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SPECTRAL EXPANSION FOR DISCONTINUOUS SINGULAR DIRAC SYSTEMS

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Abstract. In this work, a discontinuous singular Dirac system is studied. For this system, a spectral function is constructed. Finally, by using the spectral function, a spectral expansion formula is given.

1. Introduction

In the last three decades, there has been an increasing interest in the discontinuous boundary value problems that appear in various physical problems (see [14]), geophysics (see [11]), and radio science (see [15]). The discontinuous boundary value problems were studied in [8, 17, 18, 7, 23].

On the other hand, eigenfunction expansions are important in the study of various problems in mathematics. When we solve a partial differential equation, we can use the separation variables method. Then we need an eigenfunction expansion. There exists a lot of literature devoted to this subject ([12, 21, 8, 1, 2, 3, 4, 5, 6]).

Consider the discontinuous Dirac system defined as

$$\tau(y) = \lambda y, \; x \in J := J_1 \cup J_2, \tag{1}$$

where $J_1 := [a, c), \; J_2 := (c, b]; \; -\infty < a < c < b < +\infty;$

$$\tau(y) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y'(x) + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} y(x),$$

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \; \lambda \in \mathbb{C};$$

p and r are real-valued, Lebesgue measurable functions on J and $p, r \in L^1(J_k)$, $k = 1, 2$. If $p, r \in L^1[c-\epsilon, c+\epsilon]$ for some $\epsilon > 0$,then c is the regular point.

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During the last century, Dirac operators play an important role in quantum mechanics since the existence of antimatter and a description of the electron spin are governed by these operators. The study of the fundamental theory of Dirac operators has a long history (see [20, 12], and references cited therein) and their spectral theory has also been investigated intensively. Regular impulsive Dirac operators were studied in [9, 16]. But there is a few study for the singular case. In [3], Allahverdiev and Tuna studied the resolvent operator of one-dimensional singular Dirac operator with transmission conditions.

In this paper, we shall construct a spectral function for singular discontinuous one-dimensional Dirac operators on semi-unbounded intervals. Later, we will give an eigenfunction expansion. A similar problem for the impulsive Dirac system on the whole line was recently investigated by the present authors in [1].

2. Main Results

Let us consider

$$\tau(y) = \lambda y,\ x \in J, \tag{2}$$

with the boundary condition

$$y_2(a)\cos\beta + y_1(a)\sin\beta = 0, \tag{3}$$

and impulsive conditions

$$y(c+) = Cy(c-), \tag{4}$$

where $\beta \in \mathbb{R} := (-\infty, \infty)$, $\det C = \delta > 0$ and $C \in M_2(\mathbb{R})$ i.e, C is the 2×2 matrix with entries from $\mathbb{R}$.

We adjoin to the problem (2)-(4) the boundary condition

$$y_2(b)\cos\alpha + y_1(b)\sin\alpha = 0, \tag{5}$$

where $\alpha \in \mathbb{R}$.

Now, we will denote by $H = L^2(J_1 : E) \dot{+} L^2(J_2 : E)$ the Hilbert space of vector-valued functions with values in E and with the inner product (scalar product)

$$\langle u, v\rangle_H := \int_a^c (u(x), v(x))_E\, dx + \gamma \int_c^b (u(x), v(x))_E\, dx,$$

where $E := \mathbb{R}^2,\ \gamma = \frac{1}{\delta}$, $(.,.)_E$ denotes the inner product on E and

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \in H,$$

$$u_1(x) = \begin{cases} u_{11}(x), & x \in J_1 \\ u_{12}(x), & x \in J_2 \end{cases},\quad u_2(x) = \begin{cases} u_{21}(x), & x \in J_1 \\ u_{22}(x), & x \in J_2 \end{cases},$$

$$v(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} \in H,$$

$$v_1(x) = \begin{cases} v_{11}(x), & x \in J_1 \\ v_{12}(x), & x \in J_2 \end{cases},\quad v_2(x) = \begin{cases} v_{21}(x), & x \in J_1 \\ v_{22}(x), & x \in J_2. \end{cases}.$$

Denote by

$$\phi(x,\lambda) = \begin{pmatrix} \phi_1(x,\lambda) \\ \phi_2(x,\lambda) \end{pmatrix},$$

$$\phi_1(x,\lambda) = \begin{cases} \phi_{11}(x,\lambda), & x \in J_1 \\ \phi_{12}(x,\lambda), & x \in J_2 \end{cases},\ \phi_2(x,\lambda) = \begin{cases} \phi_{21}(x,\lambda), & x \in J_1 \\ \phi_{22}(x,\lambda), & x \in J_2 \end{cases}$$

and

$$\chi(t,\lambda) = \begin{pmatrix} \chi_1(x,\lambda) \\ \chi_2(x,\lambda) \end{pmatrix},$$

$$\chi_1(x,\lambda) = \begin{cases} \chi_{11}(x,\lambda), & x \in J_1 \\ \chi_{12}(x,\lambda), & x \in J_2 \end{cases},\ \chi_2(x,\lambda) = \begin{cases} \chi_{21}(x,\lambda), & x \in J_1 \\ \chi_{22}(x,\lambda), & x \in J_2 \end{cases}.$$

the solutions of Eq. (2) which satisfy the following conditions

$$\phi_{11}(a,\lambda) = -\cos\beta,\ \phi_{21}(a,\lambda) = \sin\beta,$$

$$\chi_{12}(b,\lambda) = \cos\alpha,\ \chi_{22}(b,\lambda) = -\sin\alpha. \tag{6}$$

and

$$y(c+) = Cy(c-),$$

where $C \in M_2(\mathbb{R})$ and $\det C = \delta > 0$. It is clear that the problem (2)-(5) is a regular self-adjoint problem for the Dirac system.

