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SOME SPECIAL SMARANDACHE RULED SURFACES BY FRENET FRAME IN E^3-II

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Abstract. In this study, firstly Smarandache ruled surfaces whose base curves are Smarandache curves derived from Frenet vectors of the curve, and whose direction vectors are unit vectors plotting Smarandache curves, are created. Then, the Gaussian and mean curvatures of the obtained ruled surfaces are calculated separately, and the conditions to be developable or minimal for the surfaces are given. Finally, the examples are given for each surface and the graphs of these surfaces are drawn.

1. Introduction

The main resources on the theory of curves and surfaces, which have an important place in differential geometry, are [4, 6, 7, 8, 10, 11, 15, 16, 25, 26]. The image of a function with two real variables in three-dimensional space creates a surface. The research on the curvature of a surface have gained momentum with the calculations developed by Newton and Leibniz in the 17^{th} century. The surfaces with zero Gaussian curvature at each point are called the developable surfaces, and the surfaces with zero mean curvature at each point are called the minimal surfaces. In the theory of surfaces, the surfaces formed by the movement of a line (direction vector) along a curve (base curve) are called ruled surfaces. The basic concepts of the ruled surfaces, which have an important place in this field, are available in many sources, [2, 9, 18, 23]. On the other hand, a unit vector based on the elements of the Frenet frame can be defined by following:

$$\gamma = \frac{aT + bN + cB}{\sqrt{a^2 + b^2 + c^2}}, \tag{1}$$

where a, b, c are some real valued functions. For $\forall s \in I$, the locus of the endpoints of the vector γ defines a differentiable curve. If γ is taken to be the

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position vector, then the generated curve is called as Smarandache curve, [26]. Some studies of the Smarandache curves are given in [1, 3, 5, 17, 19, 20, 21, 22, 27, 28]. In recent studies on the ruled surfaces, Smarandache ruled surfaces, whose base curves are Smarandache curves obtained using the Frenet, Darboux and Alternative frames of any curve, have been defined and some properties of these surfaces have been investigated, [12, 13, 14, 18, 23, 24]. In this study, Gaussian and mean curvatures of Smarandache ruled surfaces produced from Smarandache curves and Frenet vectors are calculated, the conditions of being developable or minimum of these surfaces are specified, the examples are given for each surface and the graphs of these surfaces are drawn.

2. Preliminaries

$\alpha : I \to E^3$ be a unit speed regular curve. The Frenet frame T, N, B, the curvatures κ, τ and the Frenet derivative formulae of the curve α are given by followings:

$$T = \alpha',\ N = \frac{\alpha''}{\|\alpha''\|},\ B = T \wedge N,$$

$$\kappa = \|\alpha''\|\,,\ \tau = \langle N', B\rangle\,,$$

and

$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = -\tau N.$$

respectively. The surface formed by a line moving depending on the parameter of a curve is called a ruled surface and its parametric expression is as follows:

$$X\,(s, v) = \alpha(s) + vr\,(s)\,. \tag{2}$$

The normal vector field, the Gaussian and the mean curvatures of $X\,(s, v)$ are given by the relations below:

$$N_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|}, \tag{3}$$

$$K = \frac{eg - f^2}{EG - F^2},\ H = \frac{Eg - 2fF + eG}{2\,(EG - F^2)}. \tag{4}$$

respectively. Here, the coefficients of the first and the second fundamental forms are defined by follows:

$$E = \langle X_s, X_s\rangle\,,\ F = \langle X_s, X_v\rangle\,,\ G = \langle X_v, X_v\rangle\,, \tag{5}$$

$$e = \langle X_{ss}, N_X\rangle\,,\ f = \langle X_{sv}, N_X\rangle\,,\ g = \langle X_{vv}, N_X\rangle\,. \tag{6}$$

3. Some special Smarandache Ruled Surfaces according to Frenet Frame in $\mathbf{E}^3$ – II

The Smarandache curves obtained for the a, b, c values of the vector γ given in the expression (1) are defined as follows:

- For $a = b = 1,\ c = 0$, the $TN-$ Smarandache curve drawn by vector γ is $\gamma_1 = \dfrac{T+N}{\sqrt{2}}$,
- For $a = c = 1,\ b = 0$, the $TB-$ Smarandache curve drawn by vector γ is $\gamma_2 = \dfrac{T+B}{\sqrt{2}}$,
- For $b = c = 1,\ a = 0$, the $NB-$ Smarandache curve drawn by vector γ is $\gamma_3 = \dfrac{N+B}{\sqrt{2}}$.

Definition 3.1. *The ruled surface generated by continuously moving the vector TN along the γ_1- Smarandache curve is defined as follows:*

$$\Pi(s,v) = \frac{1}{\sqrt{2}}(T+N) + \frac{v}{\sqrt{2}}(T+N).$$

The first and the second partial differentials of $\Pi(s,v)$ are

$$\Pi_s = \frac{1}{\sqrt{2}}(1+v)(-\kappa T + \kappa N + \tau B), \qquad \Pi_v = \frac{1}{\sqrt{2}}(T+N),$$

$$\Pi_{ss} = \frac{1}{\sqrt{2}}(1+v)\left(-\left(\kappa' + \kappa^2\right)T + \left(\kappa' - \kappa^2 - \tau^2\right)N + (\tau' + \kappa\tau)B\right),$$

$$\Pi_{sv} = \frac{1}{\sqrt{2}}(-\kappa T + \kappa N + \tau B), \qquad \Pi_{vv} = 0.$$

And the vectorial product of the vectors Π_s , Π_v and its norm are

$$\Pi_s \wedge \Pi_v = \frac{(1+v)(-\tau T + \tau N - 2\kappa B)}{2},$$

$$\|\Pi_s \wedge \Pi_v\| = \frac{(1+v)\sqrt{4\kappa^2 + 2\tau^2}}{2}.$$

If we denote the normal vector field of the surface by N_Π, then from the expression (3), we have

$$N_{\Pi} = \frac{-\tau T + \tau N - 2\kappa B}{\sqrt{4\kappa^2 + 2\tau^2}}.$$

From the expressions (5) and (6), we compute the coefficients of the first and the second fundamental forms as

$$E_{\Pi} = \frac{1}{2}(1+v)^2 \left(2\kappa^2 + \tau^2\right), \qquad F_{\Pi} = 0, \qquad G_{\Pi} = 1$$

and

$$e_{\Pi} = \frac{(1+v)\left(2\left(\tau\kappa' - \kappa\tau'\right) - \tau\left(2\kappa^2 + \tau^2\right)\right)}{2\sqrt{2\kappa^2 + \tau^2}}, \qquad f_{\Pi} = 0, \qquad g_{\Pi} = 0,$$

respectively. Finally, by using the expression (4), we get the Gaussian and the mean curvatures

$$K_{\Pi} = 0, \qquad H_{\Pi} = \frac{2\left(\tau\kappa' - \kappa\tau'\right) - \tau\left(2\kappa^2 + \tau^2\right)}{2\left(1+v\right)\left(2\kappa^2 + \tau^2\right)\sqrt{2\kappa^2 + \tau^2}},$$

respectively.

