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SOFT DECISION CONTEXTS BASED ON SOFT CONTEXTS

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Abstract. For another study of soft context and soft concept closely related to formal context and formal concept, in this paper, we propose the notions of conditional concepts, decision concepts and soft decision context based on soft contexts. Subsequently, the notions of consistent soft decision context and consistent set are introduced, and some properties for consistent set of soft decision contexts are investigated.

1. Introduction

Wille introduced the formal concept analysis in [16], a theory for the study of information structures induced by a binary relation between objects and attributes. The three important notions of formal concept analysis are formal context, formal concept, and concept lattice. In particular, a formal context is a kind of information system, which is a tabular form of an object-attribute value relationship [3, 4, 5, 15]. Formal concept analysis has been widely applied in many fields and has been generalized to other types for practical application requirements [2, 7, 11, 12, 13, 14, 15, 17, 18].

The notion of soft set was introduced by Molodtsov to deal with complex and uncertain problems. In [6], Maji et al. introduced several operators (equality, subset, superset, complement, null, and absolute soft set, etc.) for soft set theory. In [1], Ali et al. introduce new operations that modified some concepts introduced by Maji.

In [8], we studied the new notions such as soft contexts, soft concepts and soft concept lattices that are closely related to formal contexts and formal concept lattices. The main objective of studying these new notions was to deal with formal concepts in a formal context more effectively. So, we introduced the soft context combining the formal contexts and soft sets, and investigated the basic properties of soft context and the notion of soft concept.

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Based on these results, in this paper, we investigate the concept of a new type of soft context called the soft decision context. For this purpose, we propose the notions of conditional concepts, decision concepts and soft decision context based on a soft context. Subsequently, the notions of consistent soft decision context and consistent set are introduced, and some properties for consistent set of soft decision contexts are investigated.

2. Preliminaries

A formal context is a triplet (U, A, I), where U is a finite set of objects, A is a finite set of attributes, and $I \subseteq U \times A$. In a formal context (U, A, I), for an ordered pair (x, a) of elements $x \in U$ and $a \in A$, if $(x, a) \in I$, we write xIa and say x has the property a, or a is possessed by object x. The set of all attributes possessed by $x \in U$ and the set of all objects having an attribute $a \in A$ were represented as ([15,16]):

$$x^* = \{a \in A|xIa\};\;\; a^* = \{x \in U|xIa\}.$$

And, for a set of objects $X \subseteq U$, X^* is the maximal set of properties shared by all objects in X, that is, $X^* = \{a \in A|\forall x \in X, xIa\}$;

for a set of properties $B \subseteq A$, B^* is the maximal set of objects that have all properties in B, that is, $B^* = \{x \in U|\forall b \in B, xIb\}$.

A pair (X, B), $X \subseteq U$,$B \subseteq A$, is called a *formal concept* of the context (U, A, I) if $X = B^*$ and $B = X^*$ ([15,16]). X is called the *extension* of the concept, and B is called the *intention* of the concept.

We recall that the notion of soft sets introduced in [10]. Let U be an initial universe set and A be a set of parameters. In general, parameters are properties or attributes of objects in U.

A pair (F, A) is called a *soft set* [10] over U if F is a set-valued mapping of A into the set of all subsets of the set U, i.e.,

$$F : A \to 2^U.$$

We call a soft set (F, A) is *pure* [9] if $\cap_{a\in A}F(a) = \emptyset$ and $F(a) \neq \emptyset$ for every $a \in A$. From now on, we assume that all soft sets are *pure*.

Let $U = \{\mathrm{x_1, x_2, \ldots, x_n}\}$ be a non-empty finite set of *objects*, $A = \{\mathrm{a_1, a_2, \ldots, a_m}\}$ a non-empty finite set of *attributes*, and $F : A \to 2^U$ a soft set. Then the triple (U, A, F) is called *a soft context* [8].

Let (U, A, F) be a soft context. Then in [8], F^+ and F^- defined as the following:

(1) $F^+ : 2^A \to 2^U$ is a mapping defined as $F^+(B) = \cap_{b\in B}F(b)$;
(2) $F^- : 2^U \to 2^A$ is a mapping defined as $F^-(X) = \{a \in A : X \subseteq F(a)\}$.

For $a \in A$ and $x \in U$, simply, $F^+(\{a\}) = F^+(a)$ and $F^-(\{x\}) = F^-(x)$.

