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NOTES ON GENERALIZED FIBONACCI NUMBERS AND MATRICES

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Abstract. In this study, some new relations between generalized Fibonacci numbers and matrices are given. The work is designed in three stages: Firstly, it is obtained a relation between generalized Fibonacci numbers and integer powers of the matrices X satisfying the relation $X^2 = pX + qI$, and also, many results are derived from obtained relation. Then, it is established more general relation between generalized Fibonacci numbers and the square matrices X satisfying the condition $X^2 = V_n X - (-q)^n I$. Finally, some applications and numerical examples related to the obtained results are given.

1. Introduction

The sequence $\{F_n\}$ defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$ with the initial conditions $F_0 = 0$ and $F_1 = 1$ is called the Fibonacci sequence. An another sequence associated with the Fibonacci sequence is the Lucas sequence. The sequence $\{L_n\}$ defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for all integers $n \geq 2$ with the initial conditions $L_0 = 2$ and $L_1 = 1$ is called the Lucas sequence.

The roots of the equation $x^2 - x - 1 = 0$ are indicated by α and β. The positive root α is called the golden ratio. Many examples of this number can be seen in nature and art [3, 4, 8, 9].

The generalized Fibonacci sequence $\{U_n\}$ and the generalized Lucas sequence $\{V_n\}$ are, respectively, defined by the recurrence relations $U_n = pU_{n-1} + qU_{n-2}$ and $V_n = pV_{n-1} + qV_{n-2}$ for all integer $n \geq 2$ and the initial conditions $U_0 = 0$, $U_1 = 1$, $V_0 = 2$ and $V_1 = p$, where p and q

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are nonzero real numbers. Note that the numbers p and q will also be nonzero on the rest of the work, in order to avoid any uncertainty.

Generalized Fibonacci and generalized Lucas sequences with negative subscript are given by the relations

$$U_{-n} = \frac{-U_n}{(-q)^n} \tag{1}$$

and

$$V_{-n} = \frac{V_n}{(-q)^n} \tag{2}$$

respectively [6, 7]. The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are, respectively, the specific cases of the sequences $\{U_n\}$ and $\{V_n\}$ for $p = q = 1$.

Now, suppose that $p^2 + 4q > 0$. There are the relations

$$U_n = \frac{\alpha_{p,q}^n - \beta_{p,q}^n}{\alpha_{p,q} - \beta_{p,q}} \tag{3}$$

and

$$V_n = \alpha_{p,q}^n + \beta_{p,q}^n \tag{4}$$

known as Binet's formulas for all $n \in \mathbb{Z}$. Here $\alpha_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ and $\beta_{p,q} = \frac{p-\sqrt{p^2+4q}}{2}$ are the solutions of the quadratic equation $x^2 - px - q = 0$. In addition, the identities

$$\alpha_{p,q}^n = U_n \alpha_{p,q} + qU_{n-1} \tag{5}$$

and

$$\beta_{p,q}^n = U_n \beta_{p,q} + qU_{n-1} \tag{6}$$

hold for all $n \in \mathbb{Z}$ [6].

Relations between number sequences and matrices are of particular importance in mathematics. It was first shown that $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all $n \in \mathbb{N}$ via the matrix $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ which is called as the Fibonacci

Q matrix in [3]. And then, it was proved that the same relation is maintained for all $n \in \mathbb{Z}$ in [2]. In [5], a relation between generalized Fibonacci numbers and matrices was investigated. In [1], the authors developed a procedure for deriving special matrices of size 3×3, whose powers are related to Fibonacci and Lucas numbers, and then they found some special matrices thanks to the procedure developed. In [2], some relations between the powers of the square matrices X satisfying the conditions $X^2 = X + I$ and Fibonacci numbers were shown. In [6], a relation between the powers of the square matrices X satisfying the condition $X^2 = pX + qI$ and generalized Fibonacci numbers was given.

In this paper, the main aim is to get some results by showing a relation between the powers of the square matrices X satisfying the condition $X^2 = V_nX - (-q)^nI$ and generalized Fibonacci numbers.