Now we define the *Green matrix of* the boundary value problem (2)-(5)

$$G(x,t,\lambda) = \frac{1}{W(\phi,\chi)} \begin{cases} \phi(x,\lambda)\chi^T(t,\lambda), & x < t \le b,\ x \ne c, t \ne c \\ \chi(x,\lambda)\phi^T(t,\lambda), & a \le t < x,\ x \ne c, t \ne c \end{cases}$$

(see [12]).

Definition 2.1. *Let $M(x,t)$ be a matrix-valued function in E of two variables with $a \leq x, t \leq b$. If*

$$\int_a^b \int_a^b \|M(x,t)\|_E^2 \, dxdt < +\infty,$$

then $M(x,t)$ is called the Hilbert-Schmidt kernel.

Let us define the operator A by

$$A\{x_i\} = \{y_i\},$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k, i = 1, 2, ... \tag{7}$$

Theorem 2.2 ([19])**.** *If*

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < +\infty, \tag{8}$$

then A s compact operator in l^2.

There is no loss of generality in assuming that $\lambda = 0$ is not an eigenvalue of the problem (2)-(5). Thus we get

$$G(x,t) = G(x,t,0) = \frac{1}{W(\phi,\chi)} \begin{cases} \phi(x)\chi^T(t), & x < t \leq b, \ x \neq c, t \neq c \\ \chi(x)\phi^T(t), & a \leq t < x, \ x \neq c, t \neq c \end{cases}. \tag{9}$$

Then we have the following theorem.

Theorem 2.3. *$G(x,t)$ is a Hilbert-Schmidt kernel.*

Proof. It follows from (9) that

$$\int_a^b dx \int_a^x \|G(x,t)\|_E^2 \, dt < +\infty,$$

and

$$\int_a^b dx \int_x^b \|G(x,t)\|_E^2 \, dt < +\infty,$$

due to the inner integral exists and is a linear combination of the products $\phi_i(x)\chi_j(t)$ $(i,j = 1,2)$, and these products belong to $H \times H$ because each of the factors belongs to H. Hence, we have

$$\int_a^b \int_a^b \|G(x,t)\|_E^2 \, dxdt < +\infty. \tag{10}$$

□

Let us define the operator K by the formula

$$g(x) := (Kf)(x) = \int_a^b G(x,t) f(t) \, dt.$$

Theorem 2.4. *K is self-adjoint and compact operator in H.*

Proof. Let $f, g \in H$. Since $G(x,t) = G^T(t,x)$ and $G(x,t)$ is a matrix-valued function in E defined on $J \times J$, we conclude that

$$\begin{aligned}
&\langle Kf, g\rangle_H \\
&= \int_a^c ((Kf)(x), g(x))_E\, dx + \gamma \int_c^b ((Kf)(x), g(x))_E\, dx \\
&= \int_a^c \int_a^c (G(x,t) f(t), g(x))_E\, dt dx \\
&+ \gamma \int_c^b \int_c^b (G(x,t) f(t), g(x))_E\, dt dx \\
&= \int_a^c (f(t), \int_a^c G(t,x) g(x))_E dx dt \\
&+ \gamma \int_c^b (f(t), \int_c^b G(t,x) g(x))_E dx dt \\
&= \langle f, Kg\rangle_H .
\end{aligned}$$

Let us denote by $\{\phi_i(x)\}_{i=1}^{\infty}$ a complete, orthonormal basis of the space H. From Theorem 2.3, we have

$$\begin{aligned}
x_i &= \langle f, \phi_i\rangle_H \\
&= \int_a^c (f(t), \phi_i(t))_E\, dt + \gamma \int_c^b (f(t), \phi_i(t))_E\, dt, \\
y_i &= \langle g, \phi_i\rangle_H \\
&= \int_a^c (g(t), \phi_i(t))_E\, dt + \gamma \int_c^b (g(t), \phi_i(t))_E\, dt, \\
a_{ik} &= \int_a^b \int_a^b (G(x,t) \phi_i(x), \phi_k(t))_E\, dx dt \ (i, j = 1, 2, 3, ...).
\end{aligned}$$

Then, H is mapped isometrically l^2. Consequently, our integral operator transforms into the operator defined by the formula (7) in the space l^2 by this mapping, and the condition (10) is translated into the condition (8). By Theorem 2.2, this operator is compact. Therefore, the original operator is compact. □

It follows from Theorem 2.4 and the Hilbert-Schmidt theorem ([10]) that there exists an orthonormal system $\varphi_1, \varphi_2, ...$ of eigenvectors of the problem (2)-(5) with corresponding nonzero eigenvalues $\lambda_1, \lambda_2, ...$, such that

$$\int_a^c \|f(x)\|_E^2 \, dx + \gamma \int_c^b \|f(x)\|_E^2 \, dx = \sum_{n=0}^{\infty} a_n^2 \tag{11}$$

where $a_n = \langle f, \varphi_n \rangle_H$.