Theorem 2.1 ([8])**.** *Let (U, A, F) be a soft context, $X, Y \subseteq U$ and $C, D \subseteq A$. Then we have the following things:*
(1) If $X \subseteq Y$, then $F^-(Y) \subseteq F^-(X)$; if $C \subseteq D$, then $F^+(D) \subseteq F^+(C)$;
(2) $X \subseteq F^+(F^-(X)) = F^+F^-(X)$; $C \subseteq F^-(F^+(C)) = F^-F^+(C)$;
(3) $F^-(X \cup Y) = F^-(X) \cap F^-(Y)$, $F^+(C \cup D) = F^+(C) \cap F^+(D)$;
(4) $F^-(X) = F^-F^+F^-(X)$, $F^+(C) = F^+F^-F^+(C)$;
(5) $F^-(X) \cup F^-(Y) \subseteq F^-(X \cap Y)$, $F^+(C) \cup F^+(D) \subseteq F^+(C \cap D)$.

In a soft context (U, A, F), the associated operation Ψ_F in [8] was induced by F^+, F^+ as the following way:
For each $X \in 2^U$,

$\Psi_F : 2^U \to 2^U$ is a mapping defined as $\Psi_F(X) = (F^+ \circ F^-)(X) = F^+F^-(X)$:

From now on, Ψ is used instead of Ψ_F when there is no ambiguity.
Then X is called a *soft concept* in (U, A, F) if $\Psi(X) = F^+F^-(X) = X$.

The set of all soft concepts is denoted by $\mathrm{sC}(U, A, F)$.

Theorem 2.2 ([8])**.** *Let (U, A, F) be a soft context. Then*
(1) $\emptyset$, U, $\Psi(X)$ are soft concepts.
(2) For each $B \subseteq A$, $F^+(B)$ is a soft concept.
(3) For each $a \in A$, $F(a)$ is a soft concept.
(4) X is a soft concept if and only if $X = F^+(B)$ for some $B \in 2^A$.
(5) $Im(F^+) = sC(U, A, F)$.

In a soft context (U, A, F), for $D \subseteq A$, consider a set-valued mapping $F|_{\mathrm{D}} : D \to 2^U$ defined by $F|_{\mathrm{D}}(d) = F(d)$ for all $d \in D$. Then obviously $(F|_{\mathrm{D}}, D)$ is a soft set. Since $(F|_{\mathrm{D}}, D)$ is a soft set, $(U, D, F|_{\mathrm{D}})$ is also a soft context. From now on, we consider only a subset $D \subseteq A$ satisfying the soft set $(F|_{\mathrm{D}}, D)$ is pure.

Since $(F|_{\mathrm{D}}, D)$ is a soft set, we can consider two associated mappings ${F|_{\mathrm{D}}}^+$ and ${F|_{\mathrm{D}}}^-$ induced by the soft set $(F|_{\mathrm{D}}, D)$ as follows:

${F|_{\mathrm{D}}}^+ : 2^D \to 2^U$ is defined by ${F|_{\mathrm{D}}}^+(B) = \cap_{b \in B} F(b)$ for each $B \in 2^D$.
${F|_{\mathrm{D}}}^- : 2^U \to 2^D$ is defined by ${F|_{\mathrm{D}}}^-(X) = \{a \in D : X \subseteq F(a)\}$ for each $X \in 2^U$.

We also define the associated operation $\Psi_{F|_{\mathrm{D}}} : 2^U \to 2^U$ as follows: $\Psi_{F|_{\mathrm{D}}} : 2^U \to 2^U$ defined as for each $X \in 2^U$,

$$\Psi_{F|_{\mathrm{D}}}(X) = ({F|_{\mathrm{D}}}^+ \circ {F|_{\mathrm{D}}}^-)(X) = {F|_{\mathrm{D}}}^+ {F|_{\mathrm{D}}}^-(X).$$

Lemma 2.3 ([8])**.** *Let (U, A, F) be a soft context, $D \subseteq A$ and $X \subseteq U$. Then*
(1) ${F|_D}^-(X) \subseteq F^-(X)$.
(2) ${F|_D}^-(X) = F^-(X) \cap D$.
(3) ${F|_D}^+(B) = F^+(B)$ for $B \subseteq D$.

(4) $sC(U, D, F|_D) = Im(F|_D{}^+)$.
(5) $sC(U, D, F|_D) \subseteq sC(U, A, F)$.

