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DYNAMICS OF GUN VIOLENCE BY LEGAL AND ILLEGAL FIREARMS: A FRACTIONAL DERIVATIVE APPROACH

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Abstract. Crime committed by civilians and criminals using legal and illegal firearms and conversion of legal firearms into illegal ones has become a common practice around the world. As a result, policies to control civilian gun ownership have been debated in several countries. The issue arose because the linkages between firearm-related mortality, weapon accessibility, and violent crime data can imply diverse options for addressing criminality. In this paper, we have projected a mathematical model in terms of the Caputo fractional derivative to address the issues viz. input of legal guns, crime committed by legal and illegal guns, and strict government policies to monitor the license of legal guns, strict action against violent crime. The boundedness, existence and uniqueness of solutions and the stability of points of equilibrium are examined. It is observed that violent crime increases with the increase of crime committed by illegal guns, crime committed by legal guns and, decreases with the increase of legal guns, the deterrent effect of civilian gun ownership, and action of law against crime. Further, legal guns increase with the increase of the limitation of trade of illegal guns and decrease with the increase of conversion of legal guns into illegal guns and increase of the growth rate of illegal guns. Again, as crime is committed by legal guns also, the policy of illegal gun control does not assure a crime-free society. Weak gun control can lead to a society with less crime. Theoretical aspects are numerically verified in the present work.

1. Introduction

Crime is a serious sociological epidemic that has spread over our society at a rapid speed over the past few decades. Easy access to firearms by civilians legally or illegally and criminal illegally is one of the primary reasons for this. Family members, friends, and acquaintances are responsible for 80 percent of firearm killings. Greed, envy, relationship termination, revenge, divergent ideas, bigotry, and drug-related fights are all common motives. [44]. Unintentional firearm deaths, public mass shootings, and firearm suicides are the

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TABLE 1. Countries with the highest rates of Homicides per 100k residents in 2019 [20]

Country	No. of death
El Salvador	36.78
Venezuela	33.27
Guatemala	29.06
Colombia	26.36
Brazil	21.93
Bahamas	21.52
Honduras	20.15
U.S. Virgin Islands	19.40
Puerto Rico	18.14
Mexico	16.41

firearm-related deaths directly proportional to the presence of legal weapons in the home [19, 23, 30, 44]. In the United States, stricter gun-control laws were linked to lower rates of firearm homicide. [24]. In various countries, including Brazil [30, 39] and the United States [30, 38], firearm-related deaths are serious public safety and public health issue. As a result, the link between gun laws and gunshot injury mortality has been hotly debated and explored. The following are some of the arguments in favor of civil society where restrictions on the use and possession of registered firearms are discussed:(i) homicide rates can be reduced by imposing restrictions on access to firearms [19]; (ii) widespread ownership of legally obtained firearms would discourage criminal actions, known as the deterrent effect of defensive gun use [22]; (iii) criminals typically carry illegally obtained firearms and are unconcerned about gun-control laws [18]; (iv) firearm suicides outnumber firearm homicides in many countries [30]; (v) easy access to firearms can contribute to events of public mass shooting [23]. In 2020, Gun Violence Archive registered 24,090 gun suicides. According to the latest data, the United States government's state and federal prisons currently house 2.24 million Americans and it costs 80 billion dollars per year [27]. Table 1 and 2 present data of the countries with the highest rate of homicides and suicides in 2019.

Over the years, mathematical studies are carried out on criminality dynamics in the forms of a variety of mathematical modeling in terms of ordinary and partial differential equations [8,10,17,26,28,35,37,40,42]. There are population dynamics models that describe the role of correctional centers in crime reduction and infectious disease model that assesses crime as a social epidemic [15]. In [32], authors have discussed the effects of gun-control policies on crime rates via game-theoretic models. Prior discovery of illicit behavior and study of the

Table 2. Countries with the highest rates of firearm-related suicide (per 100k) in 2019 [20]

Country	No. of death
Greenland	16.36
United States	7.12
Uruguay	4.74
San Marino	4.08
Montenegro	3.40

spatial version of the inspection game [33] and the role of informants in promoting the development of a crime-free society [43] are also investigated. The spread of residential burglary has been analyzed in [16].

Although gun regulation is a hot topic in America, there has been little research to investigate the causes of gun violence. Until President Obama declared gun violence a public health issue, there was no research to find out preventive measures for gun violence. He also called for more comprehensive research regarding this issue. As a response to this, the first mathematical model to measure how legal gun availability impacts firearm-related homicide rates was proposed by Dominik Wodarz and Natalia Komarova from the University of California [46]. They developed equations to determine if policies ranging from a total firearm prohibition to "arm everyone" increase or decrease killings, based on data dating back to World War I. Wordarz and Komarova show that their model predicts that stronger gun laws are the best method to prevent gun deaths by using their best guess about values predicted or suggested from the existing research literature. In Australia, the National Firearms Agreement (NFA) of 1996 was enacted as strong gun control legislation and facilitated the purchase of over 650,000 firearms. While various studies have looked into how the NFA affects firearm deaths, none have looked into how it affects crime. In 2015, Taylor and Li came up with a difference-in-difference identification approach to answer the question "Do fewer guns lead to less crime". They found that armed robbery and attempted murder decreased significantly one and two years after the NFA was adopted. To answer the question "do more legal guns mean less crime committed by illegal guns?", in [29], the author has proposed a model in terms of differential equations. He has looked into the effects of various gun-control policies on the rate of gun-related crimes. However, the author has not considered the crime committed by the legal weapons in this model.

Fractional derivative is proved to be an effective tool to study the effect of memory on the physical system [9, 34, 47]. As crime is highly related to one's past experiences in family, society, and neighborhood, it is very relevant to analyze the crime model incorporating fractional derivatives. The notion of fractional differentiation has received a lot of attention and consideration in the last 55 years, specifically in 1967 when it was first modified as part of the Caputo investigation [11]. In recent years, we can notice rigorous applications of fractional calculus in various fields such as signal and image processing, mechanics, chemistry, biology, economics, electricity, and control theory to model numerous real-world phenomena by using fractional derivatives (FDs), including, Riemann-Liouville, Caputo, Weyl, Riesz, Grünwald–Letnikov, Marchaud and Hifler, Caputo-Fabrizio, and Atangana-Baleanu operators. The significant literature can be found in [1,4,5,21,36,41]. Due to the singularity of the power-law-based FDs, the commonly used fractional differential operators have some limitations for simulating real-world problems. To deal with this, Caputo and Fabrizio have nurtured a new operator with fractional order. They stressed that many physical occurrences are non-singular and that using singular operators to simulate non-singular events could lead to erroneous results. To tackle the problem, they devised the Caputo-Fabrizio FD, a fractional differential operator using the exponential function as the kernel [12]. In 2016, Atangana and Baleanu proposed the generalized Mittag–Leffler function as a new kernel to incorporate time non-locality into the mathematical formulation of a fractional differential operator with non-singular kernel [3].

