Honam Mathematical J. **44** (2022), No. 4, pp. 535–559
https://doi.org/10.5831/HMJ.2022.44.4.535

AN IMPLICATIVE FILTER OF BE-ALGEBRAS IN CONNECTION WITH CUBIC INTUITIONISTIC FUZZY SETS

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Abstract. The notions of cubic intuitionistic fuzzy set to filters and implicative filters of BE-algebras are introduced. Relations between cubic intuitionistic fuzzy filters with cubic intuitionistic fuzzy implicative filters of BE-algebras are investigated. The homomorphic image and inverse image of cubic intuitionistic fuzzy filters are studied and some related properties are investigated. Also, the product of cubic intuitionistic fuzzy subalgebras (implicative filters) of BE-algebras are investigated.

1. Introduction

Jun et al. [7] introduced cubic sets, and then this notion is applied to several algebraic structures (see [4, 8, 10, 5, 12, 13, 14]). Jun [9] defined the notion of a cubic intuitionistic fuzzy set, which is extending the concept of a cubic set. He investigated some related properties of it. He applied this theory to BCK/BCI-algebras and obtained some useful results. Senapati et al. [15, 16] applied to this structure to ideals of BCI-algebras and B-algebras. Kim et al. [11] introduced the notion of a BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. Ahn et al. [1, 2] introduced the notion of ideals in BE-algebras, and then stated and proved several characterizations of such ideals.

In this paper, we define the notion of cubic intuitionistic fuzzy filters and cubic intuitionistic fuzzy implicative filters of BE-algebras and

Received April 18, 2022. Accepted August 23, 2022.
2010 Mathematics Subject Classification. 06F35, 03G25, 94D05.
Key words and phrases. BE-algebra, cubic intuitionistic fuzzy filter, cubic intuitionistic fuzzy implicative filter, cubic intuitionistic fuzzy subalgebra.
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study some related properties of them. Relations between cubic intuitionistic fuzzy filters with cubic intuitionistic fuzzy implicative filters of BE-algebras are investigated. The homomorphic image and inverse image of cubic intuitionistic fuzzy filters are studied and some related properties are investigated. Also, the product of cubic intuitionistic fuzzy subalgebras (implicative filters) of BE-algebras are investigated.

2. Preliminaries

By a *BE-algebra* [11] we mean a system $(U, *, 1)$ of type $(2, 0)$ which the following axioms hold:

(BE1) $(\forall x \in U)(x * x = 1)$,
(BE2) $(\forall x \in U)(x * 1 = 1)$,
(BE3) $(\forall x \in U)(1 * x = x)$,
(BE4) $(\forall x, y, z \in U)(x * (y * z) = y * (x * z))$.

We introduce a relation "$\leq$" on U by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(U, *, 1)$ is said to be *transitive* [1] if it satisfies $y * z \leq (x * y) * (x * z)$ for any $x, y, z \in X$. A BE-algebra $(X, *, 1)$ is said to be *self distributive* [11] if it satisfies $x * (y * z) = (x * y) * (x * z)$ for any $x, y, z \in X$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [1]).

Every self distributive BE-algebra $(U, *, 1)$ satisfies the following properties:

(a) $(\forall x, y, z \in U)(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(b) $(\forall x, y, z \in U)(y * z \leq (z * x) * (y * x))$,
(c) $(\forall x, y \in U)(x * (x * y) = x * y)$,
(d) $(\forall x, y, z \in U)(x * y \leq (z * x) * (z * y))$,
(e) $(\forall x, y, z \in U)((x * y) * (x * z) \leq x * (y * z))$.

Definition 2.1. Let $(U, *, 1)$ be a BE-algebra and let F be a non-empty subset of U. Then F is said to be a *filter* of U if

(F1) $1 \in F$,
(F2) $(\forall x, y \in U)(x * y \in F, x \in F \Rightarrow y \in F)$.

F is an *implicative filter* [6] of U if it satisfies (F1) and

(F3) $(\forall x, y, z \in U)(x * (y * z) \in F, x * y \in F \Rightarrow x * z \in F)$.

Given two closed subintervals $I_1 = [I_1^-, I_1^+]$ and $I_2 = [I_2^-, I_2^+]$ of $[0, 1]$, we define the order " $\ll$ " and " $\gg$ " as follows:

$$I_1 \ll I_2 \Leftrightarrow I_1^- \le I_2^- \text{ and } I_1^+ \le I_2^+$$
$$I_1 \gg I_2 \Leftrightarrow I_1^- \ge I_2^- \text{ and } I_1^+ \ge I_2^+.$$

We define the refined maximum (briefly, rmax) and refined minimum (briefly, rmin) as

$$\begin{aligned}\mathrm{rmax}\{I_1, I_2\} =& [\max\{I_1^-, I_2^-\}, \max\{I_1^+, I_2^+\}]\\ \mathrm{rmin}\{I_1, I_2\} =& [\min\{I_1^-, I_2^-\}, \min\{I_1^+, I_2^+\}].\end{aligned}$$

Denote by $I[0, 1]$ the set of all closed subintervals of $[0, 1]$. In this paper we use the interval-valued intuitionistic fuzzy set

$$A = \{\langle x, M_A(x), N_A(x)\rangle | x \in U\}$$

in which $M_A(x), N_A(x)$ are closed intervals of $[0, 1]$ for all $x \in U$. Also, we use the notations $M_A^-(x)$ and $M_A^+(x)$ to mean the left and the right end point of the interval $M_A(x)$, respectively, and so we have $M_A(x) = [M_A^-(x), M_A^+(x)]$. For the sake of simplicity, we shall use the symbol $A(x) = \langle M_A(x), N_A(x)\rangle$ or $A = \langle M_A, N_A\rangle$ for the interval-valued intuitionistic fuzzy set $A = \{\langle x, M_A(x), N_A(x)\rangle | x \in U\}$.

Definition 2.2. Let U be a non-empty set. By a cubic intuitionistic fuzzy set in U we mean a structure $\widetilde{A} = \{\langle x, A(x), \mu(x)\rangle | x \in U\}$ in which A is an interval-valued intuitionistic fuzzy set in U and μ is an intuitionistic fuzzy set in U.

A cubic intuitionistic fuzzy set $\widetilde{A} = \{\langle x, A(x), \mu(x)\rangle | x \in U\}$ is simply denoted by $\widetilde{A} = \langle A, \mu\rangle$.

3. Cubic intuiiotnistic fuzzy implicative filters in BE-algebras

Definition 3.1. A cubic intuitionistic fuzzy set $\widetilde{A} = \langle A, \mu\rangle$ in a BE-algebra U is called a *cubic intuitionistic fuzzy filter* of U if it satisfies the following conditions: for all $x, y \in U$,

(CF1) $M_A(1) \gg M_A(x)$ and $N_A(1) \ll N_A(x)$,
(CF2) $\lambda_\mu(1) \le \lambda_\mu(x)$ and $\nu_\mu(1) \ge \nu_\mu(x)$,
(CF3) $M_A(y) \gg \mathrm{rmin}\{M_A(x * y), M_A(x)\}$,
(CF4) $N_A(y) \ll \mathrm{rmax}\{N_A(x * y), N_A(x)\}$,
(CF5) $\lambda_\mu(y) \le \max\{\lambda_\mu(x * y), \lambda_\mu(x)\}$,
(CF6) $\nu_\mu(y) \ge \min\{\nu_\mu(x * y), \nu_\mu(x)\}$.