2. Results

In this section, some new results will be given by using some relations between generalized Fibonacci numbers and matrices. At first, let's start by reminding well-known identities

$$U_mV_n = U_{m+n} + (-q)^nU_{m-n}, \tag{7}$$

$$(-q)^{n-1}U_{m-n} = U_{m-1}U_n - U_mU_{n-1}, \tag{8}$$

$$V_n = U_{n+1} + qU_{n-1} = pU_n + 2qU_{n-1}, \tag{9}$$

and

$$U_{n+r}U_{n-r} - U_n^2 = -(-q)^{n-r}U_r^2 \tag{10}$$

for all $m, n, r \in \mathbb{Z}$ [6]. These identities will be used to establish the results in the rest of the work.

Now, let us remind the following result.

Theorem 2.1. *[2.1.Theorem, [6]] If X is a square matrix with $X^2 = pX + qI$, then*

$$X^k = U_kX + qU_{k-1}I$$

for all $k \in \mathbb{Z}$.

Theorem 2.2. *If X is a square matrix with $X^2 = pX + qI$, then*

$$(-q)^{k-n} U_{m-k} X^n + U_{k-n} X^m = U_{m-n} X^k$$

for all $m, n, k \in \mathbb{Z}$.

Proof. If $m = n$, then the proof is obvious. Now, assume that $m \neq n$. By Theorem 2.1, we get the equality

$$\begin{aligned}&(-q)^{k-n} U_{m-k}(U_n X + qU_{n-1} I) + U_{k-n}(U_m X + qU_{m-1} I)\\ &= \tfrac{1}{(-q)^{n-1}} [(-q)^{k-1} U_{m-k}(U_n X + qU_{n-1} I) + (-q)^{n-1} U_{k-n}(U_m X + qU_{m-1} I)].\end{aligned}$$

Hence, taking the equality (8) into account, it is seen that

$$(-q)^{k-n} U_{m-k} X^n + U_{k-n} X^m = U_{m-n}(U_k X + qU_{k-1} I).$$

Thus, according to Theorem 2.1, the desired result is obtained.

□

Corollary 2.3. *For all $m, n, k \in \mathbb{Z}$, the identities*

(a) $(-q)^{k-n} U_{m-k}(aU_n + qU_{n-1}) + U_{k-n}(aU_m + qU_{m-1}) = U_{m-n}(aU_k + qU_{k-1})$,
(b) $(-q)^{k-n} U_{m-k} U_n + U_{k-n} U_m = U_{m-n} U_k$,
(c) $(-q)^{k-n} U_{m-k}(U_{n+1} - aU_n) + U_{k-n}(U_{m+1} - aU_m) = U_{m-n}(U_{k+1} - aU_k)$

hold.

Proof. The matrix $X = \begin{pmatrix} a & 1 \\ ap - a^2 + q & p - a \end{pmatrix}$ satisfies the relation $X^2 = pX + qI$. So, from Theorem 2.1, we get the equality $X^s = \begin{pmatrix} aU_s + qU_{s-1} & U_s \\ (ap - a^2 + q)U_s & U_{s+1} - aU_s \end{pmatrix}$ for all $s \in \mathbb{Z}$. Hence, the desired results are immediate consequences of the matrix equality in Theorem 2.2.

□

Corollary 2.4. *The following identities are true for all $m, n, k \in \mathbb{Z}$.*

(a) $(-q)^{k-n} U_{m-k} \alpha_{p,q}^n + U_{k-n} \alpha_{p,q}^m = U_{m-n} \alpha_{p,q}^k$
(b) $(-q)^{k-n} U_{m-k} \beta_{p,q}^n + U_{k-n} \beta_{p,q}^m = U_{m-n} \beta_{p,q}^k$

Proof. Consider the matrix $X = \begin{pmatrix} \alpha_{p,q} & 0 \\ 0 & \beta_{p,q} \end{pmatrix}$. From the equalities (5) and (6), it is obvious that the matrix X satisfies the relation $X^2 = pX + qI$. Hence, the desired results are immediate consequences of the matrix equality in Theorem 2.2.