Let $\lambda_{m,b}$ $(m = 1, 2, ...)$ denote the (real) eigenvalues of this problem and by

$$\phi^{m,b}(x) = \begin{pmatrix} \phi_1^{m,b}(x) \\ \phi_2^{m,b}(x) \end{pmatrix},$$

$$\phi^{m,b}(x) := \phi(x, \lambda_m),$$

$$\phi_i^{m,b}(x) := \phi_i(x, \lambda_m) \; (i = 1, 2),$$

$$\phi_1^{m,b}(x) = \begin{cases} \phi_{11}^{m,b}(x), & x \in J_1 \\ \phi_{12}^{m,b}(x), & x \in J_2 \end{cases},$$

$$\phi_2^{m,b}(x) = \begin{cases} \phi_{21}^{m,b}(x), & x \in J_1 \\ \phi_{22}^{m,b}(x), & x \in J_2 \end{cases}$$

the corresponding real-valued eigenfunction which satisfies the conditions (3)-(5). If

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

$$f_1(x) = \begin{cases} f_{11}(x), & x \in J_1 \\ f_{12}(x), & x \in J_2 \end{cases}, \; f_2(x) = \begin{cases} f_{21}(x), & x \in J_1 \\ f_{22}(x), & x \in J_2 \end{cases},$$

$f(.) \in H$ and

$$\alpha_{m,b}^2 = \int_a^c \left(\left(\phi_{11}^{m,b}(x) \right)^2 + \left(\phi_{21}^{m,b}(x) \right)^2 \right) dx$$

$$+ \gamma \int_c^b \left(\left(\phi_{12}^{m,b}(x) \right)^2 + \left(\phi_{22}^{m,b}(x) \right)^2 \right) dx,$$

then we have

$$\|f\|_H^2$$

$$= \int_a^c \left(f_{11}^2(x) + f_{21}^2(x) \right) dx + \gamma \int_c^b \left(f_{12}^2(x) + f_{22}^2(x) \right) dx$$

$$(12) \qquad = \sum_{m=1}^{\infty} \frac{1}{\alpha_{m,b}^2} \left\{ \begin{array}{c} \int_a^c \left(f_{11}(x)\, \phi_{11}^{m,b}(x) + f_{21}(x)\, \phi_{21}^{m,b}(x) \right) dx \\ +\gamma \int_c^b \left(f_{12}(x)\, \phi_{12}^{m,b}(x) + f_{22}(x)\, \phi_{22}^{m,b}(x) \right) dx \end{array} \right\}^2 .$$

which is called the *Parseval equality.*

Define the function ϱ_b on $[a, \infty)$ by the formula

$$\varrho_b(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_{m,b} < 0} \frac{1}{\alpha_{m,b}^2}, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_{m,b} < \lambda} \frac{1}{\alpha_{m,b}^2} & \text{for } \lambda \geq 0 \end{cases} .$$

From (12), we have

$$(13) \qquad \|f\|_H^2 = \int_{-\infty}^{\infty} F^2(\lambda)\, d\varrho_b(\lambda),$$

where

$$F(\lambda) = \int_a^c \left(f_{11}(x)\, \phi_{11}(x,\lambda) + f_{21}(x)\, \phi_{21}(x,\lambda) \right) dx$$

$$+ \gamma \int_c^b \left(f_{12}(x)\, \phi_{12}(x,\lambda) + f_{22}(x)\, \phi_{22}(x,\lambda) \right) dx.$$

Lemma 2.5. *Let $h > 0$. Then the following relation holds*

$$(14) \qquad \bigvee_{-h}^{h} \{\varrho_b(\lambda)\} = \sum_{-h \leq \lambda_{m,b} < h} \frac{1}{\alpha_{m,b}^2} = \varrho_b(h) - \varrho_b(-h) < C,$$

where $C = C(h)$ is a positive constant $C = C(h)$ not depending on b.

Proof. Let $\sin\beta \neq 0$. It follows from the condition $\phi_{21}(a,\lambda) = \sin\beta$ that there exists a positive number k and nearby a such that

$$\frac{1}{k^2}\left(\int_a^k \phi_{21}(x,\lambda)\,dx\right)^2 > \frac{1}{2}\sin^2\beta. \tag{15}$$

due to $\phi_{21}(x,\lambda)$ is continuous on the region

$$\{(t,\lambda) : -h \leq \lambda \leq h,\ a \leq x \leq c\}.$$

Let us define $f_k(x) = \begin{pmatrix} f_{k_1}(x) \\ f_{k_2}(x) \end{pmatrix}$ by

$$f_{k_1}(x) = 0,\ f_{k_2}(x) = \begin{cases} \frac{1}{k}, & a \leq x < k \\ 0, & x \geq k. \end{cases}.$$

From (13) and (15), we get

$$\int_a^k \left(f_{k_1}^2(x) + f_{k_2}^2(x)\right)dx$$

$$= \frac{k-a}{k^2}$$

$$= \int_{-\infty}^{\infty}\left(\frac{1}{k}\int_a^k \phi_{21}(x,\lambda)\,dx\right) d\varrho_b(\lambda)$$

$$\geq \int_{-h}^{h}\left(\frac{1}{k}\int_a^k \phi_{21}(x,\lambda)\,dx\right)^2 d\varrho_b(\lambda)$$

$$> \frac{1}{2}\sin^2\beta\,\{\varrho_b(h) - \varrho_b(-h)\}.$$

Now, let $\sin\beta = 0$. Then we will define

$$f_k(x) = \begin{pmatrix} f_{k_1}(x) \\ f_{k_2}(x) \end{pmatrix}$$

by

$$f_{k,1}(x) = \begin{cases} \frac{1}{k^2}, & a \leq x < k \\ 0, & x \geq k. \end{cases},\quad f_{k,2}(x) = 0.$$

Applying the Parseval equality, we deduce that (14). □

We present below for the convenience of the reader.