Definition 3.2. A cubic intuitionistic fuzzy set $\widetilde{A} = \langle A, \mu \rangle$ in a BE-algebra U is called a *cubic intuitionistic fuzzy implicative filter* of U if it satisfies (CF1), (CF2) and the following conditions: for all $x, y, z \in U$,

(CF7) $M_A(x * z) \gg \text{rmin}\{M_A(x * (y * z)), M_A(x * y)\}$,
(CF8) $N_A(x * z) \ll \text{rmax}\{N_A(x * (y * z)), N_A(x * y)\}$,
(CF9) $\lambda_\mu(x * z) \leq \max\{\lambda_\mu(x * (y * z)), \lambda_\mu(x * y)\}$,
(CF10) $\nu_\mu(x * z) \geq \min\{\nu_\mu(x * (y * z)), \nu_\mu(x * y)\}$.

Example 3.3. Let $U := \{1, a, b, c, d, 0\}$ be a BE-algebra with the following Table 1.

TABLE 1. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Define a cubic intuitionistic fuzzy set $\widetilde{A} = \langle A, \mu \rangle$ in U as the following Table 2. It is easy to check that $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy

TABLE 2. Cayley table for $\widetilde{A} = \langle A, \mu \rangle$

U	$A = \langle M_A, N_A \rangle$	$\mu = \langle \lambda_\mu, \nu_\mu \rangle$
1	$\langle [0.6, 0.7], [0.1, 0.3] \rangle$	$(0.2, 0.8)$
a	$\langle [0.4, 0.6], [0.2, 0.4] \rangle$	$(0.3, 0.6)$
b	$\langle [0.4, 0.6], [0.2, 0.4] \rangle$	$(0.3, 0.6)$
c	$\langle [0.6, 0.4], [0.4, 0.6] \rangle$	$(0.4, 0.5)$
d	$\langle [0.6, 0.4], [0.4, 0.6] \rangle$	$(0.4, 0.5)$
0	$\langle [0.6, 0.4], [0.4, 0.6] \rangle$	$(0.4, 0.5)$

implicative filter of U.

Proposition 3.4. *Every cubic intuitionistic fuzzy implicative filter of a BE-algebra U is a cubic intuitionistic fuzzy filter of U.*

Proof. Let $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy implicative filter of U. Putting $x := 1$ in (CF7), (CF8), (CF9) and (CF10), respectively, and using (BE3), we have

$$\begin{aligned}
M_A(1*z) =& M_A(z) \gg \mathrm{rmin}\{M_A(1*(y*z)), M_A(1*y)\}\\
=& \mathrm{rmin}\{M_A(y*z), M_A(y)\},\\
N_A(1*z) =& N_A(z) \ll \mathrm{rmax}\{N_A(1*(y*z)), N_A(1*y)\}\\
=& \mathrm{rmax}\{N_A(y*z), N_A(y)\},\\
\lambda_\mu(1*z) =& \lambda_\mu(z) \leq \max\{\lambda_\mu(1*(y*z)), \lambda_\mu(1*y)\}\\
=& \max\{\lambda_\mu(y*z), \lambda_\mu(y)\} \text{ and}\\
\nu_\mu(1*z) =& \nu_\mu(z) \geq \min\{\nu_\mu(1*(y*z)), \nu_\mu(1*y)\}\\
=& \min\{\nu_\mu(y*z), \nu_\mu(y)\}.
\end{aligned}$$

Hence $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy filter of U. □

The converse of Proposition 3.4 may not true in general (see Example 3.5).

Example 3.5. Let $U := \{1, a, b, c\}$ be a BE-algebra with the following Table 3.

TABLE 3. Cayley table for the binary operation "$*$"

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Define a cubic intuitionistic fuzzy set $\widetilde{A} = \langle A, \mu\rangle$ in U as the following Table 4. It is easy to show that $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic

TABLE 4. Cayley table for $\widetilde{A} = \langle A, \mu\rangle$

U	$A = \langle M_A, N_A\rangle$	$\mu = \langle \lambda_\mu, \nu_\mu\rangle$
1	$\langle[0.5, 0.8], [0.1, 0.2]\rangle$	$(0.3, 0.7)$
a	$\langle[0.2, 0.4], [0.4, 0.5]\rangle$	$(0.6, 0.4)$
b	$\langle[0.2, 0.4], [0.4, 0.5]\rangle$	$(0.6, 0.4)$
c	$\langle[0.3, 0.6], [0.2, 0.3]\rangle$	$(0.4, 0.5)$

fuzzy filter of U. But it not a cubic intuitionistic fuzzy implicative filter of U, since $\mathrm{rmin}\{M_A(b*(a*c)), M_A(b*a)\} = \mathrm{rmin}\{M_A(1), M_A(1)\} = [0.5, 0.8] \gg M_A(b*c) = M_A(a) = [0.2, 0.4]$.

Proposition 3.6. *Let $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy filter of a BE-algebra U. Then the following assertions are valid:*

(i) $(\forall x, y \in U)(x \leq y \Rightarrow M_A(x) \ll M_A(y), N_A(x) \gg N_A(y), \lambda_\mu(x) \geq \lambda_\mu(y)$ *and* $\nu_\mu(x) \leq \nu_\mu(y))$,

(ii) $(\forall x, y, z \in U)(\mathrm{rmin}\{M_A(x*(y*z)), M_A(y)\} \ll M_A(x*z), \mathrm{rmax}\{N_A(x*(y*z)), N_A(y)\} \gg N_A(x*z), \max\{\lambda_\mu(x*(y*z)), \lambda_\mu(y)\} \geq \lambda_\mu(x*z)$ *and* $\min\{\nu_\lambda(x*(y*z)), \nu_\mu(y)\} \leq \nu_\mu(x*z))$.

Proof. Straightforward. □

The sets $\{x \in U | M_A(x) = M_A(1)\}, \{x \in U | N_A(x) = N_A(1)\}, \{x \in U | \lambda_\mu(x) = \lambda_\mu(1)\}$ and $\{x \in U | \nu_\mu(x) = \nu_\mu(1)\}$ are denoted by $U_{M_A}, U_{N_A}, U_{\lambda_\mu}$ and U_{ν_μ} , respectively.

Proposition 3.7. *Let F be a non-empty subset of a BE-algebra U and $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy set in U defined by*

$$M_A(x) = \begin{cases} [a_1, b_1], & \text{if } x \in F \\ [a_2, b_2], & \text{otherwise,} \end{cases}$$

$$N_A(x) = \begin{cases} [c_1, d_1], & \text{if } x \in F \\ [c_2, d_2], & \text{otherwise,} \end{cases}$$

$$\lambda_\mu(x) = \begin{cases} \alpha_1, & \text{if } x \in F \\ \alpha_2, & \text{otherwise,} \end{cases}$$

$$\nu_\mu(x) = \begin{cases} \beta_1, & \text{if } x \in F \\ \beta_2, & \text{otherwise} \end{cases}$$

for all $[a_1, b_1], [a_2, b_2], [c_1, d_1], [c_2, d_2] \in I[0, 1]$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$ with $[a_1, b_1] \gg [a_2, b_2], [c_1, d_1] \ll [c_2, d_2], b_1 + d_1 \leq 1, b_2 + d_2 \leq 1, \alpha_1 \leq \alpha_2, \beta_1 \geq \beta_2$ and $\alpha_1 + \beta_1 \leq 1, \alpha_2 + \beta_2 \leq 1$. Then $\widetilde{A}$ is a cubic intuitionistic fuzzy filter of U if and only if F is a filter of U. Moreover, $U_{M_A} = U_{N_A} = U_{\lambda_\mu} = U_{\nu_\mu} = F$.

Proof. Suppose that $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy filter of U. Let $x \in U$ be such that $x \in F$. Then we obtain

$$M_A(1) \gg M_A(x) = [a_1, b_1], N_A(1) \ll N_A(x) = [c_1, d_1],$$
$$\lambda_\mu(1) \leq \lambda_\mu(x) = \alpha_1 \text{ and } \nu_\mu(1) \geq \nu_\mu(x) = \beta_1.$$

Hence $M_A(1) = [a_1, b_1], N_A(1) = [c_1, d_1], \lambda_\mu(1) = \alpha_1$ and $\nu_\mu(1) = \beta_1$. Therefore $1 \in F$.