In many physical events, the current state of the system is influenced by both current and prior conditions, meaning that past events have an impact on current dynamics. The FDs are significant since the past is seen to be the source of the present. In [9, 47], a system of fractional differential equations is used to examine the effect of memory on epidemic evolution. Since past events have huge impacts on happening of any sort of crime, in this paper, we aim to examine a mathematical model involving violent crimes, legal weapons owned by civilians, and illegal weapons in the frame of the Caputo fractional derivative. The analysis of nonlinear models requires the most efficient technique to study the related behaviors [6, 7]. We have evaluated the influence of various parameters theoretically as well as numerically. The numerical results are computed by using the generalized fractional Adams-Bashforth-Moultan technique compatible with fractional AB derivative [13, 14].

2. Preliminaries

In the present work, we have used the Caputo fractional derivatives (denoted by ${}^{C}D$) because it supports the integer order initial condition. In this section, we have presented certain theorems that have been applied to determine the theoretical results corresponding to the solution of the projected model.

Definition 2.1. *[36] (Caputo Fractional Derivative) Suppose $g(\mathfrak{t})$ is k times continuously differentiable function and $g^{(k)}(\mathfrak{t})$ is integrable in $[\mathfrak{t}_0, T]$.*

The fractional derivative of the order α established by Caputo sense for $g(\mathsf{t})$,is

$$ {}^{C}_{\mathsf{t}_0}D^{\alpha}_{\mathsf{t}} g(\mathsf{t}) = \frac{1}{\Gamma(k-\alpha)} \int_{\mathsf{t}_0}^{\mathsf{t}} \frac{g^{(k)}(\tau)}{(\mathsf{t}-\tau)^{\alpha+1-k}} d\tau $$

where $\Gamma(\cdot)$ refers to Gamma function, $\mathsf{t} > a$ and k is a positive integer with the property that $k-1 < \alpha < k$.

Lemma 2.2. *[45] Consider the system*

$$ {}^{C}_{\mathsf{t}_0}D^{\alpha}_{\mathsf{t}} v(\mathsf{t}) = g(\mathsf{t}, v),\ \mathsf{t} > \mathsf{t}_0, \tag{1} $$

choosing the initial condition as $v(\mathsf{t}_0)$, where $0 < \alpha \leq 1$ and $g : [\mathsf{t}_0, \infty) \times \Omega \to \mathbb{R}^n, \Omega \in \mathbb{R}^n$. When $g(\mathsf{t}, v)$ holds the locally Lipchitz conditions concerning to v, Eq.1 has a unique solution on $[\mathsf{t}_0, \infty) \times \Omega$.

Lemma 2.3. *[25] We assume that $g(\mathsf{t})$ is a continuous function on $[\mathsf{t}_0, +\infty)$ satisfying*

$$ {}^{C}_{\mathsf{t}_0}D^{\alpha}_{\mathsf{t}} g(\mathsf{t}) \leq -\epsilon g(\mathsf{t}) + \xi,\ g(\mathsf{t}_0) = g_0, $$

where $\mathsf{t}_0 \geq 0$ is the initial time, $0 < \alpha \leq 1,\ \epsilon \neq 0,\ \ (\epsilon, \xi) \in \mathbb{R}^2$. Then,

$$ g(\mathsf{t}) \leq (g(\mathsf{t}_0) - \frac{\xi}{\epsilon}) E_{\alpha}[-\epsilon(\mathsf{t} - \mathsf{t}_0)^{\alpha}] + \frac{\xi}{\epsilon} $$

3. Model formulation

Monterio [29], came up with a mathematical model presenting the evolution of violent crimes such as robbery and assault assuming that crimes are committed only by illegal firearms. He has represented the concept that lawfully armed individuals intimidate criminals who are illegally armed. However, this firearm's defensive value relating to gun ownership is debatable, as studies reveal that gun owners are more likely to be homicide victims. In the past few decades, we have observed an increasing number of violent crimes committed by legal weapons too. It includes mass shootings, suicides, drug-related crimes, a crime for personal revenge, and others. If we consider V_c as the prevalence of violent crimes per capita, G_l the number of legal firearms owned by civilians per capita, and G_i the number of illegal firearms per capita, then Monterio's model may be modified as a model where the term $\gamma_1 G_l$ representing the crime committed by legal firearms. Moreover, committing a crime has a long-term relationship with memory. Whenever crimes happen behind that there always be the impact of some past incidents. Therefore, we aim at evaluating the influence of the Caputo fractional derivative on the dynamics of crimes and firearms evolution. With the above modification, the proposed model representing the crimes, legal firearms, and illegal firearms dynamics is presented in the form of

TABLE 3. Brief description of the model parameters.

Parameter	Assumption/fact
δ	deterrent effect of civilian gun ownership
γ_1	Violent crimes committed by legal guns
μ	Action of law against violent crime
γ_2	Violent crimes committed by illegal guns
σ	Constant supply of legal guns
β	Stimulated acquisition of legal guns
θ	Government policy in monitoring and limiting the license of legal guns
ρ	Conversion of legal guns into illegal guns
ϵ	Growth rate of illegal guns
ν	Limiting factor in the trade of illegal guns

the following system of fractional differential equations:

$$\begin{aligned} {}^C D^\alpha V_c &= \gamma_1 G_l + \gamma_2 G_i - \delta G_l G_i - \mu V_c, \\ {}^C D^\alpha G_l &= \sigma + \beta V_c - \rho G_l G_i - \theta G_l, \\ {}^C D^\alpha G_i &= \rho G_l G_i + \epsilon G_i - \nu G_i^2. \end{aligned} \tag{2}$$

The meaning of the symbols used in the model formation is described in the Table 3.

4. Boundedness

In this section we establish that the solutions of the system (2) are bounded.

Theorem 4.1. *The solutions of the system* (2) *are uniformly bounded.*

Proof. Let us define a function, $A(t) = V_c(t) + G_l(t) + G_i(t)$. Taking the Caputo fractional derivative, we get

$$\begin{aligned} {}^C D^\alpha A(t) + \xi A(t) &= {}^C D^\alpha [V_c(t) + G_l(t) + G_i(t)] + \xi\left(V_c(t) + G_l(t) + G_i(t)\right) \\ &= \gamma_1 G_l + \gamma_2 G_i - \delta G_l G_i - \mu V_c + \sigma \\ &\quad + \beta V_c - \rho G_l G_i - \theta G_l + \rho G_l G_i + \epsilon G_i - \nu G_i^2 \\ &\quad + \xi V_c + \xi G_l(t) + \xi G_i(t) \\ &\leq -(\mu - \beta - \xi) V_c - (\theta - \gamma_1 - \xi) G_l + (\gamma_2 + \epsilon + \xi - \nu G_i) G_i + \sigma. \end{aligned}$$

Here, ξ is a positive real number. If $\xi = min\{\mu-\beta, \theta-\gamma_1\}$, then

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}A(t)+\xi A(t) &\leq (\gamma_2+\epsilon+\xi-\nu G_i)G_i+\sigma \\ &= -\nu\left(G_i-\frac{\gamma_2+\epsilon+\xi}{2\nu}\right)^2+\frac{(\gamma_2+\epsilon+\xi)^2}{4\nu}+\sigma \\ &\leq \frac{(\gamma_2+\epsilon+\xi)^2}{4\nu}+\sigma. \end{aligned}$$