Theorem 2.6 ([10]). *Let* $(w_n)_{n\in\mathbb{N}}$ $(\mathbb{N} := \{1, 2, ...\})$ *be a uniformly bounded sequence of real nondecreasing function on a finite interval* $a \le \lambda \le b$. *Then there exists a subsequence* $(w_{n_k})_{k\in\mathbb{N}}$ *and a nondecreasing function* w *such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda), \ a \le \lambda \le b.$$

Theorem 2.7 ([10]). *Assume* $(w_n)_{n\in\mathbb{N}}$ *is a real, uniformly bounded, sequence of non-decreasing function on a finite interval* $a \le \lambda \le b$, *and suppose*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda), \ a \le \lambda \le b.$$

If f *is any continuous function on* $a \le \lambda \le b$, *then*

$$\lim_{n\to\infty} \int_a^b f(\lambda)\, dw_n(\lambda) = \int_a^b f(\lambda)\, dw(\lambda).$$

Let $\mathcal{H} = L^2(J_1; E) \dot{+} L^2(J_3; E)$ be the Hilbert space with the inner product

$$\langle u, v\rangle_{\mathcal{H}} := \int_a^c (u(x), v(x))_E\, dx + \gamma \int_c^\infty (u(x), v(x))_E\, dx,$$

where $J_3 := (c, \infty)$, $\gamma = \frac{1}{\delta}$ and

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \in \mathcal{H}$$

$$u_1(x) = \begin{cases} u_{11}(x), & x \in J_1 \\ u_{12}(x), & x \in J_2 \end{cases}, \ u_2(x) = \begin{cases} u_{21}(x), & x \in J_1 \\ u_{22}(x), & x \in J_2 \end{cases},$$

$$v(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} \in \mathcal{H},$$

$$v_1(x) = \begin{cases} v_{11}(x), & x \in J_1 \\ v_{12}(x), & x \in J_2 \end{cases}, \ v_2(x) = \begin{cases} v_{21}(x), & x \in J_1 \\ v_{22}(x), & x \in J_2. \end{cases}.$$

Let ϱ be any non-decreasing function on $\mathbb{R}$. Denote by $L^2_\varrho(\mathbb{R})$ the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{R}$ which are measurable with respect to the Lebesque-Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^2(\lambda)\, d\varrho(\lambda) < \infty,$$

with the inner product

$$(f, g)_\varrho := \int_{-\infty}^{\infty} f(\lambda)\, g(\lambda)\, d\varrho(\lambda).$$

Theorem 2.8. *For the Dirac system (2)-(4), there exists a non-decreasing function* $\varrho(\lambda)$ *on* $\mathbb{R}$ *with the following properties.*

(i) If

$$f(x)=\left(\begin{array}{c} f_1(x) \\ f_2(x) \end{array}\right),$$

$$f_1(x)=\begin{cases} f_{11}(x), & x\in J_1 \\ f_{12}(x), & x\in J_2 \end{cases},\ f_2(x)=\begin{cases} f_{21}(x), & x\in J_1 \\ f_{22}(x), & x\in J_2 \end{cases},$$

and $f(.)\in\mathcal{H}$, *then there exists a function* $F\in L^2_\varrho(\mathbb{R})$ *such that*

(16)

$$\lim_{b\to\infty}\int_{-\infty}^{\infty}\left\{\begin{array}{c} F(\lambda)-\int_a^c (f_{11}(x)\phi_{11}(x,\lambda)+f_{21}(x)\phi_{21}(x,\lambda))\,dx \\ -\gamma\int_c^b (f_{12}(x)\phi_{12}(x,\lambda)+f_{22}(x)\phi_{22}(x,\lambda))\,dx \end{array}\right\}^2 d\varrho(\lambda)=0,$$

and the Parseval equality

$$\|f\|^2_{\mathcal{H}}=\int_a^c \left(f_{11}^2(x)+f_{21}^2(x)\right)dx$$

$$+\gamma\int_c^\infty \left(f_{12}^2(x)+f_{22}^2(x)\right)dx=\int_{-\infty}^{\infty}F^2(\lambda)\,d\varrho(\lambda) \tag{17}$$

holds.