Let $x, y \in U$ be such that $x * y, x \in F$. Then we have

$$\begin{aligned} M_A(y) \gg& \mathrm{rmin}\{M_A(x*y), M_A(x)\} = \mathrm{rmin}\{[a_1,b_1],[a_1,b_1]\} = [a_1,b_1], \\ N_A(y) \ll& \mathrm{rmax}\{N_A(x*y), N_A(x)\} = \mathrm{rmax}\{[c_1,d_1],[c_1,d_1]\} = [c_1,d_1], \\ \lambda_\mu(y) \leq& \max\{\lambda_\mu(x*y), \lambda_\mu(x)\} = \max\{\alpha_1,\alpha_1\} = \alpha_1 \text{ and} \\ \nu_\mu(y) \geq& \min\{\nu_\mu(x*y), \nu_\mu(y)\} = \min\{\beta_1,\beta_1\} = \beta_1. \end{aligned}$$

Hence $y \in F$. Therefore F is a filter of U.

Conversely, assume that F is a filter of U. Then $1 \in F$. Hence we have $M_A(1) = [a_1, b_1] \gg M_A(x), N_A(1) = [c_1, d_1] \ll N_A(x), \lambda_\mu(1) = \alpha_1 \leq \lambda_\mu(x)$, and $\nu_\mu(x) \leq \nu_\mu(1) = \beta_1$, for all $x \in U$.

Let $x, y \in U$. Then we consider the following two cases:
Case 1. If $x * y \in F$ and $x \in F$, then $y \in F$. Hence

$$\begin{aligned} M_A(y) =& [a_1,b_1] = \mathrm{rmin}\{M_A(x*y), M_A(x)\}, \\ N_A(y) =& [c_1,d_1] = \mathrm{rmax}\{N_A(x*y), N_A(x)\}, \\ \lambda_\mu(y) =& \alpha_1 = \max\{\lambda_\mu(x*y), \lambda_\mu(x)\} \text{ and} \\ \nu_\mu(y) =& \beta_1 = \min\{\nu_\mu(x*y), \nu_\mu(x)\}. \end{aligned}$$

Case 2. If $x * y \notin F$ or $x \notin F$, then we get

$$\begin{aligned} M_A(y) \gg& [a_2,b_2] = \mathrm{rmin}\{M_A(x*y), M_A(x)\}, \\ N_A(y) \ll& [c_2,d_2] = \mathrm{rmax}\{N_A(x*y), N_A(x)\}, \\ \lambda_\mu(y) \leq& \alpha_2 = \max\{\lambda_\mu(x*y), \lambda_\mu(x)\} \text{ and} \\ \nu_\mu(y) \geq& \beta_2 = \min\{\nu_\mu(x*y), \nu_\mu(x)\}. \end{aligned}$$

It follows from Case 1 and Case 2 that $\widetilde{A}$ is a cubic intuitionistic fuzzy filter of U

Now, $U_{M_A} = \{x \in U | M_A(x) = M_A(1)\} = \{x \in U | M_A(x) = [a_1, b_1]\} = F$, $U_{N_A} = \{x \in U | N_A(x) = N_A(1)\} = \{x \in U | N_A(x) = [c_1, d_1]\} = F$, $U_{\lambda_\mu} = \{x \in U | \lambda_\mu(x) = \lambda_\mu(1)\} = \{x \in U | \lambda_\mu(x) = \alpha_1\} = F$ and $U_{\nu_\mu} = \{x \in U | \nu_\mu(x) = \nu_\mu(1)\} = \{x \in U | \nu_\mu(x) = \beta_1\} = F$. $\square$

Proposition 3.8. *Let $\widetilde{A} = \langle A, \mu \rangle$ be a cubic intuitionistic fuzzy filter of a self distributive BE-algebra U satisfying the following conditions:*

(3.1)
$$\begin{aligned}(\forall x, y, z \in U)(\text{rmin}\{M_A(x*(y*(y*z))), M_A(y*x)\} &\ll M_A(y*z),\\ \text{rmax}\{N_A(x*(y*(y*z))), N_A(y*x)\} &\gg N_A(y*z),\\ \max\{\lambda_\mu(x*(y*(y*z))), \lambda_\mu(y*x)\} &\geq \lambda_\mu(y*z),\\ \min\{\nu_\mu(x*(y*(y*z))), \nu_\mu(y*x)\} &\leq \nu_\mu(y*z)).\end{aligned}$$

Then $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy implicative filter of U.

Proof. Since $x*(y*z) = y*(x*z) \leq (x*y)*(x*(x*z)) = x*(y*(x*z)) = y*(x*(x*z))$ for all $x, y, z \in U$, it follows from Proposition 3.6(i) that $M_A(x*(y*z)) \ll M_A(y*(x*(x*z)))$, $N_A(x*(y*z)) \gg N_A(y*(x*(x*z)))$, $\lambda_\mu(x*(y*z)) \geq \lambda_\mu(y*(x*(x*z)))$ and $\nu_\mu(x*(y*z)) \leq \nu_\mu(y*(x*(x*z)))$. Using (3.1), we have $M_A(x*z) \gg \text{rmin}\{M_A(y*(x*(x*z))), M_A(x*y)\} \gg \text{rmin}\{M_A(x*(y*z)), M_A(x*y)\}$, $N_A(x*z) \ll \text{rmax}\{N_A(y*(x*(x*z))), N_A(x*y)\} \ll \text{rmax}\{N_A(x*(y*z)), N_A(x*y)\}$, $\lambda_\mu(x*z) \leq \max\{\lambda_\mu(y*(x*(x*z))), \lambda_\mu(x*y)\} \leq \max\{\lambda_\mu(x*(y*z)), \lambda_\mu(x*y)\}$ and $\nu_\mu(x*z) \geq \min\{\nu_\mu(y*(x*(x*z))), \nu_\mu(x*y)\} \geq \min\{\nu_\mu(x*(y*z)), \nu_\mu(x*y)\}$. Therefore $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy implicative filter of U. □

Theorem 3.9. *Let $\widetilde{A} = \langle A, \mu \rangle$ be a cubic intuitionistic fuzzy implicative filter of a transitive BE-algebra U. Then the following are equivalent:*

(i) $\widetilde{A} = \langle A, \mu \rangle$ *is a cubic intuitionistic fuzzy implicative filter of* U,
(ii) $(\forall x, y \in U)(M_A(x*(x*y)) \ll M_A(x*y), N_A(x*(x*y)) \gg N_A(x*y), \lambda_\mu(x*(x*y)) \geq \lambda_\mu(x*y), \nu_\mu(x*(x*y)) \leq \nu_\mu(x*y))$,
(iii) $(\forall x, y, z \in U)(M_A(x*(y*z)) \ll M_A((x*y)*(x*z)), N_A(x*(y*z)) \gg N_A((x*y)*(x*z)), \lambda_\mu(x*(y*z)) \geq \lambda_\mu((x*y)*(x*z)), \nu_\mu(x*(y*z)) \leq \nu_\mu((x*y)*(x*z)))$.