By the Lemma 2.3, we get

$$A(t)\leq \frac{1}{\xi}\left(\frac{(\gamma_2+\epsilon+\xi)^2}{4\nu}+\sigma\right)+\left(A(t_0)-\frac{1}{\xi}\left(\frac{(\gamma_2+\epsilon+\xi)^2}{4\nu}+\sigma\right)\right)E_\alpha[-\xi(t-t_0)^\alpha].$$

Clearly, $E_\alpha[-\xi(t-t_0)^\alpha]\to 0$ as $t\to\infty$. Therefore, all the solution of the system (2) that initiates in $\mathbb{R}^3_+$ remained bounded in

$$\Theta=\left\{(V_c,\,G_l,\,G_i)\in\mathbb{R}^3_+:\,V_c(t)+G_l(t)+G_i(t)\leq\frac{1}{\xi}\left(\frac{(\gamma_2+\epsilon+\xi)^2}{4\nu}+\sigma\right)+\epsilon,\,\epsilon>0\right\}.$$

□

5. Existence and Uniqueness

The existence of the solutions of the proposed model, (2) is demonstrated using the fixed-point Theorem in this Section. Since model (2) is complex and non-local, there are no specific algorithms or approaches for evaluating its exact solutions. However, the existence is guaranteed if certain conditions are met. The system (2) can be rewritten as:

$$\begin{aligned} {}^{C}D^{\alpha}V_c &= A_1(t,V_c(t)), \\ {}^{C}D^{\alpha}G_l &= A_2(t,G_l(t)), \\ {}^{C}D^{\alpha}G_i &= A_3(t,G_i(t)), \end{aligned} \tag{3}$$

where

$$\begin{aligned} A_1(t,V_c(t)) &= \gamma_1G_l+\gamma_2G_i-\delta G_lG_i-\mu V_c, \\ A_2(t,G_l(t)) &= \sigma+\beta V_c-\rho G_lG_i-\theta G_l, \\ A_3(t,G_i(t)) &= \rho G_lG_i+\epsilon G_i-\nu G_i^2. \end{aligned}$$

and

$$\begin{aligned} V_c(t)-V_c(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t A_1(\tau,V_c(\tau))(t-\tau)^{\alpha-1}d\tau, \\ G_l(t)-G_l(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t A_2(\tau,V_c(\tau))(t-\tau)^{\alpha-1}d\tau, \\ G_i(t)-G_i(0) &= \frac{1}{\Gamma(\alpha)}\int_0^t A_3(\tau,V_c(\tau))(t-\tau)^{\alpha-1}d\tau. \end{aligned} \tag{4}$$

We shall show that the kernels $A_1,\, A_2,\, A_3$ satisfy Lipschitz condition and contraction.

Theorem 5.1. *In the region* $\Omega \times [0,T]$, *where*

$$\Omega = \{(V_c,\, G_l,\, G_i) \in \mathbb{R}^3 : max\{|V_c|,\, |G_l|,\, |G_i|\} \le m\},$$

and $T < +\infty$, *the kernel* $A_1,\, A_2,\, A_3$ *satisfies Lipschitz condition if the following inequalities hold respectively:*

$$0 < 2m < 1,\; 0 < (\rho m + \theta) < 1,\; 0 < \rho m + \epsilon + 2\nu m < 1.$$

Proof. For V_c and $\bar{V_c}$

$$\begin{aligned} ||A_1(t, V_c) - A_1(t, \bar{V_c})|| &= || - \mu V_c + \mu \bar{V_c}|| \\ &\le \mu(V_c + \bar{V_c})||V_c - \bar{V_c}|| \\ &\le \mu||V_c - \bar{V_c}||. \end{aligned} \tag{5}$$

Similarly, $||A_2(t, G_l) - A_2(t, \bar{G_l})|| \le (m\rho + 2\theta)||G_l - \bar{G_l}||$ and $||A_3(t, G_i) - A_3(t, \bar{G_i})|| \le (\rho m + \epsilon + 2\nu m)||G_i - \bar{G_i}||$ Lipschitz conditions are met for $A_1,\, A_2,\, A_3$ and these follow contraction if

$$0 < \mu < 1,\; 0 < (\rho m + \theta) < 1,\; 0 < \rho m + \epsilon + 2\nu m < 1,$$

respectively. □

Theorem 5.2. *The solution of the fractional model* (2) *exists and will be unique, if we acquire some* t_α *such that*

$$\frac{1}{\Gamma(\alpha)} \xi_i t_\alpha < 1,$$

for $i = 1, 2, 3,$ *where* $\xi_1 = \mu,\; \xi_2 = (\rho m + \theta),\; \xi_3 = \rho m + \epsilon + 2\nu m.$

Proof. The proof of this theorem is categorized in three parts:

1. Using system 4 we can write the recursive form as:

$$\begin{aligned} K_{1,n} &= V_{c_n}(t) - V_{c_{n-1}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \big(A_1(\tau, V_{c_{n-1}}(\tau)) - A_1(\tau, V_{c_{n-2}}(\tau))\big)\, (t-\tau)^{\alpha-1} d\tau, \\ K_{2,n} &= G_{l_n}(t) - G_{l_{n-1}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \big(A_2(\tau, G_{l_{n-1}}(\tau)) - A_2(\tau, G_{l_{n-2}}(\tau))\big)\, (t-\tau)^{\alpha-1} d\tau, \\ K_{3,n} &= G_{i_n}(t) - G_{i_{n-1}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \big(A_3(\tau, G_{i_{n-1}}(\tau)) - A_2(\tau, G_{i_{n-2}}(\tau))\big)\, (t-\tau)^{\alpha-1} d\tau. \end{aligned} \tag{6}$$

Here,

$$V_{c_0} = V_c(0),\; G_{l_0} = G_l(0),\; G_{i_0} = G_i(0).$$

Applying norm on each equation of system (6) and using Lipschitz condition, we obtain respectively

(7)

$$||K_{1,n}|| \leq ||V_{c_n}(t) - V_{c_{n-1}}(t)|| = \frac{1}{\Gamma(\alpha)} \int_0^t || \left(A_1(\tau, V_{c_{n-1}}(\tau)) - A_1(\tau, V_{c_{n-2}}(\tau))\right) (t-\tau)^{\alpha-1} || d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)} \xi_1 \int_0^t ||K_{1,n-1}(\tau)|| d\tau,$$

$$||K_{2,n}|| = ||G_{l_n}(t) - G_{l_{n-1}}(t)|| \leq \frac{1}{\Gamma(\alpha)} \xi_2 \int_0^t ||K_{2,n-1}(\tau)|| d\tau,$$

$$||K_{3,n}|| = ||G_{i_n}(t) - G_{i_{n-1}}(t)|| \leq \frac{1}{\Gamma(\alpha)} \xi_3 \int_0^t ||K_{3,n-1}(\tau)|| d\tau.$$

As a result, we can write

(8)
$$V_{c_n}(t) = \sum_{i=1}^{n} K_{1,i},\ G_{l_n}(t) = \sum_{i=1}^{n} K_{2,i},\ G_{i_n}(t) = \sum_{i=1}^{n} K_{3,i}.$$

Applying Eq. (7) recursively, we have

(9)
$$\begin{aligned} ||K_{1,i}(t)|| &\leq ||V_{c_n}(0)|| \left[\frac{1}{\Gamma(\alpha)} \xi_1 t\right]^n, \\ ||K_{2,i}(t)|| &\leq ||G_{l_n}(0)|| \left[\frac{1}{\Gamma(\alpha)} \xi_2 t\right]^n, \\ ||K_{3,i}(t)|| &\leq ||G_{i_n}(0)|| \left[\frac{1}{\Gamma(\alpha)} \xi_3 t\right]^n. \end{aligned}$$

As a result, the existence and continuity are established.