Corollary 3.2. *The surface $\Sigma(s,v)$ is a developable surface.*

Definition 3.3. *The ruled surface generated by continuously moving the vector TB along the γ_1– Smarandache curve is defined as follows:*

$$\Phi\left(s,v\right) = \frac{1}{\sqrt{2}}\left(T + N\right) + \frac{v}{\sqrt{2}}\left(T + B\right).$$

The first and the second partial differentials of $\Phi\left(s,v\right)$ are

$$\Phi_s = \frac{1}{\sqrt{2}}\left(-\kappa T + \left(\kappa + v\left(\kappa - \tau\right)\right)N + \tau B\right), \qquad \Phi_v = \frac{1}{\sqrt{2}}\left(T + B\right),$$

$$\Phi_{ss} = \frac{1}{\sqrt{2}}\left[\begin{array}{l} -\left(\kappa' + \kappa^2 + v\kappa^2 - v\kappa\tau\right)T + \left(\kappa' + v\kappa' - v\tau' + \kappa^2 - \tau^2\right)N \\ +\ \left(\tau' + \kappa\tau + v\kappa\tau - v\tau^2\right)B \end{array}\right],$$

$$\Phi_{sv} = \frac{1}{\sqrt{2}}\left(\kappa - \tau\right)N, \qquad \Phi_{vv} = 0.$$

And the vectorial product of the vectors Φ_s , Φ_v and its norm are

$$\Phi_s \wedge \Phi_v = \frac{1}{2}\left(\left(\kappa + v\kappa - v\tau\right)T + \left(\kappa + \tau\right)N - \left(\kappa + v\kappa - v\tau\right)B\right),$$

$$\|\Phi_s \wedge \Phi_v\| = \frac{1}{2}\sqrt{2(\kappa + v\kappa - v\tau)^2 + (\kappa + \tau)^2}.$$

Thus, from the expression (3), the normal of the surface N_Φ is given as

$$N_\Phi = \frac{(\kappa + v\kappa - v\tau)\,T + (\kappa + \tau)\,N - \ (\kappa + v\kappa - v\tau)\,B}{\sqrt{2(\kappa + v\kappa - v\tau)^2 + (\kappa + \tau)^2}}.$$

By following the expressions (5) and (6), the coefficients of the first and the second fundamental forms are

$$E_\Phi = \frac{1}{2}\left(\kappa^2 + \tau^2 + (\kappa + v\kappa - v\tau)^2\right), \qquad F_\Phi = \frac{1}{2}\,(\tau - \kappa)\,, \qquad G_\Phi = 1$$

and

$$e_\Phi = \frac{\tau^2\left(\left(\frac{\kappa}{\tau}\right)' - \kappa - \tau\right) - v\left(\kappa^2\,(\kappa + \tau) - 2\tau^2\left(\kappa - \left(\frac{\kappa}{\tau}\right)'\right)\right) - v^2\left(\kappa^2 - \tau^2\right)(\kappa + \tau)}{\sqrt{2}\sqrt{2(\kappa + v\kappa - v\tau)^2 + (\kappa + \tau)^2}},$$

$$f_\Phi = \frac{\kappa^2 - \tau^2}{\sqrt{2}\sqrt{2(\kappa + v\kappa - v\tau)^2 + (\kappa + \tau)^2}}, \qquad g_\Phi = 0,$$

respectively. Finally, from the expression (4), the Gaussian and mean curvatures are obtained as

$$K_\Phi = -2\left(\frac{\left(\kappa^2 - \tau^2\right)}{2(\kappa + v\,(\kappa - \tau))^2 + (\kappa + \tau)^2}\right)^2,$$

$$H_\Phi = \frac{\begin{bmatrix} \tau^2\left(\left(\dfrac{\kappa}{\tau}\right)' - \kappa - \tau\right) + (\kappa - \tau)\left(\kappa^2 - \tau^2\right) \\ -v\left(\kappa^2\,(\kappa + \tau) - 2\tau^2\left(\kappa - \left(\dfrac{\kappa}{\tau}\right)'\right)\right) - v^2\left(\kappa^2 - \tau^2\right)(\kappa + \tau) \end{bmatrix}}{2\sqrt{2}\left((\kappa + v\,(\kappa - \tau))^2 + (\kappa + \tau)^2\right)^{\frac{3}{2}}},$$

respectively.

Corollary 3.4. *If $\kappa = \tau$, the surface $\Phi(s, v)$ is a developable surface.*

Definition 3.5. *The ruled surface generated by continuously moving the vector NB along the γ_1- Smarandache curve is defined as follows:*

$$\Sigma\,(s, v) = \frac{1}{\sqrt{2}}\,(T + N) + \frac{v}{\sqrt{2}}\,(N + B)\,.$$

The first and the second partial differentials of $\Sigma(s,v)$ are

$$\Sigma_s=\frac{1}{\sqrt{2}}\left(-\kappa(1+v)T+(\kappa-v\tau)N+\tau(1+v)B\right),\qquad \Sigma_v=\frac{1}{\sqrt{2}}(N+B),$$

$$\Sigma_{ss}=\frac{1}{\sqrt{2}}\left(\begin{array}{l}-\left[(1+v)\kappa'-(\kappa+v\tau)\kappa\right]T-\left[(1+v)\left(\kappa^2+\tau^2\right)-(\kappa'-v\tau')\right]N\\ +\left[(\kappa-v\tau)\tau+(1+v)\tau'\right]B\end{array}\right),$$

$$\Sigma_{sv}=\frac{1}{\sqrt{2}}\left(-\kappa T-\tau N+\ \tau B\right),\qquad \Sigma_{vv}=0.$$

And the vectorial product of the vectors Σ_s , Σ_v and its norm are

$$\Sigma_s\wedge\Sigma_v=\frac{1}{2}\left((\kappa-\tau-2v\tau)T+\kappa(1+v)N-\kappa(1+v)B\right),$$

$$\|\Sigma_s\wedge\Sigma_v\|=\frac{1}{2}\sqrt{(\kappa-\tau-2v\tau)^2+2(1+v)^2\kappa^2}.$$

From the expression (3), the normal of the surface N_Σ is

$$N_\Sigma=\frac{(\kappa-\tau-2v\tau)T+\kappa(1+v)N-\kappa(1+v)B}{\sqrt{(\kappa-\tau-2v\tau)^2+2(1+v)^2\kappa^2}}.$$

From the expressions (5) and (6) to compute the coefficients of fundamental forms, we get

$$E_\Sigma=\frac{1}{2}\left((1+v)^2\left(\kappa^2+\tau^2\right)+(\kappa-v\tau)^2\right),\qquad F_\Sigma=\frac{1}{2}(\kappa+\tau),\qquad G_\Sigma=1$$

and

$$e_\Sigma=\frac{\tau(\kappa'-\kappa\tau)-v\left(2\kappa\left(\kappa^2+\tau^2\right)-3\tau\kappa'+\kappa\tau'\right)}{\sqrt{(\kappa-\tau-2v\tau)^2+2(1+v)^2\kappa^2}},$$

$$f_\Sigma=\frac{\kappa(\tau-\kappa)-2v\tau\kappa}{\sqrt{(\kappa-\tau-2v\tau)^2+2(1+v)^2\kappa^2}},\qquad g_\Sigma=0,$$

respectively. Finally, from the expression (4), we obtain the Gaussian and mean curvatures as

$$K_{\Sigma} = -4\left(\frac{\kappa(\tau-\kappa)-2v\tau\kappa}{(\kappa-\tau-2v\tau)^2+2(1+v)^2\kappa^2}\right)^2,$$

$$H_{\Sigma} = \frac{\tau\kappa'-\kappa\left(2\tau^2-\kappa^2\right)-v\left(2\kappa\left(\kappa^2+\tau^2\right)-3\tau\kappa'+\kappa\tau'+2\kappa\tau(\kappa+\tau)\right)}{2^{-1}\left((\kappa-\tau-2v\tau)^2+2(1+v)^2\kappa^2\right)^{\frac{3}{2}}}.$$

Definition 3.6. *The ruled surface generated by continuously moving the vector TNB along the γ_1- Smarandache curve is defined as follows:*

$$\Upsilon(s,v) = \frac{1}{\sqrt{2}}(T+N)+\frac{v}{\sqrt{3}}(T+N+B).$$

The first and second partial differentials of $\Upsilon(s,v)$ are

$$\Upsilon_s = \frac{1}{\sqrt{6}}\left(-\left(\sqrt{3}\kappa+\sqrt{2}v\kappa\right)T+\left(\sqrt{3}\kappa-\sqrt{2}v(\kappa-\tau)\right)N+\left(\sqrt{3}\tau+\sqrt{2}v\tau\right)B\right),$$

$$\Upsilon_v = \frac{1}{\sqrt{3}}(T+N+B),$$

$$\Upsilon_{ss} = \frac{1}{\sqrt{6}}\begin{pmatrix} -\left[\sqrt{3}\left(\kappa'+\kappa^2\right)+\sqrt{2}v\left(\kappa'+\kappa^2-\tau\kappa\right)\right]T \\ +\left[\sqrt{3}\left(\tau'+\kappa\tau\right)-\sqrt{2}v\left(-\tau'+\kappa\tau-\tau^2\right)\right]B \\ +\left[\sqrt{3}\left(\kappa'-\kappa^2-\tau^2\right)-\sqrt{2}v\left(\kappa'-\tau'+\kappa^2+\tau^2\right)\right]N \end{pmatrix},$$

$$\Upsilon_{sv} = \frac{1}{\sqrt{3}}\left(-\kappa T+(\kappa-\tau)N+\tau B\right), \qquad \Upsilon_{vv}=0.$$