Proof. (i) ⇒ (ii) Assume that $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy implicative filter of U. Setting $z := y, y := x$ in (CF7), (CF8), (CF9) and (CF10), we get

$$\begin{aligned}M_A(x*y) \gg &\text{rmin}\{M_A(x*(x*y)), M_A(x*x)\}\\ =&\text{rmin}\{M_A(x*(x*y)), M_A(1)\}\\ =&M_A(x*(x*y)),\end{aligned}$$

$$\begin{aligned} N_A(x*y) \ll &\mathrm{rmax}\{N_A(x*(x*y)), N_A(x*x)\} \\ = &\mathrm{rmax}\{N_A(x*(x*y)), N_A(1)\} \\ = &N_A(x*(x*y)), \\ \lambda_\mu(x*y) \leq &\max\{\lambda_\mu(x*(x*y)), \lambda_\mu(x*x)\} \\ = &\max\{\lambda_\mu(x*(x*y)), \lambda_\mu(1)\} \\ = &\lambda_\mu(x*(x*y)) \text{ and} \\ \nu_\mu(x*y) \geq &\min\{\nu_\mu(x*(x*y)), \nu_\mu(x*x)\} \\ = &\min\{\nu_\mu(x*(x*y)), \nu_\mu(1)\} \\ = &\nu_\mu(x*(x*y)). \end{aligned}$$

Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x*(y*z) \leq x*((x*y)*(x*z)) = x*(x*((x*y)*z))$, by Proposition 3.6(i) we have $M_A(x*((x*y)*(x*z))) = M_A(x*(x*((x*y)*z))) \gg M_A(x*(y*z))$, $N_A(x*((x*y)*(x*z))) = N_A(x*(x*((x*y)*z))) \ll N_A(x*(y*z))$, $\lambda_\mu(x*((x*y)*(x*z))) = \lambda_\mu(x*(x*((x*y)*z))) \leq \lambda_\mu(x*(y*z))$, and $\nu_\mu(x*((x*y)*(x*z))) = \nu_\mu(x*(x*((x*y)*z))) \geq \nu_\mu(x*(y*z))$. It follows from (ii) that

$$\begin{aligned} M_A((x*y)*(x*z)) = &M_A(x*((x*y)*z)) \\ \gg &M_A(x*(x*((x*y)*z))) \\ \gg &M_A(x*(y*z)), \\ N_A((x*y)*(x*z)) = &N_A(x*((x*y)*z)) \\ \ll &N_A(x*(x*((x*y)*z))) \\ \ll &N_A(x*(y*z)), \\ \lambda_\mu((x*y)*(x*z)) = &\lambda_\mu(x*((x*y)*z)) \\ \leq &\lambda_\mu(x*(x*((x*y)*z))) \\ \leq &\lambda_\mu(x*(y*z)) \text{ and} \\ \nu_\mu((x*y)*(x*z)) = &\nu_\mu(x*((x*y)*z)) \\ \geq &\nu_\mu(x*(x*((x*y)*z))) \\ \geq &\nu_\mu(x*(y*z)). \end{aligned}$$

Thus (iii) holds.
(iii)⇒(i) Assume that (iii) holds. By Definition 3.1 and and (iii), we have

$$\begin{aligned} M_A(x*z) \gg &\mathrm{rmin}\{M_A((x*y)*(x*z)), M_A(x*y)\} \\ \gg &\mathrm{rmin}\{M_A(x*(y*z)), M_A(x*y)\}, \end{aligned}$$

$$\begin{aligned}
N_A(x*z) \ll & \mathrm{rmax}\{N_A((x*y)*(x*z)), N_A(x*y)\} \\
\ll & \mathrm{rmax}\{N_A(x*(y*z)), N_A(x*y)\}, \\
\lambda_\mu(x*z) \le & \max\{\lambda_\mu((x*y)*(x*z)), \lambda_\mu(x*y)\} \\
\le & \max\{\lambda_\mu(x*(y*z)), \lambda_\mu(x*y)\}\text{and} \\
\nu_\mu(x*z) \ge & \min\{\nu_\mu((x*y)*(x*z)), \nu_\mu(x*y)\} \\
\ge & \min\{\nu_\mu(x*(y*z)), \nu_\mu(x*y)\}.
\end{aligned}$$

Therefore $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy implicative filter of U. □

Proposition 3.10. *Let $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy set of a self distributive BE-algebra U. Then $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy implicative filter of U if and only if it is a cubic intuitionistic fuzzy filter of U.*

Proof. By Proposition 3.4, every cubic intuitionistic fuzzy implicative filter of U is a cubic intuitiotnistic fuzzy filter of U.

Conversely, assume that $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy filter of U. For any $x, y, z \in U$, by Definition 3.1, we have

$$\begin{aligned}
M_A(x*z) \gg & \mathrm{rmin}\{M_A((x*y)*(x*z)), M_A(x*y)\} \\
= & \mathrm{rmin}\{M_A(x*(y*z)), M_A(x*y)\}, \\
N_A(x*z) \ll & \mathrm{rmax}\{N_A((x*y)*(x*z)), N_A(x*y)\} \\
= & \mathrm{rmax}\{N_A(x*(y*z)), N_A(x*y)\}, \\
\lambda_\mu(x*z) \le & \max\{\lambda_\mu((x*y)*(x*z)), \lambda_\mu(x*y)\} \\
= & \max\{\lambda_\mu(x*(y*z)), \lambda_\mu(x*y)\} \text{ and} \\
\nu_\mu(x*z) \ge & \min\{\nu_\mu((x*y)*(x*z)), \nu_\mu(x*y)\} \\
= & \min\{\nu_\mu(x*(y*z)), \nu_\mu(x*y)\}.
\end{aligned}$$

Hence $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy implicative filter of U. □

Let $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy set in a non-emptyset U. Given $([e_1, f_1], [e_2, f_2]) \in I[0,1] \times I[0,1]$ and $(\tau_1, \tau_2) \in [0,1] \times [0,1]$, we consider the sets

$$\begin{aligned}
M_A[e_1, f_1] = & \{x \in U | M_A(x) \gg [e_1, f_1]\}, \\
N_A[e_2, f_2] = & \{x \in U | N_A(x) \ll [e_2, f_2]\}, \\
\lambda_\mu(\tau_1) = & \{x \in U | \lambda_\mu(x) \le \tau_1\} \text{ and} \\
\nu_\mu(\tau_2) = & \{x \in U | \nu_\mu(x) \ge \tau_2\}.
\end{aligned}$$

Theorem 3.11. *Let $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy implicative filter of a BE-algebra U. Then the sets $M_A[e,f], N_A[e,f], \lambda_\mu(\tau)$ and $\nu_\mu(\tau)$ are implicative filters of U, for all $[e,f] \in I[0,1]$ and $\tau \in [0,1]$.*

Proof. For any $[e,f] \in I[0,1]$ and $\tau \in [0,1]$, let $x \in U$ be such that $x \in M_A[e,f] \cap N_A[e,f] \cap \lambda_\mu(\tau) \cap \nu_\mu(\tau)$. Then $M_A(x) \gg [e,f], N_A(x) \ll [e,f], \lambda_\mu(x) \leq \tau, \nu_\mu(x) \geq \tau$. Putting $x := 1, z := 1$ and $y := x$ in (CF7)$\sim$(CF10), respectively, we have

$$\begin{aligned}
M_A(1) = M_A(1*1) \gg & \mathrm{rmin}\{M_A(1*(x*1)), M_A(1*x)\} \\
= & \mathrm{rmin}\{M_A(1), M_A(x)\} = M_A(x) \gg [e,f] \\
N_A(1) = N_A(1*1) \ll & \mathrm{rmax}\{N_A(1*(x*1)), N_A(1*x)\} \\
= & \mathrm{rmax}\{N_A(1), N_A(x)\} = N_A(x) \ll [e,f] \\
\lambda_\mu(1) = \lambda_\mu(1*1) \leq & \max\{\lambda_\mu(1*(x*1)), \lambda_\mu(1*x)\} \\
= & \max\{\lambda_\mu(1), \lambda_\mu(x)\} = \lambda_\mu(x) \leq \tau \\
\nu_\mu(1) = \nu_\mu(1*1) \geq & \min\{\nu_\mu(1*(x*1)), \nu_\mu(1*x)\} \\
= & \min\{\nu_\mu(1), \nu_\mu(x)\} \geq \tau.
\end{aligned}$$

Hence $1 \in M_A[e,f] \cap N_A[e,f] \cap \lambda_\mu(\tau) \cap \nu_\mu(\tau)$.