2. To illustrate that the relation (9) formulate the solution for Eq. (1), we assume the following:

(10)
$$\begin{aligned} V_{c_n}(t) - x(0) &= V_{c_n}(t) - W_{1n}(t), \\ G_{l_n}(t) - y(0) &= G_{l_n}(t) - W_{2n}(t), \\ G_{i_n}(t) - z(0) &= G_{i_n}(t) - W_{3n}(t). \end{aligned}$$

In order to achieve the desired outcomes, we set

$$||W_{1n}(t)|| = ||\frac{1}{\Gamma(\alpha)} \int_0^t (A_1(\tau, x) - A_1(\tau, x_{n-1})) d\tau||,$$

$$||W_{2n}(t)|| = ||\frac{1}{\Gamma(\alpha)} \int_0^t (A_2(\tau, x) - A_2(\tau, x_{n-1})) d\tau||,$$

$$||W_{3n}(t)|| = ||\frac{1}{\Gamma(\alpha)} \int_0^t (A_3(\tau, x) - A_3(\tau, x_{n-1})) d\tau||.$$

This yields

$$||W_{1n}(t)|| \le \frac{1}{\Gamma(\alpha)}\xi_1||V_c - V_{c,n-1}||t,$$
$$||W_{2n}(t)|| \le \frac{1}{\Gamma(\alpha)}\xi_2||G_l - G_{l,n-1}||t,$$
$$||W_{3n}(t)|| \le \frac{1}{\Gamma(\alpha)}\xi_3||G_i - G_{i,n-1}||t.$$

Continuing the same procedure recursively, we get

$$||W_{jn}(t)|| \le \left(\frac{1}{\Gamma(\alpha)}\xi_j t\right)^{n+1} m,\ j = 1, 2, 3.$$

At t_α, we have

$$||W_{jn}(t)|| \le \left(\frac{1}{\Gamma(\alpha)}\xi_j t_\alpha\right)^{n+1} m,\ j = 1, 2, 3. \tag{11}$$

From Eq. (11), it results that as n tends to ∞, $||W_{jn}(t)||$ tends to 0 provided $\frac{1}{\Gamma(\alpha)}\xi_j t_\alpha < 1,\ j = 1, 2, 3.$

3. We will now demonstrate the uniqueness for the solution of the system (2). Suppose that there is a different set of solution of the system (2), namely $\hat{V}_c$, $\hat{G}_l$, $\hat{G}_i$. Then, from the first equation of Eq. (4) we write

$$V_c(t) - \hat{V}_c(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (A_1(\tau, V_c) - A_1(\tau, \hat{V}_c))d\tau.$$

Using the norm, the equation above becomes

$$||V_c(t) - \hat{V}_c(t)|| = \frac{1}{\Gamma(\alpha)}\int_0^t ||(A_1(\tau, V_c) - A_1(\tau, \hat{V}_c))d\tau||. \tag{12}$$

By applying the Lipschitz condition, we get

$$||V_c(t) - \hat{V}_c(t)|| \le \frac{1}{\Gamma(\alpha)}\xi_1 t||V_c - \hat{V}_c||.$$

At some t_α this results yields

$$||V_c(t) - \hat{V}_c(t)||\left(1 - \frac{1}{\Gamma(\alpha)}\xi_1 t_\alpha\right) \le 0.$$

Since $\left(1 - \frac{1}{\Gamma(\alpha)}\xi_1 t_\alpha\right) > 0$, we must have $||V_c(t) - \hat{V}_c(t)|| = 0$. This implies $V_c(t) = \hat{V}_c(t)$.

□

6. Stability Analysis

The Jacobian matrix of the system (2) is

$$J = \begin{pmatrix} -2V\mu & \gamma_1 - G\delta & \gamma_2 - U\delta \\ \beta_1 & -2U\theta - G\rho & -U\rho \\ \beta_2 & G\rho & \epsilon - 2G\nu + U\rho \end{pmatrix}.$$

If all the eigenvalues λ_i, $i = 1, 2, ..., \beta$, of the Jacobian matrix $J(E)$, E being the point of equilibrium, satisfy the condition

$$|arg(eig(J(E)))| = |arg(\lambda_i)| > \frac{\alpha\pi}{2}, i = 1, 2, 3, \quad 0 < \alpha < 1. \tag{13}$$

Then the E is a stable point of equilibrium. By solving the characteristic equation $|J(E) - \lambda_i I| = 0$, we evaluate these eigenvalues.

Lemma 6.1. *[2] Define the following characteristic equation*

$$P(\lambda) = \lambda^\beta + A_1\lambda^{\beta-1} + A_2\lambda^{\beta-2} + ... + A_\beta = 0. \tag{14}$$

The following conditions make all the roots of the characteristic equation (14) *satisfy the Eq.* (13)*:*

1. *For $\beta = 1$, the condition for Eq.* (13) *is $A_1 > 0$.*
2. *For $\beta = 2$, the conditions for Eq.* (13) *are either Routh-Hurwitz conditions or $A_1 > 0$, $4A_2 > A_1^2$, $|tan^{-1}\frac{\sqrt{4A_2 - A_1^2}}{A_1}| > \frac{\alpha\pi}{2}$.*
3. *For $\beta = 3$, if the discriminant of the polynomial $P(\Theta)$ is positive then necessary and sufficient conditions to satisfy the Eq.* (13) *are*
$$A_1 > 0,\ A_2 > 0,\ A_1A_2 > A_3.$$
If the discriminant of the polynomial $P(\lambda)$ is negative then necessary and sufficient conditions to satisfy the Eq. (13) *are*
$$A_1 > 0,\ A_2 > 0,\ A_1A_2 = A_3.$$
4. *For general β, $A_\beta > 0$ is the necessary condition for Eq.* (13) *to be satisfied.*

Theorem 6.2. *Crime free equilibrium point $E_1 = \left(0, \frac{\sigma}{\theta}, 0\right)$ is always unstable.*