And the vectorial product of the vectors Υ_s , Υ_v and its norm are

$$\Upsilon_s\wedge\Upsilon_v = \frac{1}{3\sqrt{2}}\begin{pmatrix} \left(\sqrt{3}(\kappa-\tau)+\sqrt{2}v\kappa\right)T+\left(\sqrt{3}(\kappa+\tau)+\sqrt{2}v(\kappa-\tau)\right)N \\ -\left(2\sqrt{3}\kappa+\sqrt{2}v(2\kappa-\tau)\right)B \end{pmatrix},$$

$$\|\Upsilon_s\wedge\Upsilon_v\| = \frac{1}{3\sqrt{2}}\begin{pmatrix} \left(\sqrt{3}(\kappa-\tau)+\sqrt{2}v\kappa\right)^2+\left(\sqrt{3}(\kappa+\tau)+\sqrt{2}v(\kappa-\tau)\right)^2 \\ +\left(\sqrt{3}(\kappa+\tau)+\sqrt{2}v(2\kappa-\tau)\right)^2 \end{pmatrix}^{\frac{1}{2}}$$

From the expression (3), the normal of the ruled surface $\Upsilon(s,v)$ is

$$N_{\Upsilon}=\frac{\left(\begin{array}{l}\left(\sqrt{3}\left(\kappa-\tau\right)+\sqrt{2}v\kappa\right)T+\left(\sqrt{3}\left(\kappa+\tau\right)+\sqrt{2}v\left(\kappa-\tau\right)\right)N\\ -\left(2\sqrt{3}\kappa+\sqrt{2}v\left(2\kappa-\tau\right)\right)B\end{array}\right)}{\left(\begin{array}{l}\left(\sqrt{3}\left(\kappa-\tau\right)+\sqrt{2}v\kappa\right)^{2}+\left(\sqrt{3}\left(\kappa+\tau\right)+\sqrt{2}v\left(\kappa-\tau\right)\right)^{2}\\ +\left(\sqrt{3}\left(\kappa+\tau\right)+\sqrt{2}v\left(2\kappa-\tau\right)\right)^{2}\end{array}\right)^{\frac{1}{2}}}$$

From the expressions (5) and (6) to compute the coefficients of fundamental forms, we get

$$E_{\Upsilon}=\frac{1}{6}\left(6\kappa^{2}+3\tau^{2}-2\sqrt{6}v\left(2\kappa^{2}-\kappa\tau+\tau^{2}\right)+4v^{2}\left(\kappa^{2}-\kappa\tau+\tau^{2}\right)\right),$$

$$F_{\Upsilon}=\frac{1}{6}\left(\sqrt{6}\tau+4v\left(\tau-\kappa\right)\right),\qquad G_{\Upsilon}=1$$

and

$$e_{\Upsilon}=\frac{\left(\begin{array}{l}-3\left(\kappa+\tau\right)\left(2\kappa^{2}+\tau^{2}\right)+6\left(\tau\kappa'-\kappa\tau'\right)\\ -\sqrt{6}v\left(\left(\kappa^{2}+\tau^{2}\right)'+4\kappa^{3}+2\kappa\tau\left(2\tau-\kappa\right)+3\kappa\tau'\right)\\ -2v^{2}\left(\kappa'\left(\kappa-\tau\right)+2\left(\kappa^{3}-\tau^{3}\right)+4\tau\kappa\left(\tau-\kappa\right)-3\kappa\tau'\right)\end{array}\right)}{6\sqrt{\left(3\kappa^{2}+3\tau^{2}+4\tau\kappa+4\sqrt{6}v\left(2\kappa^{2}-\tau^{2}\right)+4v^{2}\left(3\kappa^{2}+\tau^{2}-3\kappa\tau\right)\right)}},$$

$$f_{\Upsilon}=\frac{2\left(\sqrt{3}\left(2\kappa^{2}-\tau^{2}-3\kappa\tau\right)+\sqrt{2}v\left(2\kappa^{2}+\tau^{2}-3\kappa\tau\right)\right)}{\sqrt{\left(3\kappa^{2}+3\tau^{2}+4\tau\kappa+4\sqrt{6}v\left(2\kappa^{2}-\tau^{2}\right)+4v^{2}\left(3\kappa^{2}+\tau^{2}-3\kappa\tau\right)\right)}},$$

$$g_{\Upsilon}=0,$$

respectively. Finally, from the expression (4), we compute the Gaussian and mean curvatures as

$$K_{\Upsilon}=-\frac{\left(\sqrt{3}\left(2\kappa^{2}-\tau^{2}-3\kappa\tau\right)+\sqrt{2}v\left(2\kappa^{2}+\tau^{2}-3\kappa\tau\right)\right)^{2}}{\left(3\kappa^{2}+3\tau^{2}+4\tau\kappa+4\sqrt{6}v\left(2\kappa^{2}-\tau^{2}\right)+4v^{2}\left(3\kappa^{2}+\tau^{2}-3\kappa\tau\right)\right)^{2}},$$

$$H_{\Upsilon}=\frac{\left(\begin{array}{l}-3\left(\kappa+\tau\right)\left(2\kappa^2+\tau^2\right)+6\left(\tau\kappa'-\kappa\tau'\right)\\ -\sqrt{6}v\left(\left(\kappa^2+\tau^2\right)'+4\kappa^3+2\kappa\tau\left(2\tau-\kappa\right)+3\kappa\tau'\right)\\ -2v^2\left(\kappa'\left(\kappa-\tau\right)+2\left(\kappa^3-\tau^3\right)+4\tau\kappa\left(\tau-\kappa\right)-3\kappa\tau'\right)\\ -8\left(\sqrt{3}\left(2\kappa^2-\tau^2-3\kappa\tau\right)+\sqrt{2}v\left(2\kappa^2+\tau^2-3\kappa\tau\right)\right)\left(\sqrt{6}\tau+4v\left(\tau-\kappa\right)\right)\end{array}\right)}{\frac{3}{2}\left(3\kappa^2+3\tau^2+4\tau\kappa+4\sqrt{6}v\left(2\kappa^2-\tau^2\right)+4v^2\left(3\kappa^2+\tau^2-3\kappa\tau\right)\right)^{\frac{3}{2}}}.$$

Definition 3.7. *The ruled surface generated by continuously moving the vector TN along the γ_2- Smarandache curve is defined as follows:*

$$\Psi\left(s,v\right)=\frac{1}{\sqrt{2}}\left(T+B\right)+\frac{v}{\sqrt{2}}\left(T+N\right).$$

The first and second partial differentials of $\Psi\left(s,v\right)$ are

$$\Psi_s=\frac{1}{\sqrt{2}}\left(-\kappa vT+\left(\kappa\left(1+v\right)-\tau\right)N+v\tau B\right),\qquad \Psi_v=\frac{1}{\sqrt{2}}\left(T+N\right),$$

$$\Psi_{ss}=\frac{1}{\sqrt{2}}\left(\begin{array}{l}\left[\kappa\tau-\kappa^2-v\left(\kappa'+\kappa^2\right)\right]T+\left[\kappa'-\tau'-v\left(\kappa^2+\tau^2-\kappa'\right)\right]N\\ +\left[\kappa\tau-\tau^2+v\left(\tau'+\tau\kappa\right)\right]B\end{array}\right),$$

$$\Psi_{sv}=\frac{1}{\sqrt{2}}\left(-\kappa T+\kappa N+\tau B\right),\qquad \Psi_{vv}=0.$$