For any $[e,f] \in I[0,1]$ and $\tau \in [0,1]$, let $x, y, z \in U$ be such that $x*(y*z), x*y \in M_A[e,f] \cap N_A[e,f] \cap \lambda_\mu(\tau) \cap \nu_\mu(\tau)$. Then $M_A(x*(y*z)) \gg [e,f], N_A(x*(y*z))) \ll [e,f], \lambda_\mu(x*(y*z)) \leq \tau, \nu_\mu(x*(y*z)) \geq \tau$, $M_A(x*y) \gg [e,f], N_A(x*y) \ll [e,f], \lambda_\mu(x*y) \leq \tau$ and $\nu_\mu(x*y) \geq \tau$. Since $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy implicative filter of a BE-algebra U, we have

$$\begin{aligned}
M_A(x*z) \gg & \mathrm{rmin}\{M_A(x*(y*z)), M_A(x*y)\} \gg \mathrm{rmin}\{[e,f],[e,f]\} = [e,f] \\
N_A(x*z) \ll & \mathrm{rmax}\{N_A(x*(y*z)), N_A(x*y)\} \ll \mathrm{rmax}\{[e,f],[e,f]\} = [e,f] \\
\lambda_\mu(x*z) \leq & \max\{\lambda_\mu(x*(y*z)), \lambda_\mu(x*y)\} \leq \max\{\tau,\tau\} = \tau \\
\nu_\mu(x*z) \geq & \min\{\nu_\mu(x*(y*z)), \nu_\mu(x*y)\} \geq \min\{\tau,\tau\} = \tau.
\end{aligned}$$

Hence $x*z \in M_A[e,f] \cap N_A[e,f] \cap \lambda_\mu(\tau) \cap \nu_\mu(\tau)$. Therefore the sets $M_A[e,f], N_A[e,f], \lambda_\mu(\tau)$ and $\nu_\mu(\tau)$ are implicative filters of U, for all $[e,f] \in I[0,1]$ and $\tau \in [0,1]$. □

Theorem 3.12. *Let $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy set in a BE-algebra U such that $M_A[e_1,f_1], N_A[e_2,f_2]$, $\lambda_\mu(\tau_1)$ and $\nu_\mu(\tau_2)$ are implicative filters of U, for all $([e_1,f_1],[e_2,f_2]) \in I[0,1] \times I[0,1]$ and $(\tau_1,\tau_2) \in [0,1] \times [0,1]$. Then $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy implicative filter of U.*

Proof. Suppose that for every $([e_1, f_1], [e_2, f_2]) \in I[0,1] \times I[0,1]$ and $(\tau_1, \tau_2) \in [0,1] \times [0,1]$, $M_A[e_1, f_1], N_A[e_2, f_2]$, $\lambda_\mu(\tau_1)$ and $\nu_\mu(\tau_2)$ are implicative filters of U. Let $x \in U$ be such that $M_A(1) \ll M_A(x)$. Put $[x_0, y_0] := \frac{1}{2}[M_A(1) + M_A(x)]$ for $[x_0, y_0] \in I[0,1]$. Then we have $M_A(1) \ll [x_0, y_0] \ll M_A(x)$. Hence $1 \notin M_A[e_1, f_1]$, but $x \in M_A[e_1, f_1]$, which is a contradiction. Let $y \in U$ be such that $N_A(1) \gg M_A(y)$. Put $[s_0, t_0] := \frac{1}{2}[N_A(1) + N_A(y)]$ for $[s_0, t_0] \in I[0,1]$. Then we have $N_A(1) \gg [s_0, t_0] \gg N_A(y)$. Hence $1 \notin N_A[e_2, f_2]$, but $y \in N_A[e_2, f_2]$, which is a contradiction. Similarly we can prove λ_μ and ν_μ satisfies (CF2).

Let $x, y, z \in U$ be such that $M_A(x*z) \ll \text{rmin}\{M_A(x*(y*z)), M_A(x*y)\}$. Assume that $[a_0, b_0] = \frac{1}{2}[M_A(x*z) + \text{rmin}\{M_A(x*(y*z)), M_A(x*y)\}]$ for $[a_0, b_0] \in I[0,1]$. Then we have $M_A(x*z) \ll [a_0, b_0] \ll \text{rmin}\{M_A(x*(y*z)), M_A(x*y)\}$ and so $x*z \notin M_A[e_1, f_1]$, but $x*(y*z), x*y \in M_A[e_1, f_1]$. This is a contradiction. Let $x, y, z \in U$ be such that $N_A(x*z) \gg \text{rmax}\{N_A(x*(y*z)), N_A(x*y)\}$. Assume that $[a_1, b_1] = \frac{1}{2}[N_A(x*z) + \text{rmax}\{N_A(x*(y*z)), N_A(x*y)\}]$ for $[a_1, b_1] \in I[0,1]$. Then we have $N_A(x*z) \ll [a_1, b_1] \ll \text{rmax}\{N_A(x*(y*z)), N_A(x*y)\}$ and so $x*z \notin N_A[e_1, f_1]$, but $x*(y*z), x*y \in N_A[e_2, f_2]$. This is a contradiction. Hence M_A satisfies (CF7) and N_A satisfies (CF8).

Similarly, we can prove that λ_μ and ν_μ satisfies (CF9) and (CF10). Therefore $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy implicative filter of U. □

Theorem 3.13. *Let $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy implicative filter of a BE-algebra U. Then the sets $U_{M_A}, U_{N_A}, U_{\lambda_\mu}$ and U_{ν_μ} are implicative filters of U.*

Proof. Let $x, y, z \in U$ be such that $x*(y*z) \in U_{M_A}$ and $x*y \in U_{M_A}$. Then $M_A(x*(y*z)) = M_A(1) = M_A(x*y)$. Hence $M_A(x*z) \gg \text{rmin}\{M_A(x*(y*z)), M_A(x*y)\} = M_A(1)$. By (CF1), we deduce that $M_A(x*z)) \ll M_A(1)$. Therefore $M_A(x*z) = M_A(1)$ and so $x*z \in U_{M_A}$. Thus U_{M_A} is an implicative filter of U. By a similar way, we can prove that U_{N_A} is an implicative filter of U.

Again, let $x, y, z \in U$ be such that $x*(y*z) \in U_{\lambda_\mu}$ and $x*y \in U_{\lambda_\mu}$. Then $\lambda_\mu(x*(y*z)) = \lambda_\mu(1) = \lambda_\mu(x*y)$. Hence $\lambda_\mu(x*z) \leq \max\{\lambda_\mu(x*(y*z)), \lambda_\mu(x*y)\} = \lambda_\mu(1)$. By (CF2), we deduce that $\lambda_\mu(1) \leq \lambda_\mu(x*z)$. Therefore $\lambda_\mu(x*z) = \lambda_\mu(1)$ and so $x*z \in U_{\lambda_\mu}$. Thus U_{λ_μ} is a implicative filter of U. By a similar way, we can prove that U_{ν_μ} is an implicative filter of U. It is complete the proof. □

For any element x and y of a BE-algebra U and a positive integer n, let $x^n * y$ denote $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = y$.