Proof. Jacobian matrix at the crime free equilibrium point E_1 is

$$J(E_1) = \begin{pmatrix} -\mu & \gamma_1 & \gamma_2 - \frac{\delta\sigma}{\theta} \\ \beta & -\theta & -\frac{\rho\sigma}{\theta} \\ 0 & 0 & \frac{\rho\sigma}{\theta} + \epsilon \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ is $\lambda_{1,1} = \frac{\theta\epsilon+\rho\sigma}{\theta}$, $\lambda_{1,2} = \frac{-\theta\sqrt{4\beta\gamma_1+\theta^2-2\theta\mu+\mu^2}-\theta^2-\theta\mu}{2\theta}$, $\lambda_{1,3} = \frac{\theta\sqrt{4\beta\gamma_1+\theta^2-2\theta\mu+\mu^2}-\theta^2-\theta\mu}{2\theta}$. Since $\lambda_{1,1} > 0$, therefore the Crime-free equilibrium point is always unstable. This is because crimes are committed by legal guns also. □

Theorem 6.3. *Illegal gun free equilibrium point*

$$E_2 = \left(\frac{\gamma_1 \sigma}{\theta\mu - \beta_1\gamma_1}, \frac{\mu\sigma}{\theta\mu - \beta_1\gamma_1}, 0 \right),$$

exists if $\theta\mu - \beta_1\gamma_1 > 0$. E_2 *is always unstable.*

Proof. Jacobian matrix at the illegal gun free equilibrium point E_2 is

$$J(E_2) = \begin{pmatrix} -\mu & \gamma_1 & \gamma_2 - \frac{\delta\mu\sigma}{\theta\mu-\beta\gamma_1} \\ \beta & -\theta & -\frac{\mu\rho\sigma}{\theta\mu-\beta\gamma_1} \\ 0 & 0 & \frac{\mu\rho\sigma}{\theta\mu-\beta\gamma_1} + \epsilon \end{pmatrix}.$$

The eigenvalues of $J(E_2)$ are $\lambda_{2,1} = \frac{\epsilon(\theta\mu-\beta\gamma_1)+\mu\rho\sigma}{\theta\mu-\beta\gamma_1}$, $\lambda_{2,2} = \frac{1}{2}\left(-\sqrt{4\beta\gamma_1 + (\theta-\mu)^2} - \theta - \mu\right)$, $\lambda_{2,3} = \frac{1}{2}\left(\sqrt{4\beta\gamma_1 + (\theta-\mu)^2} - \theta - \mu\right)$. Since $\lambda_{2,1} > 0$, the illegal gun free equilibrium point is unstable. □

This implies that the society has to sustain with the all the three categories. In this situation, it is very important to find the measures to keep the crime in control.

Theorem 6.4. *Co-existence equilibrium point is stable if* γ_1, γ_2, *and* ϵ *are limited.*

Proof. To find co-existence equilibrium point we solve the system of equations

$$\gamma_1 G_l + \gamma_2 G_i - \delta G_l G_i - \mu V_c = 0, \tag{15}$$

$$\sigma + \beta V_c - \rho G_l G_i - \theta G_l = 0, \tag{16}$$

$$\rho G_l G_i + \epsilon G_i - \nu G_i^2 = 0. \tag{17}$$

□

From Equations (15), (16) and (17), we get

$$V_c^* = \frac{G_i^*(\gamma_2 - \delta G_l^*) + \gamma_1 G_l^*}{\mu}, \tag{18}$$

$$G_l^* = \frac{\sqrt{P_1} + Q_1}{2\rho(\beta\delta + \mu\rho)} \tag{19}$$

$$= \frac{G_i\nu - \epsilon}{\rho}, \tag{20}$$

$$G_i^* = \frac{\sqrt{P_2} + Q_2}{2\nu(\beta\delta + \mu\rho)}.$$

where, $P_1 = (-\beta\gamma_1\nu - \beta\gamma_2\rho + \beta\delta\epsilon + \theta\mu\nu + \epsilon\mu\rho)^2 + 4\rho(\beta\delta + \mu\rho)(\beta\gamma_2\epsilon + \mu\nu\sigma)$,
$Q_1 = \beta\gamma_1\nu + \beta\gamma_2\rho - \beta\delta\epsilon - \theta\mu\nu - \epsilon\mu\rho$,
$P_2 = (-\beta\gamma_1\nu - \beta\gamma_2\rho + \beta\delta\epsilon + \theta\mu\nu + \epsilon\mu\rho)^2 + 4\rho(\beta\delta + \mu\rho)(\beta\gamma_2\epsilon + \mu\nu\sigma)$,
$Q_2 = \beta\gamma_1\nu + \beta\gamma_2\rho + \beta\delta\epsilon - \theta\mu\nu + \epsilon\mu\rho$.

Clearly, G_l^* and G_i^* are positive. Co-existence equilibrium point exists if $G_i^*(\gamma_2 - \delta G_l^*) + \gamma_1 G_l^* > 0$. That is, for the existence of a coexistence point of equilibrium, crime committed at an equilibrium point must be greater than the deterrent effect of civilian gun ownership. We observe that violent crime (V_c) increases with the increase of crime committed by illegal guns ($\gamma_2 G_i$), crime committed by legal guns ($\gamma_1 G_l$), and decrease with the increase of legal guns (G_l), the deterrent effect of civilian gun ownership (δ), and action of law against crime (μ). Further, legal guns increase with the increase of the limitation of trade of illegal guns and decrease with the increase of conversion of legal guns into illegal guns and increase of the growth rate of illegal guns.
The characteristic equation at the coexistence equilibrium point is

$$Q(\lambda, C_1, C_2, C_3) = \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0, \tag{21}$$

where
$C_1 = G_i\nu + G_i\rho + \theta + \mu > 0$,
$C_2 = 2G_i^2\nu\rho + \beta\delta G_i + G_i\theta\nu + G_i\mu\nu + G_i\mu\rho + \theta\mu - \beta\gamma_1 - G_i\epsilon\rho > 0$,
$C_3 = 2\beta\delta G_i^2\nu + 2G_i^2\mu\nu\rho + G_i\theta\mu\nu - G_i\epsilon\mu - \beta\gamma_1 G_i\nu - \beta\gamma_2 G_i\rho - \beta\delta G_i\epsilon$.
The positivity of C_2 depends on limiting the impact of crime committed by legal guns, and the growth of illegal guns. Therefore, the necessary condition for stability of the coexistence equilibrium point is to limit these factors. Again, sufficient condition for stability of coexistence equilibrium point is $C_1C_2 > C_3$. These conditions are sufficient for affirmation of condition (13).
To describe this set, we employ the boundary locus technique which, as a first step, requires a description of its boundary in the (C_1, C_2, C_3)-space. For this purpose, we define the boundary locus in the form We set $BL(\alpha) = (C_1, C_2, C_3) \in \mathbb{R}^3 : \exists\, \lambda \in \mathbb{C}$ with $|arg(\lambda)| = \frac{\alpha\pi}{2}$ and $Q(\lambda; C_1, C_2, C_3) = 0$, where $0 < \alpha < 1$. In other words, $BL(\alpha)$ consists of all real triplets (C_1, C_2, C_3) such that Eq. (21) admits a zero

$$\lambda = \kappa e^{\frac{i\alpha\pi}{2}}, \text{ for suitable real } \kappa \geq 0. \tag{22}$$