(ii) The integrals

$$\int_{-\infty}^{\infty}F(\lambda)\phi_1(x,\lambda)\,d\varrho(\lambda),\ \text{and}\ \int_{-\infty}^{\infty}F(\lambda)\phi_2(x,\lambda)\,d\varrho(\lambda)$$

converge to f_1 *and* f_2 *in* $\mathcal{H}$, *respectively. That is,*

$$\lim_{n\to\infty}\left\{\begin{array}{c} \int_a^c\left\{f_{11}(x)-\int_{-\infty}^{\infty}F(\lambda)\phi_{11}(x,\lambda)\,d\varrho(\lambda)\right\}^2 dx \\ +\gamma\int_c^n\left\{f_{12}(x)-\int_{-\infty}^{\infty}F(\lambda)\phi_{12}(x,\lambda)\,d\varrho(\lambda)\right\}^2 dx \end{array}\right\}=0,$$

$$\lim_{n\to\infty}\left\{\begin{array}{c} \int_a^c\left\{f_{21}(x)-\int_{-\infty}^{\infty}F(\lambda)\phi_{21}(x,\lambda)\,d\varrho(\lambda)\right\}^2 dx \\ +\gamma\int_c^n\left\{f_{22}(x)-\int_{-\infty}^{\infty}F(\lambda)\phi_{22}(x,\lambda)\,d\varrho(\lambda)\right\}^2 dx \end{array}\right\}=0.$$

We note that the function ϱ is called a spectral function for the system (2)-(4).

Proof. Assume that the vector valued function

$$f_\xi(x)=\left(\begin{array}{c} f_{\xi_1}(x) \\ f_{\xi_2}(x) \end{array}\right),$$

$$f_{\xi_1}(x)=\begin{cases} f_{\xi_{11}}(x), & x\in J_1 \\ f_{\xi_{12}}(x), & x\in J_2 \end{cases},f_{\xi_2}(x)=\begin{cases} f_{\xi_{21}}(x), & x\in J_1 \\ f_{\xi_{22}}(x), & x\in J_2 \end{cases}$$

satisfies the following conditions.

1) $f_\xi(x) = \begin{cases} f_\xi(x), & x \in [a,c) \cup (c,\xi] \\ 0, & \text{otherwise} \end{cases}$, where $\xi < b$.

2) $f_\xi(x)$ has a continuous derivative.

3) $f_\xi(x)$ satisfies the conditions (3)-(4).

Applying (13) to $f_\xi(x)$, we conclude that

$$\int_a^c \left(f_{\xi_{11}^2}(x) + f_{\xi_{21}}^2(x) \right) dx$$

$$+\gamma \int_c^\xi \left(f_{\xi_{12}^2}(x) + f_{\xi_{22}}^2(x) \right) dx = \int_{-\infty}^{\infty} F_\xi^2(\lambda)\, d\varrho(\lambda), \tag{18}$$

where

$$F_\xi(\lambda) = \int_a^c \left(f_{\xi_{11}}(x)\,\phi_{11}(x,\lambda) + f_{\xi_{21}}(x)\,\phi_{21}(x,\lambda) \right) dx$$

$$+\gamma \int_c^\xi \left(f_{\xi_{12}}(x)\,\phi_{12}(x,\lambda) + f_{\xi_{22}}(x)\,\phi_{22}(x,\lambda) \right) dx. \tag{19}$$

Since $\phi(x,\lambda)$ satisfies the system (2), we see that

$$\phi_1(x,\lambda) = \frac{1}{\lambda}\left[-\phi_2'(x,\lambda) + p(x)\,\phi_1(x,\lambda)\right],$$
$$\phi_2(x,\lambda) = \frac{1}{\lambda}\left[\phi_1'(x,\lambda) + r(x)\,\phi_2(x,\lambda)\right].$$

By (19), we get

$$F_\xi(\lambda) = \frac{1}{\lambda}\int_a^c f_{\xi_{11}}(x)\left[-\phi_{21}'(x,\lambda) + p(x)\,\phi_{11}(x,\lambda)\right] dx$$

$$+\frac{1}{\lambda}\int_a^c f_{\xi_{21}}(x)\left[\phi_{11}'(x,\lambda) + r(x)\,\phi_{21}(x,\lambda)\right] dx$$

$$+\frac{1}{\lambda}\gamma\int_c^\xi f_{\xi_{12}}(x)\left[-\phi_{22}'(x,\lambda) + p(x)\,\phi_{12}(x,\lambda)\right] dx$$

$$+\frac{1}{\lambda}\gamma\int_c^\xi f_{\xi_{22}}(x)\left[\phi_{12}'(x,\lambda) + r(x)\,\phi_{22}(x,\lambda)\right] dx.$$

Since $f_{\xi}(x)$ vanishes in a neighborhood of the point b and $f_{\xi}(x)$ and $\phi(x,\lambda)$ satisfy the boundary conditions (3), (4), (5), we obtain

$$F_{\xi}(\lambda)=\frac{1}{\lambda}\int_a^c \phi_{11}(x,\lambda)\left[-f'_{\xi_{11}}(x)+p(x)f_{\xi_{11}}(x)\right]dx$$

$$+\frac{1}{\lambda}\int_a^c \phi_{21}(x,\lambda)\left[f'_{\xi_{21}}(x)+r(x)f_{\xi_{21}}(x)\right]dx$$

$$+\frac{1}{\lambda}\gamma\int_c^b \phi_{12}(x,\lambda)\left[-f'_{\xi_{12}}(x)+p(x)f_{\xi_{12}}(x)\right]dx$$

$$+\frac{1}{\lambda}\gamma\int_c^b \phi_{22}(x,\lambda)\left[f'_{\xi_{22}}(x)+r(x)f_{\xi_{22}}(x)\right]dx,$$

by integration by parts.