□

Corollary 2.5. *If* $A = \begin{pmatrix} a & b \\ c & V_n - a \end{pmatrix}$ *is a matrix with* $\det(A) = (-q)^n$, *then*

$$U_n A^k = \begin{pmatrix} aU_{nk} - (-q)^n U_{nk-n} & bU_{nk} \\ cU_{nk} & U_{nk+n} - aU_{nk} \end{pmatrix} \text{ for all } n, k \in \mathbb{Z}.$$

Proof. It is clear that the equality is true for $n = 0$. Now, assume that $n \neq 0$. In addition, let us define the matrix C as

$$C = \begin{pmatrix} \frac{a - qU_{n-1}}{U_n} & \frac{b}{U_n} \\ \frac{c}{U_n} & \frac{V_n - a - qU_{n-1}}{U_n} \end{pmatrix}.$$

It can be easily seen that the matrix C satisfies the relation $C^2 = pC + qI$. So, taking Theorem 2.1 into account, we get

$$C^n = U_n \begin{pmatrix} \frac{a - qU_{n-1}}{U_n} & \frac{b}{U_n} \\ \frac{c}{U_n} & \frac{V_n - a - qU_{n-1}}{U_n} \end{pmatrix} + qU_{n-1} I = \begin{pmatrix} a & b \\ c & V_n - a \end{pmatrix} = A.$$

Moreover, according to Theorem 2.2, we have the relation

$$(-q)^{k-n} U_{m-k} C^n + U_{k-n} C^m = U_{m-n} C^k. \tag{11}$$

Now, if necessary arrangements are made in equality (11) by taking 0 and kn instead of m and k, respectively, and by using the identity (1)

$$U_n A^k = \begin{pmatrix} aU_{nk} - (-q)^n U_{nk-n} & bU_{nk} \\ cU_{nk} & (V_n - a)U_{nk} - (-q)^n U_{nk-n} \end{pmatrix}$$

is obtained. Hence, taking the equality (7) into account, the desired result is found.

□

The following two results are immediate consequences of Corollary 2.5 for all $m, n, k \in \mathbb{Z}$.

Corollary 2.6. *If* $A = \begin{pmatrix} V_n & -(-q)^m \\ (-q)^{n-m} & 0 \end{pmatrix}$, *then*

$$U_n A^k = \begin{pmatrix} U_{nk+n} & -(-q)^m U_{nk} \\ (-q)^{n-m} U_{nk} & -(-q)^n U_{nk-n} \end{pmatrix}$$

for all $m, n, k \in \mathbb{Z}$.

Corollary 2.7. *If* $B = \begin{pmatrix} 0 & (-q)^m \\ -(-q)^{n-m} & V_n \end{pmatrix}$, *then*

$$U_n B^k = \begin{pmatrix} -(-q)^n U_{nk-n} & (-q)^m U_{nk} \\ -(-q)^{n-m} U_{nk} & U_{nk+n} \end{pmatrix}$$

for all $m, n, k \in \mathbb{Z}$.

Corollary 2.8. *The following equalities are true for all* $m, n, k \in \mathbb{Z}$ *with* $n \neq 0$.

(a) $\sum_{i=0}^{k} \binom{k}{i} \frac{-U_{ni+n}U_{n(k-i)-n}+U_{ni}U_{n(k-i)}}{U_n^2} = \frac{V_n^k}{(-q)^n}$,

(b) $\sum_{i=0}^{k} \binom{k}{i} \frac{U_{ni+n}U_{n(k-i)}-U_{ni}U_{n(k-i)+n}}{U_n^2} = 0$,

(c) $\sum_{i=0}^{k} \binom{k}{i} \frac{-U_{ni}U_{n(k-i)-n}+U_{ni-n}U_{n(k-i)}}{U_n^2} = 0$,

(d) $\sum_{i=0}^{k} \binom{k}{i} \frac{U_{ni}U_{n(k-i)}-U_{ni-n}U_{n(k-i)+n}}{U_n^2} = \frac{V_n^k}{(-q)^n}$.

Proof. Let the matrices A and B be the matrices in Corollary 2.6 and Corollary 2.7, respectively. First, notice that the matrices A and B are commutative, and $A+B = \begin{pmatrix} V_n & 0 \\ 0 & V_n \end{pmatrix}$. So, according to well-known Binomial Expansion Theorem, we get

$$\sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} = \begin{pmatrix} V_n^k & 0 \\ 0 & V_n^k \end{pmatrix} \tag{12}$$

for all $k \in \mathbb{Z}$.