Substituting (22) into (21) and separating real and imaginary parts, one gets

$$\kappa^3 cos(\frac{3\alpha\pi}{2}) + C_1\kappa^2 cos(\alpha\pi) + \kappa C_2 cos(\frac{\alpha\pi}{2}) + C_3 = 0, \tag{23}$$

and

$$\kappa^3 sin(\frac{3\alpha\pi}{2}) + C_1\kappa^2 sin(\alpha\pi) + \kappa C_2 sin(\frac{\alpha\pi}{2}) = 0. \tag{24}$$

This system has the solution $\kappa = C_3 = 0$ (C_1, C_2 being arbitrary) and the solution

$$C_2 = -2\kappa C_1 cos\frac{\alpha\pi}{2} - 4\kappa^2 cos^2(\frac{\alpha\pi}{2}) + \kappa^2, \tag{25}$$

$$C_3 = 2\kappa^3 cos(\frac{\alpha\pi}{2}) + C_1\kappa^2. \tag{26}$$

Therefore from Eq. 25 we obtain
(27)

$$\kappa_{\pm}(C_1, C_2; \alpha) = \frac{-C_1 cos\left(\frac{\alpha\pi}{2}\right) \pm \sqrt{C_1^2 cos^2(\frac{\alpha\pi}{2}) - C_2\left(4cos^2\left(\frac{\alpha\pi}{2}\right) - 1\right)}}{4cos^2(\frac{\alpha\pi}{2}) - 1}, \alpha \neq \frac{2}{3}$$

It holds $\kappa_{\pm}(C_1, C_2; \alpha) > 0$ if and only if either

$$0 < \alpha < \frac{2}{3},\ C_2 \leq \frac{C_1^2 cos^2(\frac{\alpha\pi}{2})}{4cos^2(\frac{\alpha\pi}{2}) - 1}, \tag{28}$$

or

$$\frac{2}{3} < \alpha \leq 1,\ C_1 \geq 0,\ C_2 < 0, \tag{29}$$

or

$$\frac{2}{3} < \alpha \leq 1,\ C_2 \geq \frac{C_1^2 cos^2(\frac{\alpha\pi}{2})}{4cos^2(\frac{\alpha\pi}{2}) - 1}. \tag{30}$$

With these assumption, by replacing κ into (26) we get,

$$\begin{aligned} C_3 \quad = \quad & (-2C_1^2 cos(\frac{\alpha\pi}{2}) + 8C_2 cos^3(\frac{\alpha\pi}{2}) \\ & -2C_2 cos(\frac{\alpha\pi}{2})\sqrt{C_1^2 cos^2(\frac{\alpha\pi}{2}) - 4C_2 cos(\frac{\alpha\pi}{2}) + C_2} \\ & -2C_1^3 cos^2(\frac{\alpha\pi}{2}) + 8C_1C_2 cos^4(\frac{\alpha\pi}{2}) + 2C_1C_2 cos^2(\frac{\alpha\pi}{2}) - C_1C_2)/\left(4cos^2(\frac{\alpha\pi}{2}) - 1\right)^3. \end{aligned} \tag{31}$$

Thus we conclude that if $\frac{2}{3} < \alpha < 1$, then $(C_1, C_2, C_3) \in BL(\alpha)$ if and only if

$$C_2 \geq \frac{C_1^2 cos^2(\frac{\alpha\pi}{2})}{4cos^2(\frac{\alpha\pi}{2}) - 1},$$

and C_3 is satisfied in Eq. (31). If $\alpha = \frac{2}{3}$, then from Eq. (25) and (26), we have $C_2 = -C_1\kappa,\ C_3 = \kappa^3 + C_1\kappa^2$. By setting $S = \cos\left(\frac{\pi\alpha}{2}\right)$ from Eq. (31), we get the inequality

$$\begin{aligned} & C_3\left(4S^2 - 1\right)^3 + 2C_1^3 S^2 + C_1C_2\left(-8S^4 - 2S^2 + 1\right) \\ & > 2S\left(2C_2\left(4S^2 - 1\right) - C_1^2\right)\sqrt{C_1^2 S^2 + C_2\left(1 - 4S^2\right)}. \end{aligned} \tag{32}$$

The relation (32) gives the facility to analyze the stability in presence of fractional derivative.

7. Numerical Methods

To solve the crime model (2), we have presented the generalized Adams-Bashforth-Moulton technique in this section. Consider the nonlinear equation below.

$$
\begin{aligned}
&{}^{C}D_t^{\alpha}x(t) = \phi(t, x(t)), \quad 0 \le t \le T,\\
&x^{(m)}(0) = x_0^{(m)}, \quad m = 0, 1, 2, 3, \cdot, \nu,\ \nu = \lceil \alpha \rceil.
\end{aligned} \tag{33}
$$

The Volterra integral equation corresponding to this case is as follows:

$$
x(t) = \sum_{m=0}^{\nu-1} x_0^{(m)} \frac{t^m}{m!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s, x(s)) ds. \tag{34}
$$

To integrate Eq. (34), Diethelm et al. [13,14] have used Adams-Bashforth Moultan method by setting $h = \frac{T}{N}$, $t_n = nh$, $n = 0, 1, 2, \ldots, N \in Z^+$. As a result, finite difference form of the system (2) can be presented as:

$$
\begin{aligned}
V_{c_{n+1}} &= V_{c_0} + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\gamma_1 G^P_{l_{n+1}} + \gamma_2 G^P_{i_{n+1}} - \delta G^P_{l_{n+1}} G^P_i - \mu V^P_{c_{n+1}} \right)\\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{k=0}^{n} a_{k,n+1} \left(\gamma_1 G_{l_k} + \gamma_2 G_{i_k} - \delta G_{l_k} G_i - \mu V_{c_k} \right),\\
G_{l_{n+1}} &= G_{l_0} + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\sigma + \beta V^P_{c_{n+1}} - \rho G^P_{l_{n+1}} G^P_{i_{n+1}} - \theta G^P_{l_{n+1}} \right)\\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{k=0}^{n} a_{k,n+1} \left(\sigma + \beta V_{c_k} - \rho G_{l_k} G_{i_k} - \theta G_{l_k} \right),\\
G_{i_{n+1}} &= G_{i_0} + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\rho G^P_{l_{n+1}} G^P_{i_{n+1}} + \epsilon G^P_{i_{n+1}} - \nu G^{P^2}_{i_{n+1}} \right)\\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{k=0}^{n} a_{k,n+1} \left(\rho G_{l_k} G_{i_k} + \epsilon G_{i_k} - \nu G^2_{i_k} \right),
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
V^P_{c_{n+1}} &= V_{c_0} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{n} b_{k,n+1} \left(\gamma_1 G_{l_k} + \gamma_2 G_{i_k} - \delta G_{l_k} G_i - \mu V_{c_k} \right),\\
G^P_{l_{n+1}} &= G_{l_0} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{n} b_{k,n+1} \left(\sigma + \beta V_{c_k} - \rho G_{l_k} G_{i_k} - \theta G_{l_k} \right),\\
G^P_{i_{n+1}} &= G_{i_0} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{n} b_{k,n+1} \left(\rho G_{l_k} G_{i_k} + \epsilon G_{i_k} - \nu G^2_{i_k} \right),
\end{aligned} \tag{36}
$$

in which

(37)

$$
a_{k,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & k = 0,\\ (n-k+2)^{\alpha+1} + (n-k)^{\alpha+1} - 2(n-k+1)^{\alpha+1}, & 1 \le k \le n,\\ 1, & k = n+1, \end{cases}
$$

and

$$b_{k,n+1} = ((n-k+1)^{\alpha} - (n-k)^{\alpha}), \quad 0 \le k \le n. \tag{38}$$