Definition 3.14. Let $\widetilde{A} = \langle A, \mu \rangle$ be a cubic intuitionistic fuzzy set on a BE-algebra U. Then $\widetilde{A} = \langle A, \mu \rangle$ is called a *cubic intuitionistic fuzzy n-fold implicative filter* of U if it satisfies (CF1), (CF2) and the following conditions: for all $x, y, z \in U$,

(CF11) $M_A(x^n * z) \gg \mathrm{rmin}\{M_A(x^n * (y * z)), M_A(x^n * y)\}$,
(CF12) $N_A(x^n * z) \ll \mathrm{rmax}\{N_A(x^n * (y * z)), N_A(x^n * y)\}$,
(CF13) $\lambda_\mu(x^n * z) \leq \max\{\lambda_\mu(x^n * (y * z)), \lambda_\mu(x^n * y)\}$,
(CF14) $\nu_\mu(x^n * z) \geq \min\{\nu_\mu(x^n * (y * z)), \nu_\mu(x^n * y)\}$.

Note that every cubic intuitionistic fuzzy 1-fold implicative filter of U is a cubic intuitionistic fuzzy implicative filter of U.

Example 3.15. Let $U := \{1, a, b, c, d, 0\}$ be a BE-algebra with the following Table 5.

TABLE 5. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	d
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Define a cubic intuitionistic fuzzy set $\widetilde{A} = \langle A, \mu \rangle$ in U as the following Table 6. It is easy to check that $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy n-fold implicative filter of U.

Proposition 3.16. *Every cubic intuitionistic fuzzy n-fold implicative filter of a BE-algebra U is a cubic intuitionistic fuzzy filter of U.*

Proof. Taking $x := 1$ in Definition 3.14 and (BE3), we have $M_A(z) \gg \mathrm{rmin}\{M_A(y * z), M_A(y)\}$, $N_A(z) \ll \mathrm{rmax}\{N_A(y * z), N_A(y)\}$, $\lambda_\mu(z) \leq \max\{\lambda_\mu(y * z), \lambda_\mu(y)\}$ and $\nu_\mu(z) \geq \min\{\nu_\mu(y * z), \nu_\mu(y)\}$ for all $y, z \in U$.
Hence $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy filter of U. $\square$

The converse of Proposition 3.16 may not be true in general (see Example 3.17).

TABLE 6. Cayley table for $\widetilde{A} = \langle A, \mu \rangle$

U	$A = \langle M_A, N_A \rangle$	$\mu = \langle \lambda_\mu, \nu_\mu \rangle$
1	$\langle [0.5, 0.8], [0.1, 0.2] \rangle$	$(0.1, 0.9)$
a	$\langle [0.1, 0.4], [0.4, 0.6] \rangle$	$(0.7, 0.2)$
b	$\langle [0.3, 0.7], [0.2, 0.3] \rangle$	$(0.2, 0.7)$
c	$\langle [0.3, 0.7], [0.2, 0.3] \rangle$	$(0.2, 0.7)$
d	$\langle [0.1, 0.4], [0.4, 0.6] \rangle$	$(0.7, 0.2)$
0	$\langle [0.1, 0.4], [0.4, 0.6] \rangle$	$(0.7, 0.2)$

Example 3.17. Let $U := \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.15. Define a cubic intuitionistic fuzzy set $\widetilde{A} = \langle A, \mu \rangle$ in U as the following Table 7. It is easy to check that $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuition-

TABLE 7. Cayley table for $\widetilde{A} = \langle A, \mu \rangle$

U	$A = \langle M_A, N_A \rangle$	$\mu = \langle \lambda_\mu, \nu_\mu \rangle$
1	$\langle [0.5, 0.8], [0.1, 0.2] \rangle$	$(0.1, 0.9)$
a	$\langle [0.3, 0.7], [0.2, 0.3] \rangle$	$(0.2, 0.7)$
b	$\langle [0.3, 0.7], [0.2, 0.3] \rangle$	$(0.2, 0.7)$
c	$\langle [0.3, 0.7], [0.2, 0.3] \rangle$	$(0.2, 0.7)$
d	$\langle [0.3, 0.7], [0.2, 0.3] \rangle$	$(0.2, 0.7)$
0	$\langle [0.3, 0.7], [0.2, 0.7] \rangle$	$(0.2, 0.7)$

istic fuzzy filter of U. But it is not a cubic 1-fold implicative filter of U, since $M_A(d*c) = M_A(b) = [0.3, 0.7] \ll \text{rmin}\{M_A(d*(b*c)), M_A(d*b)\} = M_A(1) = [0.5, 0.8]$.

Theorem 3.18. *For any cubic intuitionistic fuzzy filter $\widetilde{A} = \langle A, \mu \rangle$ of a transitive BE-algebra U, the following are equivalent:*

(i) *$\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy n-fold implicative filter of U;*

(ii) *$(\forall x, y \in U)(M_A(x^{n+1} * y) \ll M_A(x^n * y), N_A(x^{n+1} * y) \gg N_A(x^n * y), \lambda_\mu(x^{n+1} * y) \geq \lambda_\mu(x^n * y), \nu_\mu(x^{n+1} * y) \leq \nu_\mu(x^n * y))$;*

(iii) *$(\forall x, y, z \in U)(M_A(x^n * (y * z)) \ll M_A((x^n * y) * (x^n * z)), N_A(x^n * (y * z)) \gg N_A((x^n * y) * (x^n * z)), \lambda_\mu(x^n * (y * z)) \geq \lambda_\mu((x^n * y) * (x^n * z)), \nu_\mu(x^n * (y * z)) \leq \nu_\mu((x^n * y) * (x^n * z)))$.*

Proof. (i)⇒(ii) Assume that $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy n-fold implicative filter of U. Setting $z := y, y := x$ in Definition 3.14, we have

$$\begin{aligned}
M_A(x^n * y) &\gg \text{rmin}\{M_A(x^n * (x * y)), M_A(x^n * x)\} \\
&= \text{rmin}\{M_A(x^{n+1} * y), M_A(1)\} \\
&= M_A(x^{n+1} * y), \\
N_A(x^n * y) &\ll \text{rmax}\{N_A(x^n * (x * y)), N_A(x^n * x)\} \\
&= \text{rmax}\{N_A(x^{n+1} * y), N_A(1)\} \\
&= N_A(x^{n+1} * y), \\
\lambda_\mu(x^n * y) &\leq \max\{\lambda_\mu(x^n * (x * y)), \lambda_\mu(x^n * x)\} \\
&= \max\{\lambda_\mu(x^{n+1} * y), \lambda_\mu(1)\} \\
&= \lambda_\mu(x^{n+1} * y) \text{ and} \\
\nu_\mu(x^n * y) &\geq \min\{\nu_\mu(x^n * (x * y)), \nu_\mu(x^n * x)\} \\
&= \min\{\nu_\mu(x^{n+1} * y), \nu_\mu(1)\} \\
&= \nu_\mu(x^{n+1} * y).
\end{aligned}$$

Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x^n*(y*z) \leq x^n*((x^n*y)*(x^n*z))$ for any $x, y, z \in U$, we have $M_A(x^n*((x^n*y)*(x^n*z))) \gg M_A(x^n*(y*z))$, $N_A(x^n*((x^n*y)*(x^n*z))) \ll N_A(x^n*(y*z))$, $\lambda_\mu(x^n*((x^n*y)*(x^n*z))) \leq \lambda_\mu(x^n*(y*z))$ and $\nu_\mu(x^n*((x^n*y)*(x^n*z))) \geq \nu_\mu(x^n*(y*z))$. Since $x^{n+1}*(x^{n-1}*((x^n*y)*z)) = x^n*(x^n*((x^n*y)*z)) = x^n*((x^n*y)*(x^n*z))$ and using (ii), we have