And the vectorial product of the vectors Ψ_s , Ψ_v and its norm are

$$\Psi_s\wedge\Psi_v=\frac{1}{2}\left(-v\tau T+v\tau N+\left(\tau-\kappa-2v\kappa\right)B\right),$$

$$\left\|\Psi_s\wedge\Psi_v\right\|=\frac{1}{2}\sqrt{2v^2\tau^2+\left(\tau-\kappa-2v\kappa\right)^2}.$$

From the expression (3), we compute the normal of the surface denoted by N_Ψ as

$$N_\Psi=\frac{-v\tau T+v\tau N+\left(\tau-\kappa-2v\kappa\right)B}{\sqrt{2v^2\tau^2+\left(\tau-\kappa-2v\kappa\right)^2}}.$$

By the expressions (5) and (6), the coefficients of fundamental forms are given as

$$E_\Psi=\frac{1}{2}\left(v^2\left(\kappa^2+\tau^2\right)+\left(\kappa+v\kappa-\tau\right)^2\right),\qquad F_\Psi=\frac{1}{2}\left(\kappa-\tau\right),\qquad G_\Psi=1$$

and

$$e_\Psi = \frac{\begin{pmatrix} \tau(\tau-\kappa)^2 + v\left(4\kappa^2\tau + 2\kappa\tau^2 + \tau\kappa' - \kappa\tau'\right) \\ +v^2\left(2\tau\kappa' - \tau^3 - 2\kappa\tau' - 2\kappa^2\tau\right) \end{pmatrix}}{\sqrt{2}\sqrt{2v^2\tau^2 + (\tau - 2\kappa - v\kappa)^2}},$$

$$f_\Psi = \frac{\tau(\tau-\kappa)}{\sqrt{2}\sqrt{2v^2\tau^2 + (\tau-\kappa-2v\kappa)^2}}, \qquad g_\Psi = 0,$$

respectively. Finally, from the expression (4), we compute the Gaussian and mean curvatures as below:

$$K_\Psi = -2\left(\frac{\tau(\tau-\kappa)}{2v^2\tau^2 + (\tau-\kappa-2v\kappa)^2}\right)^2,$$

$$H_\Psi = \frac{\begin{pmatrix} \tau(\tau-\kappa)^2\left((\tau-\kappa)^2 - 1\right) + v\left(\tau\kappa' - \kappa\tau' + 2\kappa\tau\left(2\kappa + \tau + (\tau-\kappa)^3\right)\right) \\ +v^2\left(2\tau\kappa' - 2\kappa\tau' - \tau\left(\tau^2 + 2\kappa^2\right)\left(1 + (\tau-\kappa)^2\right)\right) \end{pmatrix}}{\left(\sqrt{2}\right)^{-1}\left(2v^2\tau^2 + (\tau-\kappa-2v\kappa)^2\right)^{\frac{3}{2}}}.$$

Corollary 3.8. *If $\tau = 0$ or $\kappa = \tau$, the ruled surface $\Psi(s,v)$ is developable.*

Definition 3.9. *The ruled surface generated by continuously moving the vector TB along the γ_2– Smarandache curve is defined as follows:*

$$\mathrm{P}(s,v) = \frac{1}{\sqrt{2}}(T+B) + \frac{v}{\sqrt{2}}(T+B).$$

The first and the second partial differentials of $\mathrm{P}(s,v)$ are

$$\mathrm{P}_s = \frac{(1+v)}{\sqrt{2}}(\kappa-\tau)N, \qquad \mathrm{P}_v = \frac{1}{\sqrt{2}}(T+B),$$

$$\mathrm{P}_{ss} = \frac{(1+v)}{\sqrt{2}}\left(\left(-\kappa^2 + \kappa\tau\right)T + (\kappa' - \tau')N + \left(\kappa\tau - \tau^2\right)B\right),$$

$$\mathrm{P}_{sv} = \frac{(\kappa-\tau)}{\sqrt{2}}N, \qquad \mathrm{P}_{vv} = 0.$$

And the vectorial product of the vectors P_s , P_v and its norm are

$$\mathrm{P}_s \wedge \mathrm{P}_v = \frac{1}{2}(1+v)(\kappa-\tau)(T-B),$$

$$\|\mathrm{P}_s \wedge \mathrm{P}_v\| = \frac{1}{\sqrt{2}}(1+v)(\kappa-\tau).$$

If we denote the normal vector of the surface by N_{P}, then from the expression (3), we get

$$N_{\mathrm{P}} = \frac{1}{\sqrt{2}}(T-B).$$

By using the expressions (5) and (6), the coefficients of the first and the second fundamental forms are given as in the followings:

$$E_{\mathrm{P}} = \frac{1}{2}(1+v)^2(\kappa-\tau)^2, \qquad F_{\mathrm{P}} = 0, \qquad G_{\mathrm{P}} = 1$$

and

$$e_{\mathrm{P}} = \frac{1}{\sqrt{2}}(1+v)\left(\tau^2-\kappa^2\right), \qquad f_{\mathrm{P}} = 0, \qquad g_{\mathrm{P}} = 0,$$

respectively. Finally, from the expression (4), the Gaussian and mean curvatures are obtained as:

$$K_{\mathrm{P}} = 0, \qquad H_{\mathrm{P}} = \frac{\tau+\kappa}{\sqrt{2}(1+v)(\tau-\kappa)}, \qquad \tau \neq \kappa.$$

Corollary 3.10. *The ruled surface* $\mathrm{P}(s,v)$ *is always developable, also if* $\kappa+\tau=0$, *the surface is minimal.*

Definition 3.11. *The ruled surface generated by continuously moving the vector* NB *along the* γ_2*– Smarandache curve is defined as follows:*

$$\mathrm{Z}(s,v) = \frac{1}{\sqrt{2}}(T+B) + \frac{v}{\sqrt{2}}(N+B).$$

The first and the second partial differentials of $\mathrm{Z}(s,v)$ are

$$\mathrm{Z_s} = \frac{1}{\sqrt{2}}(-\kappa v T + (\kappa-\tau-v\tau)N + v\tau B), \qquad \mathrm{Z_v} = \frac{1}{\sqrt{2}}(N+B),$$

$$\mathrm{Z_{ss}} = \frac{1}{\sqrt{2}}\left(\begin{array}{l} \left[\kappa\tau-\kappa^2+v(\tau\kappa-\kappa')\right]T + \left[\kappa'-\tau'+v\left(\kappa^2+\tau^2+\tau'\right)\right]N \\ +\left[\kappa\tau-\tau^2+v\left(\tau'-\tau^2\right)\right]B \end{array}\right),$$

$$\mathrm{Z_{sv}} = \frac{1}{\sqrt{2}}(-\kappa T - \tau N + \tau B), \qquad \mathrm{Z_{vv}} = 0.$$

And the vectorial product of the vectors Z_s , Z_v and its norm are

$$\mathrm{Z_s} \wedge Z_v = \frac{1}{2}\left(\left(\kappa - \tau - 2v\tau\right)T + v\kappa N - v\kappa B\right),$$

$$\left\|\mathrm{Z_s} \wedge Z_v\right\| = \frac{1}{2}\sqrt{2v^2\kappa^2 + \left(\kappa - \tau - 2v\tau\right)^2}.$$

By using the expression (3), the normal vector field denoted by N_{Z} can be computed as:

$$N_{\mathrm{Z}} = \frac{\left(\kappa - \tau - 2v\tau\right)T + v\kappa N - v\kappa B}{\sqrt{2v^2\kappa^2 + \left(\kappa - \tau - 2v\tau\right)^2}}.$$