For any finite $h>0$, using (13), we have

$$\int_{|\lambda|>h}F_{\xi}^2(\lambda)\,d\varrho_b(\lambda)$$

$$\leq\frac{1}{h^2}\int_{|\lambda|>h}\left\{\begin{array}{c}\int_a^c\phi_{11}(x,\lambda)\left[-f'_{\xi_{11}}(x)+p(x)f_{\xi_{11}}(x)\right]dx\\+\int_a^c\phi_{21}(x,\lambda)\left[f'_{\xi_{21}}(x)+r(x)f_{\xi_{21}}(x)\right]dx\\+\gamma\int_c^b\phi_{12}(x,\lambda)\left[-f'_{\xi_{12}}(x)+p(x)f_{\xi_{12}}(x)\right]dx\\+\gamma\int_c^b\phi_{22}(x,\lambda)\left[f'_{\xi_{22}}(x)+r(x)f_{\xi_{22}}(x)\right]dx\end{array}\right\}^2 d\varrho_b(\lambda)$$

$$\leq\frac{1}{h^2}\int_{-\infty}^{\infty}\left\{\begin{array}{c}\int_a^c\phi_{11}(x,\lambda)\left[-f'_{\xi_{11}}(x)+p(x)f_{\xi_{11}}(x)\right]dx\\+\int_a^c\phi_{21}(x,\lambda)\left[f'_{\xi_{21}}(x)+r(x)f_{\xi_{21}}(x)\right]dx\\+\gamma\int_c^b\phi_{12}(x,\lambda)\left[-f'_{\xi_{12}}(x)+p(x)f_{\xi_{12}}(x)\right]dx\\+\gamma\int_c^b\phi_{22}(x,\lambda)\left[f'_{\xi_{22}}(x)+r(x)f_{\xi_{22}}(x)\right]dx\end{array}\right\}^2 d\varrho_b(\lambda)$$

$$=\frac{1}{h^2}\left\{\int_a^c\left[-f'_{\xi_{11}}(x)+p(x)f_{\xi_{11}}(x)\right]^2dx\right\}$$

$$+\frac{1}{h^2}\left\{\int_a^c\left[f'_{\xi_{21}}(x)+r(x)f_{\xi_{21}}(x)\right]^2dx\right\}$$

$$+\frac{\gamma}{h^2}\left\{\int_c^{\xi}\left[-f'_{\xi_{12}}(x)+p(x)f_{\xi_{12}}(x)\right]^2dx\right\}$$

$$+\frac{\gamma}{h^2}\left\{\int_c^{\xi}\left[f'_{\xi_{22}}(x)+r(x)f_{\xi_{22}}(x)\right]^2dx\right\}.$$

Then, from (18), we see that

$$| \int_a^c \left(f_{\xi_{11}^2}(x) + f_{\xi_{21}}^2(x) \right) dx$$

$$+\gamma \int_c^\xi \left(f_{\xi_{12}^2}(x) + f_{\xi_{22}}^2(x) \right) dx$$

$$- \int_{-h}^h F_\xi^2(\lambda)\, d\varrho_b(\lambda) \,|$$

$$< \frac{1}{h^2} \left\{ \int_a^c \left[-f'_{\xi_{11}}(x) + p(x) f_{\xi_{11}}(x) \right]^2 dx \right\}$$

$$+ \frac{1}{h^2} \left\{ \int_a^c \left[f'_{\xi_{21}}(x) + r(x) f_{\xi_{21}}(x) \right]^2 dx \right\}$$

$$+ \frac{\gamma}{h^2} \left\{ \int_c^\xi \left[-f'_{\xi_{12}}(x) + p(x) f_{\xi_{12}}(x) \right]^2 dx \right\}$$

$$+ \frac{\gamma}{h^2} \left\{ \int_c^\xi \left[f'_{\xi_{22}}(x) + r(x) f_{\xi_{22}}(x) \right]^2 dx \right\}. \tag{20}$$

By Lemma 2.5, the set$\{\varrho_b(\lambda)\}$ is bounded. Using Theorems 2.6 and 2.7, we can find a sequence $\{b_k\}$ such that the function $\varrho_{b_k}(\lambda)$ converges to a monotone function $\varrho(\lambda)$. Passing to the limit with respect to $\{b_k\}$ in (20), we get

$$| \int_a^c \left(f_{\xi_{11}^2}(x) + f_{\xi_{21}}^2(x) \right) dx + \gamma \int_c^\xi \left(f_{\xi_{12}^2}(x) + f_{\xi_{22}}^2(x) \right) dx$$

$$- \int_{-h}^h F_\xi^2(\lambda)\, d\varrho(\lambda) \,|$$

$$< \frac{1}{h^2}\left\{\int_a^c \left[-f'_{\xi_{11}}(x) + p(x) f_{\xi_{11}}(x)\right]^2 dx\right\}$$

$$+ \frac{1}{h^2}\left\{\int_a^c \left[f'_{\xi_{21}}(x) + r(x) f_{\xi_{21}}(x)\right]^2 dx\right\}$$

$$+ \frac{\gamma}{h^2}\left\{\int_c^\xi \left[-f'_{\xi_{12}}(x) + p(x) f_{\xi_{12}}(x)\right]^2 dx\right\}$$

$$+ \frac{\gamma}{h^2}\left\{\int_c^\xi \left[f'_{\xi_{22}}(x) + r(x) f_{\xi_{22}}(x)\right]^2 dx\right\}.$$