(3.2)
$$\begin{aligned}
M_A(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) &= M_A(x^n * (x^{n-1} * ((x^n * y) * z))) \\
&\gg M_A(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\
&= M_A(x^n * ((x^n * y) * (x^n * z))) \\
&\gg M_A(x^n * (y * z)),
\end{aligned}$$

(3.3)
$$\begin{aligned}
N_A(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) &= N_A(x^n * (x^{n-1} * ((x^n * y) * z))) \\
&\ll N_A(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\
&= N_A(x^n * ((x^n * y) * (x^n * z))) \\
&\ll N_A(x^n * (y * z)),
\end{aligned}$$

(3.4)

$$\begin{aligned}\lambda_\mu(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) =& \lambda_\mu(x^n * (x^{n-1} * ((x^n * y) * z)))\\ \leq& \lambda_\mu(x^{n+1} * (x^{n-1} * ((x^n * y) * z)))\\ =& \lambda_\mu(x^n * ((x^n * y) * (x^n * z)))\\ \leq& \lambda_\mu(x^n * (y * z)) \text{ and}\end{aligned}$$

(3.5)

$$\begin{aligned}\nu_\mu(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) =& \nu_\mu(x^n * (x^{n-1} * ((x^n * y) * z)))\\ \geq& \nu_\mu(x^{n+1} * (x^{n-1} * ((x^n * y) * z)))\\ =& \nu_\mu(x^n * ((x^n * y) * (x^n * z)))\\ \geq& \nu_\mu(x^n * (y * z)).\end{aligned}$$

By (ii), (3.2)∼(3.5), we have

$$\begin{aligned}M_A(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& M_A(x^n * (x^{n-2} * ((x^n * y) * z)))\\ \gg& M_A(x^{n+1} * (x^{n-2} * ((x^n * y) * z)))\\ \gg& M_A(x^n * (y * z)) \text{ and}\end{aligned}$$

$$\begin{aligned}N_A(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& N_A(x^n * (x^{n-2} * ((x^n * y) * z)))\\ \ll& N_A(x^{n+1} * (x^{n-2} * ((x^n * y) * z)))\\ \ll& N_A(x^n * (y * z)),\end{aligned}$$

$$\begin{aligned}\lambda_\mu(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& \lambda_\mu(x^n * (x^{n-2} * ((x^n * y) * z)))\\ \leq& \lambda_\mu(x^{n+1} * (x^{n-2} * ((x^n * y) * z)))\\ \leq& \lambda_\mu(x^n * (y * z)) \text{ and}\end{aligned}$$

$$\begin{aligned}\nu_\mu(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& \nu_\mu(x^n * (x^{n-2} * ((x^n * y) * z)))\\ \geq& \nu_\mu(x^{n+1} * (x^{n-2} * ((x^n * y) * z)))\\ \geq& \nu_\mu(x^n * (y * z)) \text{ and}\end{aligned}$$

Continuing this process, we conclude that

$$\begin{aligned}M_A((x^n * y) * (x^n * z)) =& M_A(x^n * ((x^n * y) * z))\\ &\gg M_A(x^n * (y * z)),\end{aligned}$$

$$\begin{aligned}N_A((x^n * y) * (x^n * z)) =& N_A(x^n * ((x^n * y) * z))\\ &\ll N_A(x^n * (y * z)),\end{aligned}$$

$$\begin{aligned}\lambda_\mu((x^n * y) * (x^n * z)) =& \lambda_\mu(x^n * ((x^n * y) * z))\\ &\leq \lambda_\mu(x^n * (y * z)) \text{ and}\end{aligned}$$

$$\begin{aligned}\nu_\mu((x^n * y) * (x^n * z)) =& \nu_\mu(x^n * ((x^n * y) * z))\\ &\geq \nu_\mu(x^n * (y * z)).\end{aligned}$$

(iii)⇒(i) Let $x, y, z \in U$. It follows from (iii) that

$$\begin{aligned}M_A(x^n * z) &\gg \text{rmin}\{M_A((x^n * y) * (x^n * z)), M_A(x^n * y)\}\\ &\gg \text{rmin}\{M_A(x^n * (y * z)), M_A(x^n * y)\},\end{aligned}$$

$$\begin{aligned}N_A(x^n * z) &\ll \text{rmax}\{N_A((x^n * y) * (x^n * z)), N_A(x^n * y)\}\\ &\ll \text{rmax}\{N_A(x^n * (y * z)), N_A(x^n * y)\},\end{aligned}$$

$$\begin{aligned}\lambda_\mu(x^n * z) &\leq \max\{\lambda_\mu((x^n * y) * (x^n * z)), \lambda_\mu(x^n * y)\}\\ &\leq \max\{\lambda_\mu(x^n * (y * z)), \lambda_\mu(x^n * y)\} \text{ and}\end{aligned}$$

$$\begin{aligned}\nu_\mu(x^n * z) &\geq \min\{\nu_\mu((x^n * y) * (x^n * z)), \nu_\mu(x^n * y)\}\\ &\geq \min\{\nu_\mu(x^n * (y * z)), \nu_\mu(x^n * y)\}.\end{aligned}$$

Hence $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy implicative filter of U. □

Definition 3.19. Let n be a positive integer. A BE-algebra U is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in U$.

Corollary 3.20. *In an n-fold implicative BE-algebra, the notion of cubic intuitionistic fuzzy filters and cubic intuitionistic n-fold implicative filters coincide.*

Proof. Straightforward. □

4. Direct product of cubic intuitionistic fuzzy sets in BE-algebras

Let f be a mapping form a set U into a set V and $\widetilde{A} = \langle A, \mu\rangle$ be a cubic intuitionistic fuzzy set in V. Then the inverse image of $\widetilde{A}$, is defined as

$$f^{-1}(\widetilde{A}) := \{\langle x, f^{-1}(A), f^{-1}(\mu)\rangle | x \in U\}$$

with the membership function and non-membership function respectively are given by

$$\begin{aligned}f^{-1}(A)(x) =& f^{-1}(\langle M_A(x), N_A(x)\rangle)\\ =& \langle f^{-1}(M_A)(x), f^{-1}(N_A)(x)\rangle\\ =& \langle M_A(f(x)), N_A(f(x))\rangle \quad \text{and}\end{aligned}$$

$$\begin{aligned} f^{-1}(\mu)(x) =& f^{-1}(\langle \lambda_\mu, \nu_\mu \rangle)(x) \\ =& \langle f^{-1}(\lambda_\mu)(x), f^{-1}(\nu_\mu)(x) \rangle \\ =& \langle \lambda_\mu(f(x)), \nu_\mu(f(x)) \rangle. \end{aligned}$$

Theorem 4.1. *Let $f : U \to V$ be a homomorphism of BE-algebras. If $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy implicative filter of V, then the inverse image $f^{-1}(\widetilde{A}) = \{\langle x, f^{-1}(A), f^{-1}(\mu) \rangle | x \in U\}$ of $\widetilde{A}$ under f is a cubic intuitionistic fuzzy implicative filter of U.*

Proof. Assume that $\widetilde{A} = \langle A, \mu \rangle$ is a cubic intuitionistic fuzzy implicative filter of V and let $x \in U$. Then we have

$$\begin{aligned} f^{-1}(M_A)(x) =& M_A(f(x)) \ll M_A(1_V) = M_A(f(1_U)) = f^{-1}(M_A)(1_U), \\ f^{-1}(N_A)(x) =& N_A(f(x)) \gg N_A(1_V) = N_A(f(1_U)) = f^{-1}(N_A)(1_U), \\ f^{-1}(\lambda_\mu)(x) =& \lambda_\mu(f(x)) \geq \lambda_\mu(f(1_U)) = f^{-1}(\lambda_\mu)(1_U) \text{ and} \\ f^{-1}(\nu_\mu)(x) =& \nu_\mu(f(x)) \leq \nu_\mu(f(1_U)) = f^{-1}(\nu_\mu)(1_U). \end{aligned}$$