From the expressions (5) and (6), the coefficients of fundamental forms can be given as:

$$E_{\mathrm{Z}} = \frac{1}{2}\left(v^2\left(\kappa^2 + 2\tau^2\right) + v2\left(\tau^2 - \tau\kappa\right) + \left(\kappa - \tau\right)^2\right),$$

$$F_{\mathrm{Z}} = \frac{1}{2}\left(\kappa - \tau\right), \qquad G_{\mathrm{Z}} = 1$$

and

$$e_{\mathrm{Z}} = \frac{2\kappa^2\tau - \kappa\left(\kappa^2 + \tau^2\right) + v\left(2\kappa\tau\left(\kappa - \tau\right) + \tau\kappa' - \tau'\kappa\right) + v^2\left(\kappa^2 + 2\tau\kappa'\right)}{\sqrt{2}\sqrt{2v^2\kappa^2 + \left(\kappa - \tau - 2v\tau\right)^2}},$$

$$f_{\mathrm{Z}} = \frac{\left(\tau - \kappa\right)\kappa}{\sqrt{2}\sqrt{2v^2\kappa^2 + \left(\kappa - \tau - 2v\tau\right)^2}}, \qquad g_{\mathrm{Z}} = 0,$$

respectively. Finally, from the expression (4), the Gaussian and mean curvatures are obtained as follows:

$$K_{\mathrm{Z}} = -2\left(\frac{\left(\tau - \kappa\right)\kappa}{2v^2\kappa^2 + \left(\kappa - \tau - 2v\tau\right)^2}\right)^2,$$

$$H_{\mathrm{Z}} = \frac{v\left(2\kappa\tau\left(\kappa - \tau\right) + \tau\kappa' - \tau'\kappa\right) + v^2\left(\kappa^2 + 2\tau\kappa'\right)}{2^{-1}\left(2v^2\kappa^2 + \left(\kappa - \tau - 2v\tau\right)^2\right)^{\frac{3}{2}}}.$$

Corollary 3.12. *If $\kappa = \tau$, the ruled surface $\mathrm{Z}\left(s, v\right)$ is developable.*

Definition 3.13. *The ruled surface generated by continuously moving the vector TNB along the γ_2– Smarandache curve is defined as follows:*

$$\mathrm{F}\left(s, v\right) = \frac{1}{\sqrt{2}}\left(T + B\right) + \frac{v}{\sqrt{3}}\left(T + N + B\right).$$

The first and second partial differentials of $\mathrm{F}(\mathrm{s},\mathrm{v})$ are

$$\mathrm{F}_s = \frac{1}{\sqrt{6}}\left(-\sqrt{2}\kappa v T + (\kappa-\tau)\left(\sqrt{3}+v\sqrt{2}\right)N + \sqrt{2}v\tau B\right),$$

$$\mathrm{F}_v = \frac{1}{\sqrt{3}}\left(T+N+B\right),$$

$$\mathrm{F}_{ss} = \frac{1}{\sqrt{6}}\left(\begin{array}{l} -\left[\sqrt{3}\kappa\left(\kappa-\tau\right)+\sqrt{2}v\left(\kappa^2-\kappa\tau+\kappa'\right)\right]T \\ +\left[\sqrt{3}\left(\kappa'-\tau'\right)+\sqrt{2}v\left(\kappa'-\tau'-\kappa^2-\tau^2\right)\right]N \\ +\left[\sqrt{3}\tau\left(\kappa-\tau\right)+\sqrt{2}v\left(\kappa\tau-\tau^2+\tau'\right)\right]B \end{array}\right),$$

$$\mathrm{F}_{sv} = \frac{1}{\sqrt{3}}\left(-\kappa T + (\kappa-\tau)N + \tau B\right), \qquad \mathrm{F}_{vv} = 0.$$

And the vectorial product of the vectors F_s , F_v and its norm are

$$\mathrm{F}_s \wedge \mathrm{F}_v = \frac{1}{3\sqrt{2}}\left(\begin{array}{l} \left[\sqrt{3}\left(\kappa-\tau\right)+\sqrt{2}v\left(\kappa-2\tau\right)\right]T+\sqrt{2}v\left(\kappa+\tau\right)N \\ -\left[\sqrt{3}\left(\kappa-\tau\right)+\sqrt{2}v\left(2\kappa-\tau\right)\right]B \end{array}\right),$$

$$\|\mathrm{F}_s \wedge \mathrm{F}_v\| = \frac{1}{3\sqrt{2}}\sqrt{6(\kappa-\tau)^2 + v6\sqrt{6}(\kappa-\tau)^2 + 12v^2\left(\kappa^2+\tau^2-\kappa\tau\right)}.$$

By using the expression (3), the normal vector field denoted by N_{F} can be computed as:

$$N_{\mathrm{F}} = \frac{\left(\begin{array}{l} \left(\sqrt{3}\left(\kappa-\tau\right)+\sqrt{2}v\left(\kappa-2\tau\right)\right)T+\sqrt{2}v\left(\kappa+\tau\right)N \\ -\left(\sqrt{3}\left(\kappa-\tau\right)+\sqrt{2}v\left(2\kappa-\tau\right)\right)B \end{array}\right)}{\sqrt{6(\kappa-\tau)^2 + v6\sqrt{6}(\kappa-\tau)^2 + 12v^2\left(\kappa^2+\tau^2-\kappa\tau\right)}}.$$

The coefficients of first and second fundamental form are calculated by using the expressions (5) and (6) as:

$$E_{\mathrm{F}} = \frac{1}{6}\left(3(\kappa-\tau)^2 + v2\sqrt{6}(\kappa-\tau)^2 + 4v^2\left(\kappa^2+\tau^2-\kappa\tau\right)\right),$$

$$F_{\mathrm{F}} = \frac{1}{\sqrt{6}}\left(\kappa-\tau\right), \qquad G_{\mathrm{F}} = 1$$

and

$$e_{\mathrm{F}}=\frac{\begin{pmatrix}-6\kappa(\kappa-\tau)^2+2\sqrt{6}v\left(\kappa\left(\kappa'-\tau'\right)-\tau\left(\left(\kappa-\tau\right)^2\right)\right)\\ +2v^2\left(-\kappa\left(2\kappa^2+\tau^2\right)-\tau\left(\kappa'+2\tau'\right)-3\kappa\tau'\right)\end{pmatrix}}{\sqrt{6}\sqrt{6(\kappa-\tau)^2+v6\sqrt{6}(\kappa-\tau)^2+12v^2\left(\kappa^2+\tau^2-\kappa\tau\right)}},$$

$$f_{\mathrm{F}}=\frac{\left(\kappa^2-\ \tau^2\right)}{\sqrt{6}\sqrt{6(\kappa-\tau)^2+v6\sqrt{6}(\kappa-\tau)^2+12v^2\left(\kappa^2+\tau^2-\kappa\tau\right)}},\qquad g_{\mathrm{F}}=0,$$

respectively. Finally, from the expression (4), we have the Gaussian and mean curvatures as in the following:

$$K_{\mathrm{F}}=-\frac{\left(\kappa^2-\ \tau^2\right)^2}{12\Big((\kappa-\tau)^2+v\sqrt{6}(\kappa-\tau)^2+2v^2\left(\kappa^2+\tau^2-\kappa\tau\right)\Big)^2},$$

$$H_{\mathrm{F}}=\frac{\begin{pmatrix}3\left(\kappa^2-\tau^2\right)(\kappa-\tau)+v2\sqrt{6}\kappa\left(\kappa'-\tau'-(\tau-\kappa)^2\right)\\ +2v^2\left(-4\kappa^2\left(\kappa-\tau\right)-\left(3\kappa+2\tau\right)\left(\tau+\tau'\right)-\tau\kappa'\right)\end{pmatrix}}{2\Big((\kappa-\tau)^2+v\sqrt{6}(\kappa-\tau)^2+2v^2\left(\kappa^2+\tau^2-\kappa\tau\right)\Big)^{\frac{3}{2}}}.$$

Corollary 3.14. *If $\kappa=\tau$, the ruled surface $\mathrm{F}(s,v)$ is developable.*

Definition 3.15. *The ruled surface generated by continuously moving the vector TN along the γ_3– Smarandache curve is defined as follows:*

$$\mathrm{R}(s,v)=\frac{1}{\sqrt{2}}(N+B)+\frac{v}{\sqrt{2}}(T+N).$$

The first and second partial differentials of $\mathrm{R}(s,v)$ are

$$\mathrm{R}_s=\frac{1}{\sqrt{2}}\left(-\left(\kappa+v\kappa\right)T+\left(-\tau+v\kappa\right)N+\left(\tau+v\tau\right)B\right),\qquad \mathrm{R}_v=\frac{1}{\sqrt{2}}(T+N),$$

$$\mathrm{R}_{ss}=\frac{1}{\sqrt{2}}\begin{pmatrix}\left[-\kappa'+\kappa\tau-v\left(\kappa'+\kappa^2\right)\right]T-\left[\left(\kappa^2+\tau^2+\tau'\right)+v\left(\kappa^2+\tau^2-\kappa'\right)\right]N\\ +\left[\tau'-\tau^2+v\left(\tau'+\tau\kappa\right)\right]B\end{pmatrix},$$

$$\mathrm{R}_{sv}=\frac{1}{\sqrt{2}}\left(-\kappa T+\kappa N+\tau B\right),\qquad \mathrm{R}_{vv}=0.$$