Hence, letting $h \to \infty$, we obtain

$$\int_a^c \left(f_{\xi_{11}^2}(x) + f_{\xi_{21}}^2(x)\right) dx$$

$$+\gamma \int_c^\xi \left(f_{\xi_{12}^2}(x) + f_{\xi_{22}}^2(x)\right) dx = \int_{-\infty}^{\infty} F_\xi^2(\lambda)\, d\varrho(\lambda).$$

Now, let f be an arbitrary vector valued function on $\mathcal{H}$. It is known that there exists a sequence of vector valued function $\{f_\xi(x)\}$ satisfying the condition 1-3 and such that

$$\lim_{\xi\to\infty}\left\{\int_a^\infty (f_1(x) - f_{\xi 1}(x))^2 dx + \gamma \int_a^\infty (f_2(x) - f_{\xi 2}(x))^2 dx\right\} = 0.$$

Let

$$F_\xi(\lambda) = \int_a^c \left(f_{\xi_{11}}(x)\, \phi_{11}(x,\lambda) + f_{\xi_{21}}(x)\, \phi_{21}(x,\lambda)\right) dx$$

$$+\gamma \int_c^\xi \left(f_{\xi_{12}}(x)\, \phi_{12}(x,\lambda) + f_{\xi_{22}}(x)\, \phi_{22}(x,\lambda)\right) dx.$$

Then, we have

$$\|f_\xi\|_{\mathcal{H}}^2 = \int_{-\infty}^{\infty} F_\xi^2(\lambda)\, d\varrho(\lambda).$$

Since

$$\int_a^c (f_{\xi_1 1}(x) - f_{\xi_2 1}(x))^2 dx + \gamma \int_c^\infty (f_{\xi_1 2}(x) - f_{\xi_1 2}(x))^2 dx \to 0$$

as $\xi_1, \xi_2 \to \infty$, we have

$$\int_{-\infty}^{\infty} \left(F_{\xi_1}(\lambda) - F_{\xi_2}(\lambda)\right)^2 d\varrho(\lambda)$$

$$= \int_a^c (f_{\xi_1 1}(x) - f_{\xi_2 1}(x))^2 dx + \gamma \int_c^{\infty} (f_{\xi_1 2}(x) - f_{\xi_1 2}(x))^2 dx \to 0$$

as $\xi_1, \xi_2 \to \infty$. Therefore, there exists a limit function F that satisfies

$$\|f\|_{\mathcal{H}}^2 = \int_{-\infty}^{\infty} F^2(\lambda)\, d\varrho(\lambda),$$

by the completeness of the space $L_\varrho^2(\mathbb{R})$.

Our next goal is to show that the function

$$K_\xi(\lambda) = \int_a^c f_{11}(x)\,\phi_{11}(x,\lambda) + f_{21}(x)\,\phi_{21}(x,\lambda)\, dx$$

$$+\gamma \int_c^{\xi} f_{12}(x)\,\phi_{12}(x,\lambda) + f_{22}(x)\,\phi_{22}(x,\lambda)\, dx$$

converges as $\xi \to \infty$ to F in the metric of space $L_\varrho^2(\mathbb{R})$. Let g be another vector-valued function in $\mathcal{H}$. By a similar argument, $G(\lambda)$ be defined by g. It is clear that

$$\int_a^c (f_1(x) - g_1(x))^2 dx + \gamma \int_c^{\infty} (f_2(x) - g_2(x))^2 dx$$

$$= \int_{-\infty}^{\infty} \{F(\lambda) - G(\lambda)\}^2\, d\varrho(\lambda).$$

Set

$$g(x) = \begin{cases} f(x), & x \in [a,c) \cup (c,\xi] \\ 0, & x \in (\xi,\infty). \end{cases}$$

Then we have

$$\int_{-\infty}^{\infty} \{F(\lambda) - K_\xi(\lambda)\}^2\, d\varrho(\lambda)$$

$$= \gamma \int_{\xi}^{\infty} \left(f_{12}^2(x) + f_{22}^2(x)\right) dx \to 0 \ \ (\xi \to \infty),$$

which proves that K_ξ converges to F in $L_\varrho^2(\mathbb{R})$ as $\xi \to \infty$. This proves (i).

Now, we will prove (ii). Suppose that the functions $f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix}$, $g(.) = \begin{pmatrix} g_1(.) \\ g_2(.) \end{pmatrix} \in \mathcal{H}$, and $F(\lambda)$ and $G(\lambda)$ are their generalized Fourier transforms (see (17)). Then $F \mp G$ are transforms of $f \mp g$. Consequently, by (17), we have

$$\int_a^c \left([f_{11}(x) + g_{11}(x)]^2 + [f_{21}(x) + g_{21}(x)]^2 \right) dx$$

$$+ \gamma \int_c^\infty \left([f_{12}(x) + g_{12}(x)]^2 + [f_{22}(x) + g_{22}(x)]^2 \right) dx$$