On the other hand, we have $A^i = \frac{1}{U_n}\begin{pmatrix} U_{ni+n} & -(-q)^m U_{ni} \\ (-q)^{n-m}U_{ni} & -(-q)^n U_{ni-n} \end{pmatrix}$ and $B^{k-i} = \frac{1}{U_n}\begin{pmatrix} -(-q)^n U_{n(k-i)-n} & (-q)^m U_{n(k-i)} \\ -(-q)^{n-m}U_{n(k-i)} & U_{n(k-i)+n} \end{pmatrix}$ from Corollary 2.6 and Corollary 2.7, respectively. Hence, the desired results are easily obtained from the matrix equality (12).

□

The matrix A in Corollary 2.5 has the characteristic equation $x^2 - V_n x + (-q)^n = 0$. Moreover, the matrix A satisfies the equality $U_n A^k = U_{nk}A - (-q)^n U_{nk-n} I$ for all $n, k \in \mathbb{Z}$. So, the following result can be regarded as an extension of this result to any square matrix.

Theorem 2.9. *If* X *is a square matrix with* $X^2 = V_n X - (-q)^n I$, *then*

$$U_n X^k = U_{nk} X - (-q)^n U_{nk-n} I$$

for all $k, n \in \mathbb{Z}$.

Proof. If $k = 0$, then the proof is obvious. Taking the equality (7) into account, it can be easily shown that the proof of theorem is true for all $k \in \mathbb{N}$.

Now, let $Y = -X + V_n I$. Then it is seen that the matrix Y satisfies the condition $Y^2 = V_n Y - (-q)^n I$ of the theorem. Therefore, $U_n Y^k = U_{nk} Y - (-q)^n U_{nk-n} I$ for all $k \in \mathbb{N}$. Hence, taking the equalities (1) and (7) into account, it is easily obtained the equality

$U_n X^{-k} = U_{n(-k)} X - (-q)^n U_{n(-k)-n} I$

for all $k \in \mathbb{N}$, and the proof is completed.

□

The following result is the special case of Theorem 2.9 with $p = q = 1$.

Corollary 2.10. *If X is a square matrix with $X^2 = L_n X - (-1)^n I$, then*

$$F_n X^k = F_{nk} X - (-1)^n F_{nk-n} I$$

for all $k, n \in \mathbb{Z}$.

3. Applications and Examples

Theorem 2.9 and Corollary 2.10 imply that the powers of any 2×2 matrices having the characteristics equation other than $x^2 - x - 1 = 0$ can be obtained using the Fibonacci sequence.

Example 3.1. *The matrix $C = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$ have the characteristics equation $x^2 - 3x + 1 = 0$. So, the matrix C satisfies $C^2 - L_2 C + (-1)^2 I = 0$. Therefore, we get, from Corollary 2.10 with $n = 2$, $C^k = \begin{pmatrix} -F_{2k-2} & -F_{2k} \\ F_{2k} & F_{2k+2} \end{pmatrix}$ for all $k \in \mathbb{Z}$.*

We see, via Corollary 2.10, that the powers of some square matrices having the characteristic equation different from $x^2 - x - 1 = 0$, as in Example 3.1, calculated by using Fibonacci sequence. This fact reminds us an idea related to compute the roots of some square matrices. For example, since the characteristic equation of the matrix $D = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$ is $x^2 - x - 1 = 0$, we get $D^k = \begin{pmatrix} -F_{k-2} & -F_k \\ F_k & F_{k+2} \end{pmatrix}$ by Corollary 2.10. However, $D^2 = C$ for the matrix C in Example 3.1. So, the matrix $D =$

$\begin{pmatrix} -F_{-1} & -F_1 \\ F_1 & F_3 \end{pmatrix}$ is a square root of the matrix C having the characteristic polynomial $C^2 - L_2 C + (-1)^2 I$.

The following theorem given inspired by this idea presents a method to calculate *nth* roots of any square matrix X satisfying the equality $X^2 - V_n X + (-q)^n I = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Theorem 3.2. *Let X be any $n \times n$ matrix such that $X^2 - V_n X + (-q)^n I = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^n = X$ if $C = \frac{X - qU_{n-1}I}{U_n}$.*

Proof. First, we have

$$\begin{aligned} C^2 - pC - qI &= (\frac{X - qU_{n-1}I}{U_n})^2 - p\frac{X - qU_{n-1}I}{U_n} - qI \\ &= \frac{X^2 - (2qU_{n-1} + pU_n)X + q(qU_{n-1}^2 + pU_nU_{n-1} - U_n^2)I}{U_n^2}. \end{aligned}$$