And the vectorial product of the vectors R_s , R_v and its norm are

$$\mathrm{R}_s \wedge \mathrm{R}_v = \frac{1}{2}\left(-\left(\tau + v\tau\right)T + \left(\tau + v\tau\right)N + \left(\kappa - \tau + 2v\kappa\right)B\right),$$

$$\|\mathrm{R}_s \wedge \mathrm{R}_v\| = \frac{1}{2}\sqrt{2\tau^2(1+v)^2 + (\kappa - \tau + 2v\kappa)^2}.$$

From the expression (3), the normal of this surface shown by N_{R} is given

$$N_{\mathrm{R}} = \frac{-\left(\tau + v\tau\right)T + \left(\tau + v\tau\right)N + \left(\kappa - \tau + 2v\kappa\right)B}{\sqrt{2\tau^2(1+v)^2 + (\kappa - \tau + 2v\kappa)^2}}.$$

Next, from the expressions (5) and (6), the coefficients of the first and the second fundamental forms can be calculated as shown below:

$$E_{\mathrm{R}} = \frac{1}{2}\left(\kappa^2 + 2\tau^2 + 2v\left(\kappa^2 - \kappa\tau + \tau^2\right) + v^2\left(2\kappa^2 + \tau^2\right)\right),$$

$$F_{\mathrm{R}} = -\frac{1}{2}\left(\kappa + \tau\right), \qquad G_{\mathrm{R}} = 1$$

and

$$e_{\mathrm{R}} = \frac{\begin{pmatrix} -\kappa\tau\left(2\tau + \kappa\right) - \tau(2\tau + \kappa)' + \kappa\tau' + v\left(-2\tau^3 - \kappa\tau^2 - 2\tau\tau' + 2\kappa\tau'\right) \\ +v^2\left(-\tau^3 + 2\kappa^2\tau + 2\kappa'\tau + 2\kappa\tau'\right) \end{pmatrix}}{\sqrt{2}\sqrt{2\tau^2(1+v)^2 + (\kappa - \tau + 2v\kappa)^2}},$$

$$f_{\mathrm{R}} = \frac{3\kappa\tau - \tau^2 + 2v\left(\kappa^2 + \kappa\tau\right)}{\sqrt{2}\sqrt{2\tau^2(1+v)^2 + (\kappa - \tau + 2v\kappa)^2}}, \qquad g_{\mathrm{R}} = 0,$$

respectively. Finally, from the expression (4), we have the Gaussian and mean curvatures as

$$K_{\mathrm{R}} = -2\left(\frac{3\kappa\tau - \tau^2 + 2v\left(\kappa^2 + \kappa\tau\right)}{2\tau^2(1+v)^2 + (\kappa - \tau + 2v\kappa)^2}\right)^2,$$

$$H_{\mathrm{R}} = \frac{\begin{pmatrix} \tau\left(2\kappa^2 - \tau^2 - (2\tau + \kappa)'\right) + \kappa\tau' + v\left(2\kappa^3 - 2\tau^3 - 3\kappa\tau^2 + \left(\kappa^2 - \tau^2\right)'\right) \\ +v^2\left(2\tau\left(\kappa^2 - \tau^2\right) + 2(\kappa\tau)'\right) \end{pmatrix}}{2^{-1}\sqrt{2}\left(2\tau^2(1+v)^2 + (\kappa - \tau + 2v\kappa)^2\right)^{\frac{3}{2}}}.$$

Definition 3.16. *The ruled surface generated by continuously moving the vector TB along the γ_3– Smarandache curve is defined as follows:*

$$\Omega(s,v) = \frac{1}{\sqrt{2}}(N+B) + \frac{v}{\sqrt{2}}(T+B).$$

The first and second partial differentials of $\Omega(s,v)$ are

$$\Omega_s = \frac{1}{\sqrt{2}}(-\kappa T - \tau N + \tau B) + \frac{v}{\sqrt{2}}(\kappa - \tau)N, \qquad \Omega_v = \frac{1}{\sqrt{2}}(T+B),$$

$$\Omega_{ss} = \frac{1}{\sqrt{2}}\left(\begin{array}{l} \left[-\kappa' - \kappa\tau - v\left(\kappa^2 - \tau\kappa\right)\right]T - \left[\left(\kappa^2 + \tau^2 + \tau'\right) - v\left(\kappa' - \tau'\right)\right]N \\ + \left[\tau' - \tau^2 + v\left(\kappa\tau - \tau^2\right)\right]B \end{array}\right),$$

$$\Omega_{sv} = \frac{1}{\sqrt{2}}(\kappa - \tau)N, \qquad \Omega_{vv} = 0.$$

And the vectorial product of the vectors Ω_s , Ω_v and its norm are

$$\Omega_s \wedge \Omega_v = \frac{1}{2}(-\tau T + (\kappa+\tau)N + \tau B),$$

$$\|\Omega_s \wedge \Omega_v\| = \frac{1}{2}\sqrt{2\tau^2 + (\kappa+\tau)^2}.$$

From the expression (3), the normal of this surface shown by N_Ω is given

$$N_\Omega = \frac{-\tau T + (\kappa+\tau)N + \tau B}{\sqrt{2\tau^2 + (\kappa+\tau)^2}}.$$

Next, from the expressions (5) and (6), the coefficients of the first and the second fundamental forms can be calculated as shown below:

$$E_\Omega = \frac{1}{2}\left(\kappa^2 + 2\tau^2 + v^2(\kappa-\tau)^2\right), \qquad F_\Omega = \frac{1}{2}(\tau - \kappa), \qquad G_\Omega = 1$$

and

$$e_\Omega = \frac{\tau\left(\tau' + \kappa' + \tau\kappa - \tau^2 + v\left(\kappa^2 - \tau^2\right)\right) - (\kappa+\tau)\left(\kappa^2 + \tau^2 + \tau' - v\left(\kappa' - \tau'\right)\right)}{\sqrt{2}\sqrt{2\tau^2 + (\kappa+\tau)^2}},$$

$$f_\Omega = \frac{\kappa^2 - \tau^2}{\sqrt{2}\sqrt{2\tau^2 + (\kappa+\tau)^2}}, \qquad g_\Omega = 0,$$

respectively. Finally, from the expression (4), we have the Gaussian and mean curvatures as

$$K_\Omega = -2\left(\frac{\kappa^2-\tau^2}{2\tau^2+(\kappa+\tau)^2}\right)^2, \qquad H_\Omega = \frac{e_\Omega - 2f_\Omega F_\Omega}{2\left(E_\Omega - F_\Omega^2\right)}.$$

Corollary 3.17. *If $\kappa=\tau$, the ruled surface $\Omega(s,v)$ is developable.*

Definition 3.18. *The ruled surface generated by continuously moving the vector NB along the γ_3– Smarandache curve is defined as follows:*

$$\Gamma(s,v) = \frac{1}{\sqrt{2}}(N+B) + \frac{v}{\sqrt{2}}(N+B).$$

The first and second partial differentials of $\Gamma(s,v)$ are

$$\Gamma_s = \frac{1+v}{\sqrt{2}}(-\kappa T - \tau N + \tau B), \qquad \Gamma_v = \frac{1}{\sqrt{2}}(N+B),$$

$$\Gamma_{ss} = \left(\frac{1+v}{2}\right)\left((-\kappa' + \tau\kappa)T - \left(\kappa^2+\tau^2+\tau'\right)N + \left(\tau'-\tau^2\right)B\right),$$

$$\Gamma_{sv} = \frac{1}{\sqrt{2}}(-\kappa T - \tau N + \tau B), \qquad \Gamma_{vv} = 0.$$

