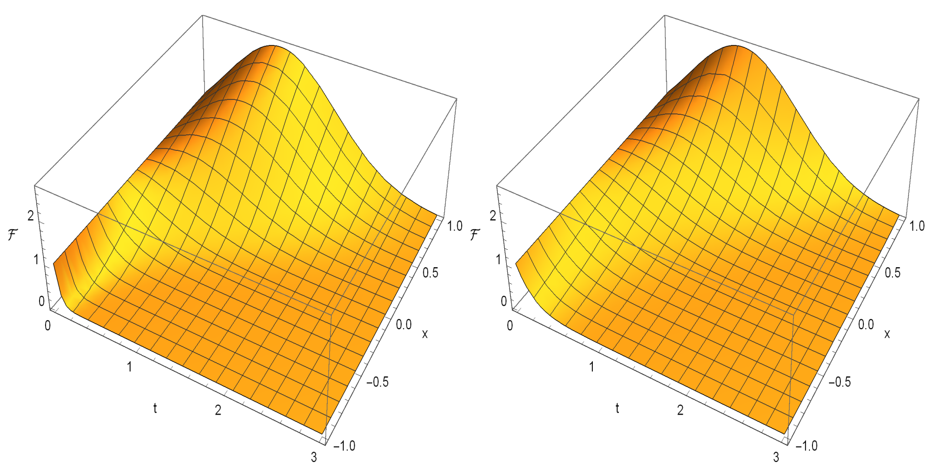

FIGURE 1. The surface for the solution $\mathcal{F}(t,[x]_{p,q})$

$-1 \le x \le 1, p = 9/10, q = 1/10,$ and $0 \le t \le 3$. In Figure 1(right), we choose $-1 \le x \le 1, p = 9/10, q = 1/2,$ and $0 \le t \le 3$. Here is a plot of the surface for this solution. In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we show a higher-resolution density plot of the solution.

The (p,q)-Hermite polynomials $\mathbf{H}_{n,p,q}(x)$ can be determined explicitly. First few examples of them are as follows.

$$\mathbf{H}_{0,p,q}(x) = 1,$$

$$\mathbf{H}_{1,p,q}(x) = \frac{2p^x}{p-q} - \frac{2q^x}{p-q},$$

$$\mathbf{H}_{2,p,q}(x) = -2 + \frac{4p^{2x}}{(p-q)^2} - \frac{8p^x q^x}{(p-q)^2} + \frac{4q^{2x}}{(p-q)^2},$$

$$\begin{aligned}\mathbf{H}_{3,p,q}(x) = {} & \frac{8p^{3x}}{(p-q)^3} - \frac{12p^{2+x}}{(p-q)^3} + \frac{24p^{1+x}q}{(p-q)^3} - \frac{12p^x q^2}{(p-q)^3} + \frac{12p^2 q^x}{(p-q)^3} - \frac{24p^{2x}q^x}{(p-q)^3} \\ & + \frac{24p^x q^{2x}}{(p-q)^3} - \frac{8q^{3x}}{(p-q)^3} - \frac{24pq^{1+x}}{(p-q)^3} + \frac{2q^{2+x}}{(p-q)^3}.\end{aligned}$$

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