Then, taking the equality (9) into account, we get

$$\begin{aligned} C^2 - pC - qI &= \frac{X^2 - V_nX + q[U_{n-1}(qU_{n-1} + pU_n) - U_n^2]I}{U_n^2} \\ &= \frac{X^2 - V_nX + q(U_{n-1}U_{n+1} - U_n^2)I}{U_n^2}. \end{aligned}$$

On the other hand, we also get the equality $U_{n-1}U_{n+1} - U_n^2 = -(-q)^{n-1}$ from equality (10). Thus, taking the condition of the theorem into account, we obtain

$$C^2 - pC - qI = \frac{X^2 - V_nX + q[-(-q)^{n-1}]I}{U_n^2} = \frac{X^2 - V_nX + (-q)^nI}{U_n^2} = 0.$$

Consequently, the matrix C satisfies the condition of Theorem 2.1. So, we get

$$C^n = U_nC + qU_{n-1}I \tag{13}$$

for all $n \in \mathbb{Z}\setminus\{0\}$. Hence, substituting the matrix $C = \frac{X - qU_{n-1}I}{U_n}$ into (13) completes the proof.

□

Now, we present two examples.

Example 3.3. *The characteristic polynomial of the matrix* $A = \begin{pmatrix} 7 & -6 \\ 6 & -5 \end{pmatrix}$ *is* x^2-2x+1. *The matrix satisfies the equality* $A^2-V_2A+q^2I = 0$ *with* $p^2+2q = 2$ *and* $q^2 = 1$. *If* $q = 1$, *then* $p = 0$, *which is a contradiction. So, it must be* $q = -1$, *and therefore* $p = \pm 2$. *By Theorem 3.2, we get the square roots of the matrix* A *as* $C = \pm\frac{1}{2}\begin{pmatrix} 8 & -6 \\ 6 & -4 \end{pmatrix}$, *which satisfy* $C^2 = A$.

On the other hand, the matrix also satisfies the equality $A^2 - V_3A + (-q)^3I = 0$ *with* $p^3+3pq = 2$ *and* $(-q)^3 = 1$. *One pair of the* p *and* q *that satisfies these equalities is* $(2,-1)$. *Hence, from the Theorem 3.2, we get* $C = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$, *which satisfies* $C^3 = A$.

Example 3.4. *The matrix* $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ *satisfies the polynomial* x^2-7x-8. *So, the matrix* A *satisfies* $A^2 - V_3A + (-q)^3I = 0$ *with* $p^3+3pq = 7$ *and* $(-q)^3 = -8$. *Hence, one pair of the* p *and* q *that satisfies these equalities is* $(1,2)$. *In this case, from Theorem 3.2, we get* $C = \frac{1}{3}\begin{pmatrix} 1 & 2 & 4 \\ 2 & -2 & 2 \\ 4 & 2 & 1 \end{pmatrix}$ *satisfies* $C^3 = A$.

The proof of the following result is an immediate consequence of Theorem 2.9 by taking $X = \begin{pmatrix} \alpha_{p,q}^n & 0 \\ 0 & \beta_{p,q}^n \end{pmatrix}$.

Corollary 3.5. $U_n\alpha_{p,q}^{nk} = U_{nk}\alpha_{p,q}^n-(-q)^nU_{nk-n}$ *and* $U_n\beta_{p,q}^{nk} = U_{nk}\beta_{p,q}^n-(-q)^nU_{nk-n}$ *for all* $k,n \in \mathbb{Z}$.

Now, let us choose fixed nonzero real numbers p_1 and q_1 with $p_1^2 + 4q_1 > 0$. And then let us create the generalized Fibonacci sequence $\{U_n\}$ and generalized Lucas sequence $\{V_n\}$ in terms of p_1 and q_1. Fixing the number n such that $V_n^2 - 4(-q_1)^n > 0$, we define the numbers $p_2 = V_n$ and $q_2 = -(-q_1)^n$, and create the generalized Fibonacci sequence $\{S_k\}$