And the vectorial product of the vectors Γ_s , Γ_v and its norm are

$$\Gamma_s \wedge \Gamma_v = \frac{1}{2}(1+v)(-2\tau T + \kappa N - \kappa B),$$

$$\|\Gamma_s \wedge \Gamma_v\| = \frac{1}{2}(1+v)\sqrt{4\tau^2+2\kappa^2}.$$

From the expression (3), the normal of this surface shown by N_Γ is given

$$N_\Gamma = \frac{-2\tau T + \kappa N - \kappa B}{\sqrt{4\tau^2+2\kappa^2}}.$$

Next, from the expressions (5) and (6), the coefficients of the first and the second fundamental forms can be calculated as shown below:

$$E_\Gamma = \frac{1}{2}(1+v)^2\sqrt{\kappa^2-2\tau^2}, \qquad F_\Gamma = 0, \qquad G_\Gamma = 1$$

and

$$e_\Gamma = \frac{(1+v)\left(\tau^3-\kappa^3-3\kappa\tau^2+\kappa'\tau-\tau'\tau\right)}{2\sqrt{2\tau^2+\kappa^2}}, \qquad f_\Gamma = \frac{\kappa\tau-\tau^2}{2\sqrt{2\tau^2+\kappa^2}}, \qquad g_\Gamma = 0,$$

respectively. Finally, from the expression (4), we have the Gaussian and mean curvatures as

$$K_\Gamma = -\frac{\left(\kappa\tau - \tau^2\right)^2}{2(1+v)^2\left(\kappa^2 + 2\tau^2\right)\sqrt{\kappa^2 - 2\tau^2}},$$

$$H_\Gamma = \frac{\tau^3 - \kappa^3 - 3\kappa\tau^2 + \kappa'\tau - \tau'\tau}{2\left(1+v\right)\sqrt{\kappa^4 - 4\tau^4}}, \qquad \kappa^2 - 2\tau^2 \neq 0.$$

Corollary 3.19. *If $\tau = 0$, the ruled surface $\Gamma\left(s,v\right)$ is developable.*

Definition 3.20. *The ruled surface generated by continuously moving the vector TNB along the γ_3– Smarandache curve is defined as follows:*

$$\Delta\left(s,v\right) = \frac{1}{\sqrt{2}}\left(N + B\right) + \frac{v}{\sqrt{3}}\left(T + N + B\right).$$

The first and second partial differentials of $\Delta\left(s,v\right)$ are

$$\Delta_s = \frac{1}{\sqrt{2}}\left(-\kappa T - \tau N + \ \tau B\right) + \frac{v}{\sqrt{3}}\left(-\kappa T + \left(\kappa - \tau\right)N + \tau B\right),$$

$$\Delta_v = \frac{1}{\sqrt{3}}\left(T + N + B\right),$$

$$\Delta_{ss} = \begin{pmatrix} -\left[\frac{1}{\sqrt{2}}\left(\kappa' + \tau\kappa\right) + \frac{v}{\sqrt{3}}\left(\kappa' + \kappa^2 - \kappa\tau\right)\right]T \\ -\left[\frac{1}{\sqrt{2}}\left(\kappa^2 + \tau^2 + \tau'\right) - \frac{v}{\sqrt{3}}\left(\kappa^2 + \tau^2 - \kappa' + \tau'\right)\right]N \\ +\left[\frac{1}{\sqrt{2}}\left(\tau' - \tau^2\right) + \frac{v}{\sqrt{3}}\left(\tau' - \tau^2 + \tau\kappa\right)\right]B \end{pmatrix},$$

$$\Delta_{sv} = \frac{1}{\sqrt{3}}\left(-\kappa T + \left(\kappa - \tau\right)N + \ \tau B\right), \qquad \Delta_{vv} = 0.$$

And the vectorial product of the vectors Δ_s , Δ_v and its norm are

$$\Delta_s \wedge \Delta_v = \frac{1}{6}\left(-2\tau T + 2\kappa N + \left(\tau - \kappa\right)B\right) + \frac{v}{3}\left(\left(\kappa - 2\tau\right)T + \left(\kappa + \tau\right)N + \left(\tau - 2\kappa\right)B\right),$$

$$\left\|\Delta_s \wedge \Delta_v\right\| = \frac{1}{6}\sqrt{5\kappa^2 + 5\tau^2 - 2\kappa\tau + 24v^2\left(\kappa^2 + \tau^2 - \kappa\tau\right)}.$$

From the expression (3), the normal of this surface shown by N_Δ is given

$$N_\Delta = \frac{2\left(-\tau + v\left(\kappa - 2\tau\right)\right)T + 2\left(\kappa + v\left(\kappa + \tau\right)\right)N + \left(\tau - \kappa\right)\left(1 + 2v\right)B}{\sqrt{5\kappa^2 + 5\tau^2 - 2\kappa\tau + 24v^2\left(\kappa^2 + \tau^2 - \kappa\tau\right)}}.$$

Next, from the expressions (5) and (6), the coefficients of the first and the second fundamental forms can be calculated as shown below:

$$E_{\Delta} = \frac{1}{2}\left(\kappa^2+\tau^2\right) + \frac{2v^2}{3}\left(\kappa^2+\tau^2-\kappa\tau\right), \qquad F_{\Delta} = -\frac{\kappa}{\sqrt{6}}, \qquad G_{\Delta} = 1$$

and

$$e_{\Delta} = \frac{\left(\begin{array}{l} 2\left(-\tau+v\left(\kappa-2\tau\right)\right)\left[\frac{1}{\sqrt{2}}\left(\kappa'+\tau\kappa\right)+\frac{v}{\sqrt{3}}\left(\kappa'+\kappa^2-\kappa\tau\right)\right] \\ -2\left(\kappa+v\left(\kappa+\tau\right)\right)\left[\frac{1}{\sqrt{2}}\left(\kappa^2+\tau^2+\tau'\right)-\frac{v}{\sqrt{3}}\left(\kappa^2+\tau^2-\kappa'+\tau'\right)\right] \\ +\left(\tau-\kappa\right)\left(1+2v\right)\left[\frac{1}{\sqrt{2}}\left(\tau'-\tau^2\right)+\frac{v}{\sqrt{3}}\left(\tau'-\tau^2+\tau\kappa\right)\right] \end{array}\right)}{\sqrt{5\kappa^2+5\tau^2-2\kappa\tau+24v^2\left(\kappa^2+\tau^2-\kappa\tau\right)}},$$

$$f_{\Delta} = \frac{-2\left(-\tau+v\left(\kappa-2\tau\right)\right)\kappa+2\left(\kappa+v\left(\kappa+\tau\right)\right)\left(\kappa-\tau\right)+\left(\tau-\kappa\right)\left(1+2v\right)\tau}{\sqrt{3}\sqrt{5\kappa^2+5\tau^2-2\kappa\tau+24v^2\left(\kappa^2+\tau^2-\kappa\tau\right)}},$$

$$g_{\Delta} = 0,$$

respectively. Finally, from the expression (4), we have the Gaussian and mean curvatures as

$$K_{\Delta} = -\frac{f_{\Delta}^2}{E_{\Delta}-{F_{\Delta}}^2}, \qquad H_{\Delta} = \frac{\sqrt{6}e_{\Delta}-2\kappa f_{\Delta}}{2\left(E_{\Delta}-{F_{\Delta}}^2\right)}.$$

Example: Let us consider the famous Viviani's curve whose parametric form is given by $\alpha(s) = \left(\cos^2(s), \cos(s)\sin(s), \sin(s)\right)$. The Frenet vectors $T(s), N(s), B(s)$ are given as follows:

$$T(s) = \left(-\frac{2\cos(s)\sin(s)}{\sqrt{\cos(s)^2+1}}, \frac{2\cos(s)^2-1}{\sqrt{\cos(s)^2+1}}, \frac{\cos(s)}{\sqrt{\cos(s)^2+1}}\right),$$

$$N(s) = \left(\begin{array}{l} -\frac{2\left(\cos(s)^4+2\cos(s)^2-1\right)}{\sqrt{3\cos(s)^2+5}\sqrt{\cos(s)^2+1}}, -\frac{\cos(s)\sin(s)\left(2\cos(s)^2+5\right)}{\sqrt{\cos(s)^2+1}\sqrt{3\cos(s)^2+5}}, \\ -\frac{\sin(s)}{\sqrt{\cos(s)^2+1}\sqrt{3\cos(s)^2+5}} \end{array}\right),$$

$$B(s) = \left(\frac{\left(2\cos(s)^2+1\right)\sin(s)}{\sqrt{3\cos(s)^2+5}}, -\frac{2\cos(s)^3}{\sqrt{3\cos(s)^2+5}}, \frac{2}{\sqrt{3\cos(s)^2+5}} \right).$$