$$= \int_{-\infty}^\infty (F(\lambda) + G(\lambda))^2 \, d\varrho(\lambda).$$

Subtracting the second relation from the first, we get

$$\int_a^c [f_{11}(x) g_{11}(x) + f_{21}(x) g_{21}(x)] \, dx$$

$$+\gamma \int_c^\infty [f_{12}(x) g_{12}(x) + f_{22}(x) g_{22}(x)] \, dx = \int_{-\infty}^\infty F(\lambda) G(\lambda) \, d\varrho(\lambda) \tag{21}$$

which is called the *generalized Parseval equality.*

Set

$$f_\tau(x) = \begin{pmatrix} f_{\tau_1}(x) \\ f_{\tau_2}(x) \end{pmatrix},$$

$$f_{\tau_1}(x) = \begin{cases} \int_{-\tau}^{\tau} F(\lambda) \phi_{11}(t,\lambda) \, d\varrho(\lambda), & x \in J_1 \\ \int_{-\tau}^{\tau} F(\lambda) \phi_{12}(t,\lambda) \, d\varrho(\lambda), & x \in J_2 \end{cases},$$

$$f_{\tau_2}(x) = \begin{cases} \int_{-\tau}^{\tau} F(\lambda) \phi_{21}(t,\lambda) \, d\varrho(\lambda), & x \in J_1 \\ \int_{-\tau}^{\tau} F(\lambda) \phi_{22}(t,\lambda) \, d\varrho(\lambda), & x \in J_2, \end{cases}$$

where F is the function defined in (16). Let $= \left\{ \begin{array}{cc} g\left(x\right), & x\in\left[a,c\right)\cup\left(c,\mu\right] \\ 0, & otherwise, \end{array} \right.$,

where $g\left(.\right)=\left(\begin{array}{c} g_1\left(.\right) \\ g_2\left(.\right) \end{array} \right)$ and $\mu>c$. Hence we have

$$\begin{aligned}
\left\langle f_\tau,g\right\rangle_{\mathcal{H}} &= \int_a^c\left\{\int_{-\tau}^{\tau}F\left(\lambda\right)\phi_{11}\left(x,\lambda\right)d\varrho\left(\lambda\right)\right\}g_{11}\left(x\right)dx \\
&+\gamma\int_c^\mu\left\{\int_{-\tau}^{\tau}F\left(\lambda\right)\phi_{12}\left(x,\lambda\right)d\varrho\left(\lambda\right)\right\}g_{12}\left(x\right)dx \\
&+\int_a^c\left\{\int_{-\tau}^{\tau}F\left(\lambda\right)\phi_{21}\left(x,\lambda\right)d\varrho\left(\lambda\right)\right\}g_{21}\left(x\right)dx \\
&+\gamma\int_c^\mu\left\{\int_{-\tau}^{\tau}F\left(\lambda\right)\phi_{22}\left(x,\lambda\right)d\varrho\left(\lambda\right)\right\}g_{22}\left(x\right)dx \\
&=\int_{-\tau}^{\tau}F\left(\lambda\right)\left\{ \begin{array}{c} \int_a^c\phi_{11}\left(x,\lambda\right)g_{11}\left(x\right)dx \\ +\gamma\int_c^\mu\phi_{12}\left(x,\lambda\right)g_{12}\left(x\right)dx \end{array} \right\}d\varrho\left(\lambda\right) \\
&+\int_{-\tau}^{\tau}F\left(\lambda\right)\left\{ \begin{array}{c} \int_a^c\phi_{21}\left(x,\lambda\right)g_{21}\left(x\right)dx \\ +\gamma\int_c^\mu\phi_{22}\left(x,\lambda\right)g_{22}\left(x\right)dx \end{array} \right\}d\varrho\left(\lambda\right) \\
&=\int_{-\tau}^{\tau}F\left(\lambda\right)G\left(\lambda\right)d\varrho\left(\lambda\right). \qquad (22)
\end{aligned}$$

From (21), we get

$$\left\langle f,g\right\rangle_{\mathcal{H}}=\int_{-\infty}^{\infty}F\left(\lambda\right)G\left(\lambda\right)d\varrho\left(\lambda\right). \qquad (23)$$

Subtracting (22) and (23), we have

$$\left\langle f_\tau-f,g\right\rangle_{\mathcal{H}}=\int_{|\lambda|>\tau}F\left(\lambda\right)G\left(\lambda\right)d\varrho\left(\lambda\right).$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left|\left\langle f_\tau-f,g\right\rangle_{\mathcal{H}}\right|^2 &\le \int_{|\lambda|>\tau}F^2\left(\lambda\right)d\varrho\left(\lambda\right)\int_{|\lambda|>\tau}G^2\left(\lambda\right)d\varrho\left(\lambda\right) \\
&\le \int_{|\lambda|>\tau}F^2\left(\lambda\right)d\varrho\left(\lambda\right)\int_{-\infty}^{\infty}G^2\left(\lambda\right)d\varrho\left(\lambda\right).
\end{aligned}$$

Apply this inequality to the function

$$g(x) = \begin{cases} f_\tau(x) - f(x), & x \in [a,c) \cup (c,\mu] \\ 0, & x \in (\mu,\infty) \end{cases},$$

we get

$$\|f_\tau - f\|_{\mathcal{H}}^2 \leq \int_{|\lambda|>\tau} F^2(\lambda)\, d\varrho(\lambda).$$

Letting $\tau \to \infty$ yields the desired result. □

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