8. Numerical simulation

We have examined the crime dynamics by assuming the suitable values of the parameters in this section. Our main focus is to analyze the behavior of the various compartments concerning gun control parameters β, σ, and θ. With the parameter values $\gamma_1 = 0.1$, $\gamma_2 = 0.5$, $\delta = 0.2$, $\beta = 0.8$, $\epsilon = 0.8$, $\nu = 5$, $\theta = 0.5$, $\rho = 2.5$, $\mu = 2, \sigma = 2$, we have shown the stability profiles of the Crime, legal guns and illegal guns in Figure 1 under the influence of fractional derivative. As α reduces it is observed that illegal gun inertia is reduced and legal gun inertia is stabilized comparatively at a higher value than illegal guns. Increase in the legal gun inertia with fractional values of α results in the conclusion that violent crime inertia is inversely proportional to α We have shown the stability of the coexistence equilibrium point of the model for various values of α. We looked at two situations to see how different gun control legislation would affect this appealing idea: (1) strong gun control and (2) weak gun control under the influence of various fractional derivatives numerically. For strong gun control we have assumed $\beta \to 0$, $\sigma \to 0$, $\theta \to \infty$. This reduces the endemic equilibrium to $(\frac{\epsilon\gamma_2}{\mu\nu}, 0, \frac{\epsilon}{\nu})$. It is noticed that strong gun control could not nullify the crime. For weak gun control we have assumed $\beta \to \infty$, $\theta \to 0$. This leads to the equilibrium point $(0, \frac{\sqrt{(-\gamma_1\nu-\gamma_2\rho+\delta\epsilon)^2+4\gamma_2\delta\rho\epsilon}+\gamma_1\nu+\gamma_2\rho-\delta\epsilon}{2\delta\rho}, \frac{\sqrt{(-\gamma_1\nu-\gamma_2\rho+\delta\epsilon)^2+4\gamma_2\delta\rho\epsilon}+\gamma_1\nu+\gamma_2\rho+\delta\epsilon}{2\delta\nu})$. This indicates that weak gun control measures can eradicate the crime. Also controlling legal guns depends on the increased deterrent effect of civilian gun ownership and the conversion factor. Whereas control of illegal guns depends on the increase of the deterrent effect of civilian gun ownership and limiting the trade of illegal guns. In Figure 2, we have analyzed the impact of a strong gun control measure under the influence of a fractional derivative. It is observed that by varying α, crime and legal guns can have a balancing nullifying effect. From Figure 2(a), 2(c), and 2(e), it can be noticed that strong gun control reduces the crime. Fractional values of α represents the situation when strong gun control influences in rapid fall in crime inertia. Also 2(b), 2(d), and 2(f) represent the effect of strong gun control on legal guns. It is observed that fractional value of α represents the situation where strong gun control delay the nullyfying effect on crime.

Relation (32) facilitates us to analyze the stability with the direct involvement of fractional derivative. For $\alpha = 0.9$, using the relation (32) for parameter values $\gamma_1 = 0.1$, $\gamma_2 = 0.5$, $\delta = 0.2$, $\beta = 0.8$, $\epsilon = 0.8$, $\nu = 5$, $\theta = 0.5$, $\rho = 5.5$, $\mu = 2$, we observe that stability range of σ is $0 < \sigma < 17.642$. We have considered $\sigma = 17.644$. The profile of the solutions of the system (2) for different α

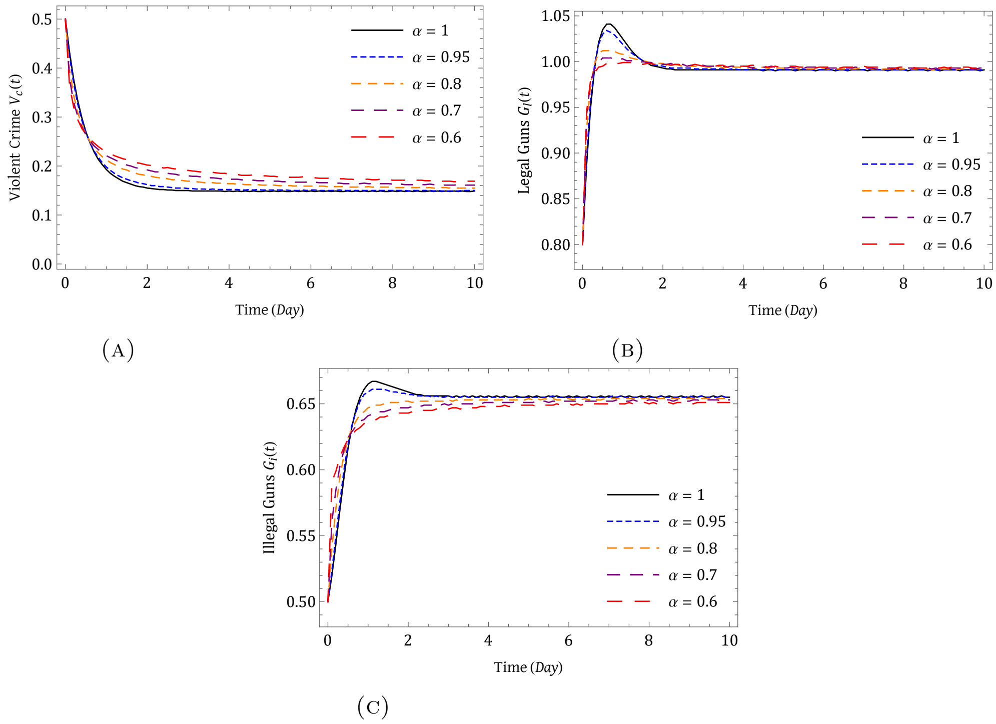

Figure 1. Profile of (A) Violent Crime, (B) Legal Guns, (C) Illegal Guns for different α.

is examined in Figure 3. It is observed that for $\alpha = 1$ the system behaves like a stable system (Figure 3(a)). But as α take fractional values the system turns into an unstable system (Figure 3(b),(c),(d)). This is due to the consideration of the σ value outside the prescribed range.