The graphs of ruled surfaces, obtained using these vectors and definitions and given the parametric equations below, are presented in FIGURE 1, 2 and 3, respectively.

$$\Pi(s,v) = \frac{1}{\sqrt{2}}(T+N) + \frac{v}{\sqrt{2}}(T+N),$$

$$\Phi(s,v) = \frac{1}{\sqrt{2}}(T+N) + \frac{v}{\sqrt{2}}(T+B),$$

$$\Sigma(s,v) = \frac{1}{\sqrt{2}}(T+N) + \frac{v}{\sqrt{2}}(N+B),$$

$$\Upsilon(s,v) = \frac{1}{\sqrt{2}}(T+N) + \frac{v}{\sqrt{3}}(T+N+B),$$

$$\Psi(s,v) = \frac{1}{\sqrt{2}}(T+B) + \frac{v}{\sqrt{2}}(T+N),$$

$$\mathrm{P}(s,v) = \frac{1}{\sqrt{2}}(T+B) + \frac{v}{\sqrt{2}}(T+B),$$

$$\mathrm{Z}(s,v) = \frac{1}{\sqrt{2}}(T+B) + \frac{v}{\sqrt{2}}(N+B),$$

$$\mathrm{F}(s,v) = \frac{1}{\sqrt{2}}(T+B) + \frac{v}{\sqrt{3}}(T+N+B),$$

$$\mathrm{R}(s,v) = \frac{1}{\sqrt{2}}(N+B) + \frac{v}{\sqrt{2}}(T+N),$$

$$\Omega(s,v) = \frac{1}{\sqrt{2}}(N+B) + \frac{v}{\sqrt{2}}(T+B),$$

$$\Gamma(s,v) = \frac{1}{\sqrt{2}}(N+B) + \frac{v}{\sqrt{2}}(N+B),$$

$$\Delta(s,v) = \frac{1}{\sqrt{2}}(N+B) + \frac{v}{\sqrt{3}}(T+N+B).$$

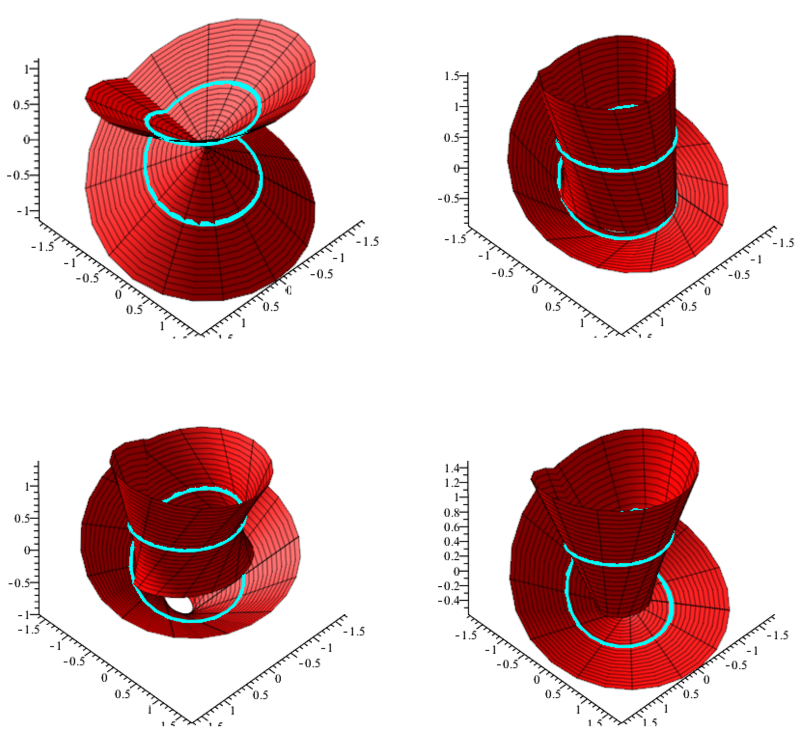
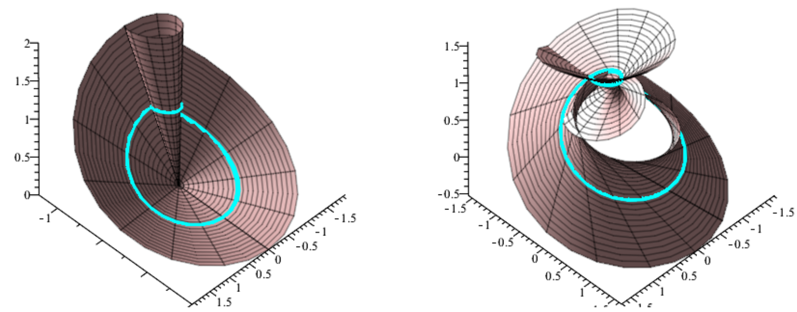


Figure 1. The ruled surfaces whose the base curve γ_1 – Smarandache curve and the direction vector TN, TB, NB, TNB respectively.



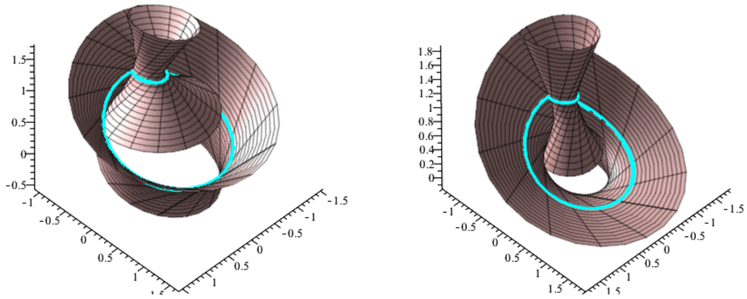


FIGURE 2. The ruled surfaces whose the base curve γ_2 – Smarandache curve and the direction vector TN, TB, NB, TNB, respectively.

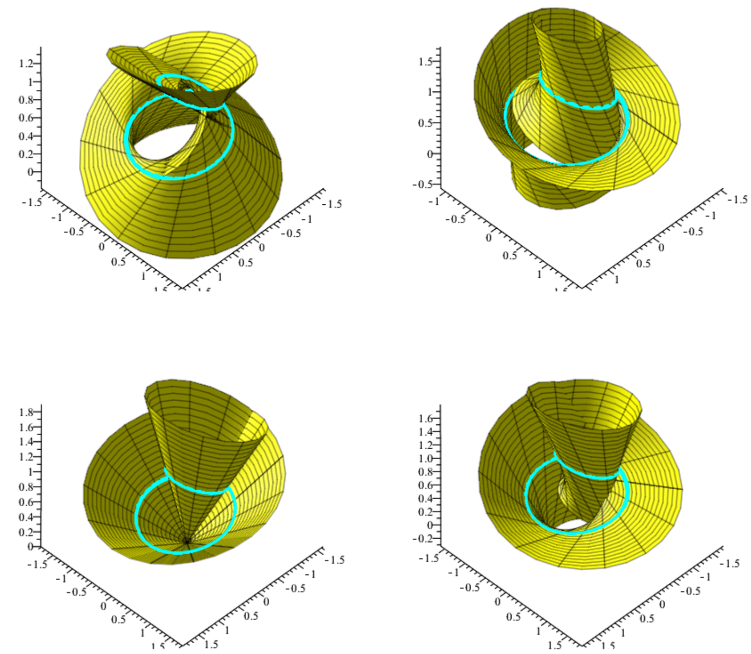


FIGURE 3. The ruled surfaces whose the base curve γ_3 – Smarandache curve and the direction vector TN, TB, NB, TNB, respectively.

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