3. Soft Decision Contexts

Definition 3.1. *Let* (U, A, F) *and* (U, D, H) *be two soft contexts. The quintuple* (U, A, F, D, H) *is called a soft decision context, where* $A \cap D = \emptyset$, (F, A) *and* (H, D) *are two soft sets. Then* A *and* D *are called condition attribute set and decision attribute set, respectively. For* $X \in sC(U, A, F)$ *(resp.,* $sC(U, D, H)$*),* X *is called the conditional concept(resp., decision concept) of* (U, A, F, D, H)*, respectively.*

Example 3.2. *Let* $U = \{1, 2, 3, 4, 5\}$, $A = \{a, b, c, d, e, f, g, h\}$ *and* $D = \{d_1, d_2, d_3\}$. *Let us consider two soft sets* (F, A) *and* (H, D) *defined by*
$F(a) = F(d) = \{1, 3, 5\}; F(b) = \{1, 2, 4, 5\}; F(c) = \{2, 4\}; F(e) = \{1, 3\}; F(f) = F(g) = \{1, 5\}; F(h) = \{2\}$.
$H(d_1) = \{1, 2, 4, 5\}; H(d_2) = \{1, 3, 5\}; H(d_3) = \{2, 4\}$.
Then (U, A, F, D, H) *is a soft decision context. Table A shows a soft decision context* (U, A, F, D, H).

Table A: A soft decision context

-	a	b	c	d	e	f	g	h	d_1	d_2	d_3
1	1	1	0	1	1	1	1	0	1	1	0
2	0	1	1	0	0	0	0	1	1	0	1
3	1	0	0	1	1	0	0	0	0	1	0
4	0	1	1	0	0	0	0	0	1	0	1
5	1	1	0	1	0	1	1	0	1	1	0

From now on, we will write *abcd* instead of a set $\{a, b, c, d\}$.

Example 3.3. *For a soft decision context* (U, A, F, D, H) *in Example 3.2,* $sC(U, A, F) = \{\emptyset, 1, 2, 13, 15, 24, 135, 1245, U\}; sC(U, D, H) = \{\emptyset, 15, 24, 135, 1245, U\}$.

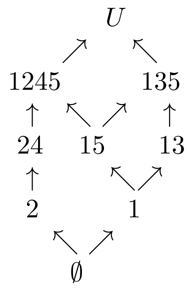
Let (U, A, F) be a soft context. In [8], we defined an order between $X_1, X_2 \in \text{sC}(U, A, F)$ as the following: For $X_1, X_2 \in \text{sC}(U, A, F)$, $X_1 \precsim X_2$ if and only if $X_1 \subseteq X_2$.

Then the *infimum* $\curlywedge$ and *supremum* $\curlyvee$ in the ordered set $(\text{sC}(U, A, F), \preceq)$ are defined as follows: $X_1 \curlywedge X_2 = X_1 \cap X_2$; $X_1 \curlyvee X_2 = \Psi(X_1 \cup X_2) = F^+F^-(X_1 \cup X_2)$.

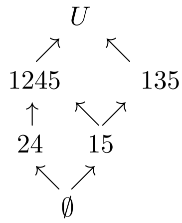
Then we showed that $(\text{sC}(U, A, F), \preceq, \curlywedge, \curlyvee)$ is a complete lattice in [8]. For a soft decision context (U, A, F, D, H), the complete lattice $(\text{sC}(U, A, F), \precsim, \curlywedge, \curlyvee)$ simply will be denoted by $\text{s}(U, A, F)$ and called *the condition concept lattice* of a soft decision context (U, A, F, D, H).

And the complete lattice $(\mathrm{sC}(U, D, H), \precsim, \curlywedge, \curlyvee)$ also simply will be denoted by $\mathrm{s}(U, D, H)$ and called *the decision concept lattice* of a soft decision context (U, A, F, D, H).

Example 3.4. *Consider a soft decision context (U, A, F, D, H) defined in Example 3.2. Then the condition concept lattice $s(U, A, F)$ and the decision concept lattice $s(U, D, H)$ are shown in the diagrams below:*



The condition concept lattice $s(U, A, F)$



The decision concept lattice $s(U, D, H)$

We introduce the notion of consistent sets on a given soft context, and investigate basic properties of consistent sets based on a soft context.

Definition 3.5. *Let (U, A, F, D, H) be a soft decision context. Then (U, A, F, D, H) is said to be consistent if $sC(U, D, H) \subseteq sC(U, A, F)$.*

When we consider a soft decision context (U, A, F, D, H) defined as Example 3.2, from Example 3.3, we know that $\mathrm{sC}(U, D, H) \subseteq \mathrm{sC}(U, A, F)$. Thus the soft decision context (U, A, F, D, H) is consistent.