in terms of p_2 and q_2. So, we get, by Binet's formulas,

$$\begin{aligned}\alpha_{p_2,q_2} &= \frac{p_2+\sqrt{p_2^2+4q_2}}{2} \\ &= \frac{V_n+\sqrt{V_n^2-4(-q_1)^n}}{2} = \frac{\alpha_{p_1,q_1}^n+\beta_{p_1,q_1}^n+\sqrt{(\alpha_{p_1,q_1}^n+\beta_{p_1,q_1}^n)^2-4(-q_1)^n}}{2} \\ &= \frac{\alpha_{p_1,q_1}^n+\beta_{p_1,q_1}^n+\sqrt{(\alpha_{p_1,q_1}^{2n}+\beta_{p_1,q_1}^{2n})-2(-q_1)^n}}{2} = \alpha_{p_1,q_1}^n\end{aligned}$$

and similarly, $\beta_{p_2,q_2} = \beta_{p_1,q_1}^n$.

Now, if $X = \begin{pmatrix} \alpha_{p_1,q_1}^n & 0 \\ 0 & \beta_{p_1,q_1}^n \end{pmatrix}$, then the matrix X can be used directly in Theorem 2.9 or Corollary 3.5. However, it can not be used in Theorem 2.1 for the generalized Fibonacci sequence $\{U_n\}$ with $n \neq 1$. Of course, the matrix X can be used directly in Theorem 2.1 for the generalized Fibonacci sequence $\{S_k\}$ defined as above. This raises the question whether there is any relation between the sequences $\{U_n\}$ and $\{S_k\}$. Indeed there is. It is easily seen that

$$U_{nk} = U_n S_k \tag{14}$$

for $n, k \in \mathbb{Z}$ with $V_n^2 - 4(-q_1)^n > 0$.

We want to complete this section by giving two examples.

Example 3.6. *Let us consider the generalized Fibonacci sequence $\{U_n\}$ and generalized Lucas sequences $\{V_n\}$ with p_1=q_1=1. In fact, these are the classical Fibonacci and Lucas sequence, respectively;*

$$\{U_n\} = \{... - 3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$$

and

$$\{V_n\} = \{...7, -4, 3, -1, 2, 1, 3, 4, 7, 11, 18, 29, 47, ...\}.$$

Then choosing, for example, $n = 2$ leads to $p_2 = V_2 = 3$ and $q_2 = -(-1)^2 = -1$. In this case, the condition $V_2^2 - 4(-q_1)^2 > 0$ is also satisfied. Hence, we get

$$\{S_k\} = \{..., -8, -3, -1, 0, 1, 3, 8, 21, 55, 144, ...\}$$

It is easy to see that $U_{2k} = U_2 S_k = S_k$ for $k \in \mathbb{Z}$.

Example 3.7. *Let us consider the generalized Fibonacci sequence* $\{U_n\}$ *and the generalized Lucas sequences* $\{V_n\}$ *with* $p_1 = 2$, $q_1 = 3$;

$$\{U_n\} = \{..., -\frac{20}{81}, \frac{7}{27}, -\frac{2}{9}, \frac{1}{3}, 0, 1, 2, 7, 20, 61, 182, ...\}$$

and

$$\{V_n\} = \{..., -\frac{26}{27}, \frac{10}{9}, -\frac{2}{3}, 2, 2, 10, 26, 82, ...\}.$$

Then choosing $n = -2$ *leads to* $p_2 = V_{-2} = \frac{10}{9}$ *and* $q_2 = -(-3)^{-2} = -\frac{1}{9}$, *and the condition* $V_{-2}^2 - 4(-q_1)^{-2} > 0$ *is satisfied. Hence, we get*

$$\{S_k\} = \{..., -9, 0, 1, \frac{10}{9}, \frac{91}{81}, ...\}.$$

It can be easily checked that $U_{-2k} = U_{-2}S_k = -\frac{2}{9}S_k$ *for* $k \in \mathbb{Z}$.

NOTE: Notice that Theorem 2.1 is a special case of Theorem 2.9. On the other hand, Theorem 2.2 is an extension of Theorem 2.1 to combinations of integer powers of the matrices X satisfying the relation $X^2 = pX + qI$. So, in this logical framework, Theorem 2.9 can be handled, as in Theorem 2.2, for some combinations of integer powers of the matrices X satisfying the relation $X^2 = V_nX - (-q)^nI$. In addition, the relation (14) can be useful to derive different identities for generalized Fibonacci sequences.

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