9. Conclusion

In this work, a crime model about violent crimes committed by registered and unregistered guns is analyzed in the frame of the Caputo fractional derivative. We have analyzed three points of equilibrium namely axial equilibrium, illegal gun-free equilibrium, and endemic equilibrium. The boundedness, existence and uniqueness of solutions are evaluated theoretically. From the endemic equilibrium point, we examined two scenarios such as strong gun-control and weak gun control. We could conclude that strong gun control can eliminate legal guns from society but can not eliminate the crimes. Whereas weak gun control can ensure less crime in society. In the projected model numerical

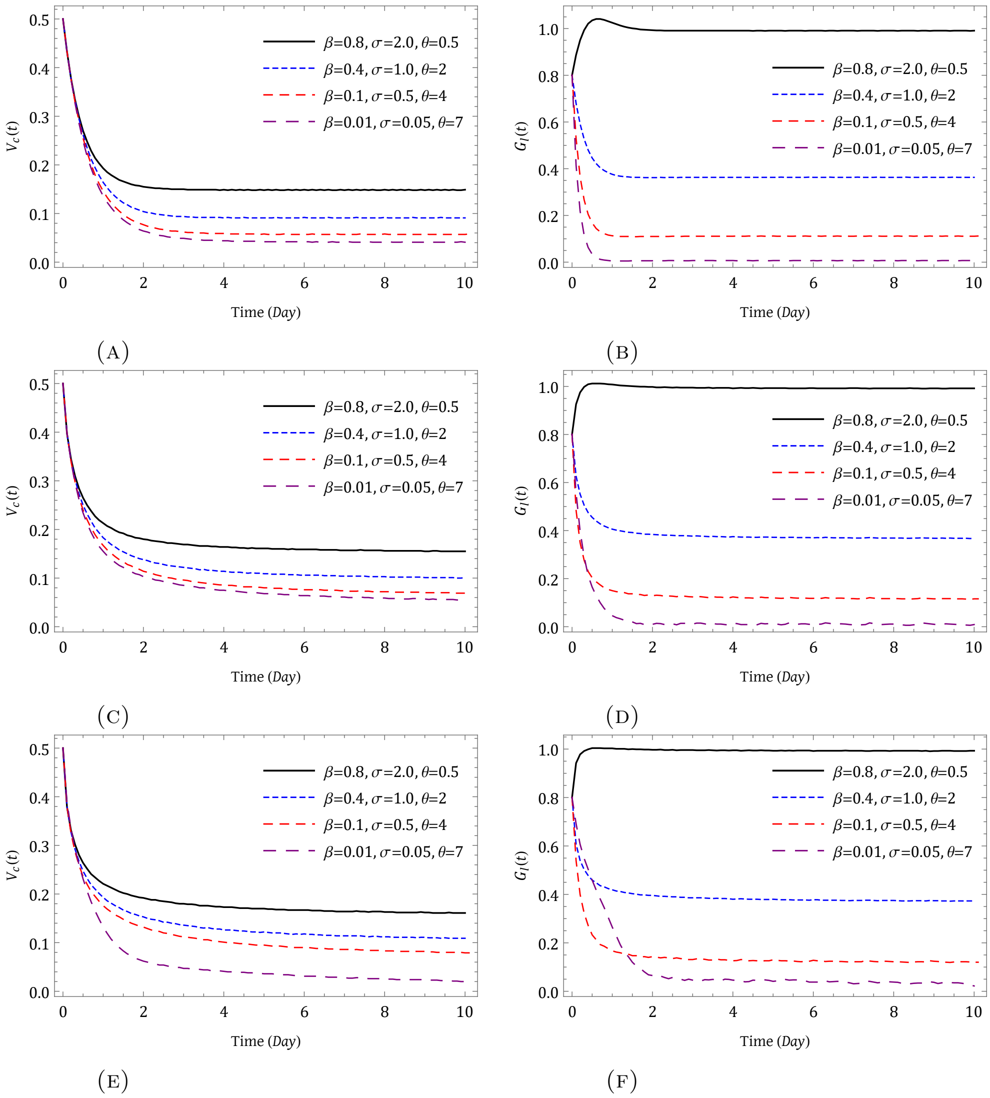

Figure 2. Effect of strong gun-control measure on (a) Violent Crime, (b) Legal Guns, for $\alpha = 1$, (c) Violent Crime,(d) Legal Guns for $\alpha = 0.9$ (e) Violent Crime,(f) Legal Guns for $\alpha = 0.8$

results obtained by incorporating fractional derivative seems more realistic because the application of fractional value ensures coexistence. As the crimes are committed by both legal and illegal firearms, eradication of anyone is practically not possible. Only controlling may be possible. The future scope of the

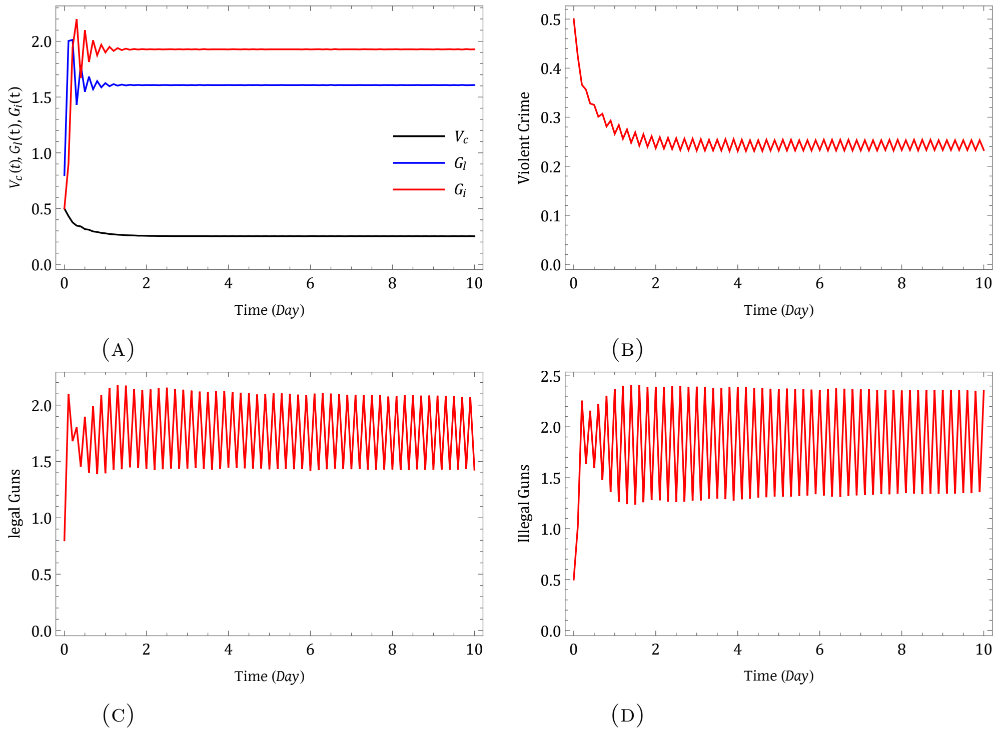

FIGURE 3. Profile of (a) Violent Crime, Legal Guns, Illegal Guns for $\alpha = 1$, (b) Violent Crime,(c) Legal Guns, (d) Illegal Guns for $\alpha = 0.95$. For parameter values $\gamma_1 = 0.1$, $\gamma_2 = 0.5$, $\delta = 0.2$, $\beta = 0.8$, $\epsilon = 0.8$, $\nu = 5$, $\theta = 0.5$, $\rho = 5.5$, $\mu = 2, \sigma = 17.644$.

study in this regard can be considered as the involvement of social factors such as poverty, unemployment, violent and permissive family, and delinquent peer groups in the crime model.

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