Definition 3.6. *Let (U, A, F, D, H) be a consistent soft decision context and $B \subseteq A$. Then B is called a consistent set of (U, A, F, D, H) if $(U, B, F|_B, D, H)$ is also a consistent soft decision context, where $F|_B : B \to 2^U$ is a set-valued map defined as $F|_B(b) = F(b)$ for $b \in B$.*

Theorem 3.7. *Let (U, A, F, D, H) be a consistent soft decision context and $B \subseteq A$. B is a consistent set if and only if for each $E \subseteq D$, there exists $C \subseteq B$ such that $F^+(E) = F^+(C)$.*

Proof. Let B be a consistent set and $E \subseteq D$. Since $(U, B, F|_{\mathrm{B}}, D, H)$ is a consistent soft decision context, $\mathrm{sC}(U, D, H) \subseteq \mathrm{sC}(U, B, F|_{\mathrm{B}})$. Since $F^+(E) \in \mathrm{sC}(U, D, H)$, $F^+(E) \in \mathrm{sC}(U, B, F|_{\mathrm{B}})$, and so by (4) of Theorem 2.2, there exists $C \subseteq B$ such that $F^+(E) = F^+(C)$.

Now, to prove the converse of the above proposition, we assume that for each $E \subseteq D$, there exists $C \subseteq B$ such that $F^+(E) = F^+(C)$. Let $X \in \mathrm{sC}(U, D, H)$. Then by (4) of Theorem 2.2, there exists $E \subseteq D$ such that $X = F^+(E)$. Thus, by assumption, there exists $C \subseteq B$ such that $F^+(E) = F^+(C)$. Then by (2) of Theorem 2.2, $F^+(C) \in \mathrm{sC}(U, B, F|_{\mathrm{B}})$, and $X \in \mathrm{sC}(U, B, F|_{\mathrm{B}})$. Thus, we can conclude $\mathrm{sC}(U, D, H) \subseteq \mathrm{sC}(U, B, F|_{\mathrm{B}})$. Finally, by Definition 3.5 and 3.6, we have B is a consistent set. □

Theorem 3.8. *Let (U, A, F, D, H) be a consistent soft decision context and $B \subseteq A$. B is a consistent set if and only if for each $E \subseteq D$, $F^+(F^+F^-(E) \cap B) = F^+(E)$.*

Proof. Let B be a consistent set and $E \subseteq D$. From Definition 3.5 and 3.6, $\mathrm{sC}(U, D, H) \subseteq \mathrm{sC}(U, B, F|_{\mathrm{B}})$ and $F^+(E) \in \mathrm{sC}(U, B, F|_{\mathrm{B}})$. Thus $F^-F^+(E) \subseteq B$. Consequently, by (4) of Theorem 2.1, $F^+(F^-F^+(E) \cap B) = F^+(E)$.

For the converse proof, we assume that for each $E \subseteq D$, $F^+(F^-F^+(E) \cap B) = F^+(E)$ holds. Set $C = F^-F^+(E) \cap B$. Then $C \subseteq B$ and for $E \subseteq D$, by assumption, $F^+(C) = F^+(F^-F^+(E) \cap B) = F^+(E)$. By the above Theorem 3.7, B is a consistent set.

□

Theorem 3.9. *Let (U, A, F, D, H) be a consistent soft decision context. If B is a consistent set of (U, A, F, D, H), then $F^+(D) \subseteq F^+(B)$.*

Proof. Since B is a consistent set of the consistent formal decision context (U, A, F, D, H), by Theorem 3.7 there exists $C \subseteq B$ such that $F^+(D) = F^+(C)$. Since $C \subseteq B$, by (1) of Theorem 2.1, it implies $F^+(B) \subseteq F^+(D)$.

□

Theorem 3.10. *Let (U, A, F, D, H) be a consistent soft decision context and $B \subseteq A$. Then B is a consistent set if and only if for each $d \in D$, there exists a nonempty subset C of B such that $F^+(C) = F^+(d)$.*

Proof. Suppose that for each $d \in D$ there exists $C \subseteq B$ such that $F^+(C) = F^+(d)$. For each $E \subseteq D$, by Theorem 2.1, $F^+(E) = \cap_{d \in E} F^+(d) = \cap_{d \in E} F^+(C) = F^+(\cup_{d \in E} C)$. Since $F^+(E) = F^+(\cup_{d \in E} C)$ and $\cup_{d \in E} C \subseteq B$, by Theorem 3.7, B is a consistent set.

The converse is obtained from Theorem 3.7.

□

Theorem 3.11. *Let (U, A, F, D, H) be a consistent soft decision context and $B \subseteq A$. B is a consistent soft decision context if and only if for each $d \in D$, $F^+(F^-F^+(d) \cap B) = F^+(d)$.*

Proof. By Theorem 3.8 and 3.10, it is directly obtained. □

4. Conclusion

Soft decision context is one of the core studies of soft context. So we studied the consistent set based on soft consistent decision contexts in this paper. We will study the notion of attribute reduction in soft decision context using the notion of consistent set in soft decision context. And the more efficient attribute reduction method in soft decision context will be further studied. Finally, we would like to study the application of this method to attribute reduction in a formal decision context.

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