Let $x, y, z \in U$. Then we get

$$\begin{aligned} &\mathrm{rmin}\{f^{-1}(M_A)(x * (y * z)), f^{-1}(M_A)(x * y)\} \\ &\quad = \mathrm{rmin}\{M_A(f(x * (y * z))), M_A(f(x * y))\} \\ &\quad = \mathrm{rmin}\{M_A(f(x) * (f(y) * f(z))), M_A(f(x) * f(y))\} \\ &\quad \ll M_A(f(x) * f(z)) = M_A(f(x * z)) = f^{-1}(M_A)(x * z), \end{aligned}$$

$$\begin{aligned} &\mathrm{rmax}\{f^{-1}(N_A)(x * (y * z)), f^{-1}(N_A)(x * y)\} \\ &\quad = \mathrm{rmax}\{N_A(f(x * (y * z))), N_A(f(x * y))\} \\ &\quad = \mathrm{rmax}\{N_A(f(x) * (f(y) * f(z))), N_A(f(x) * f(y))\} \\ &\quad \gg N_A(f(x) * f(z)) = N_A(f(x * z)) = f^{-1}(N_A)(x * z), \end{aligned}$$

$$\begin{aligned} &\max\{f^{-1}(\lambda_\mu)(x * (y * z)), f^{-1}(\lambda_\mu)(x * y)\} \\ &\quad = \max\{\lambda_\mu(f(x * (y * z))), \lambda_\mu(f(x * y))\} \\ &\quad = \max\{\lambda_\mu(f(x) * (f(y) * f(z))), \lambda_\mu(f(x) * f(y))\} \\ &\quad \geq \lambda_\mu(f(x) * f(z)) = \lambda_\mu(f(x * z)) = f^{-1}(\lambda_\mu)(x * z) \text{ and} \end{aligned}$$

$$\begin{aligned} &\min\{f^{-1}(\nu_\mu)(x * (y * z)), f^{-1}(\nu_\mu)(x * y)\} \\ &\quad = \min\{\nu_\mu(f(x * (y * z))), \nu_\mu(f(x * y))\} \\ &\quad = \min\{\nu_\mu(f(x) * (f(y) * f(z))), \nu_\mu(f(x) * f(y))\} \\ &\quad \leq \nu_\mu(f(x) * f(z)) = \nu_\mu(f(x * z)) = f^{-1}(\nu_\mu)(x * z). \end{aligned}$$

Therefore $f^{-1}(\widetilde{A}) = \{\langle x, f^{-1}(A), f^{-1}(\mu)\rangle | x \in U\}$ is a cubic intuitionistic fuzzy implicative filter of U. $\square$

Theorem 4.2. *Let $f : U \to V$ be a homomorphism from a BE-algebra U onto a BE-algebra V. If $f^{-1}(\widetilde{A})$ is a cubic intuitionistic fuzzy implicative filter of U, then $\widetilde{A} = \langle A, \mu\rangle$ is a cubic intuitionistic fuzzy implicative filter of V.*

Proof. For any $y \in V$, there exists $a \in U$ such that $f(a) = y$. Then

$$\begin{aligned} M_A(y) = M_A(f(a)) = f^{-1}(M_A)(a) \ll f^{-1}(M_A)(1_U) = M_A(f(1_U)) = M_A(1_V),\\ N_A(y) = N_A(f(a)) = f^{-1}(N_A)(a) \gg f^{-1}(N_A)(1_U) = N_A(f(1_U)) = M_A(1_V),\\ \lambda_\mu(y) = \lambda_\mu(f(a)) = f^{-1}(\lambda_\mu)(a) \geq f^{-1}(\lambda_\mu)(1_U) = \lambda_\mu(f(1_U)) = \lambda_\mu(1_V) \text{ and}\\ \nu_\mu(y) = \nu_\mu(f(a)) = f^{-1}(\nu_\mu)(a) \leq f^{-1}(\nu_\mu)(1_U) = \nu_\mu(f(1_U)) = \nu_\mu(1_V). \end{aligned}$$

Let $x, y, z \in V$. Then $f(a) = x$, $f(b) = y$ and $f(c) = z$ for some $a, b, c \in U$. It follows that

$$\begin{aligned} M_A(x * z) =& M_A(f(a) * f(c)) = M_A(f(a * c)) = f^{-1}(M_A)(a * c)\\ &\gg \text{rmin}\{f^{-1}(M_A)(a * (b * c)), f^{-1}(M_A)(a * b)\}\\ &= \text{rmin}\{M_A(f(a * (b * c)), M_A(f(a * b))\}\\ &= \text{rmin}\{M_A(f(a) * (f(b) * f(c))), M_A(f(a) * f(b))\}\\ &= \text{rmin}\{M_A(x * (y * z)), M_A(x * y)\}, \end{aligned}$$

$$\begin{aligned} N_A(x * z) =& N_A(f(a) * f(c)) = N_A(f(a * c)) = f^{-1}(N_A)(a * c)\\ &\ll \text{rmax}\{f^{-1}(N_A)(a * (b * c)), f^{-1}(N_A)(a * b)\}\\ &= \text{rmax}\{N_A(f(a * (b * c)), N_A(f(a * b))\}\\ &= \text{rmax}\{N_A(f(a) * (f(b) * f(c))), N_A(f(a) * f(b))\}\\ &= \text{rmax}\{N_A(x * (y * z)), N_A(x * y)\}, \end{aligned}$$

$$\begin{aligned} \lambda_\mu(x * z) =& \lambda_\mu(f(a) * f(c)) = \lambda_\mu(f(a * c)) = f^{-1}(\lambda_\mu)(a * c)\\ &\leq \max\{f^{-1}(\lambda_\mu)(a * (b * c)), f^{-1}(\lambda_\mu)(a * b)\}\\ &= \max\{\lambda_\mu(f(a * (b * c)), \lambda_\mu(f(a * b))\}\\ &= \max\{\lambda_\mu(f(a) * (f(b) * f(c))), \lambda_\mu(f(a) * f(b))\}\\ &= \max\{\lambda_\mu(x * (y * z)), \lambda_\mu(x * y)\}, \end{aligned}$$

$$\begin{aligned}\nu_\mu(x*z) =&\nu_\mu(f(a)*f(c)) = \nu_\mu(f(a*c)) = f^{-1}(\nu_\mu)(a*c)\\ \geq& \min\{f^{-1}(\nu_\mu)(a*(b*c)), f^{-1}(\nu_\mu)(a*b)\}\\ =& \min\{\nu_\mu(f(a*(b*c)), \nu_\mu(f(a*b))\}\\ =& \min\{\nu_\mu(f(a)*(f(b)*f(c))), \nu_\mu(f(a)*f(b))\}\\ =& \min\{\nu_\mu(x*(y*z)), \nu_\mu(x*y)\}.\end{aligned}$$

This completes the proof. □

Definition 4.3. A cubic intuitionistic fuzzy set $\widetilde{A} = \langle A, \mu\rangle$ in a BE-algebra U is called a *cubic intuitinoistic fuzzy subalgebra* of U it satisfies the following conditions: for all $a, b \in U$,

$$\begin{aligned} M_A(a*b) \gg& \text{rmin}\{M_A(a), M_A(b)\}\\ N_A(a*b) \ll& \text{rmax}\{N_A(a), N_A(b)\}\\ \lambda_\mu(a*b) \leq& \max\{\lambda_\mu(a), \lambda_\mu(b)\}\\ \nu_\mu(a*b) \geq& \min\{\nu_\mu(a), \nu_\mu(y)\}.\end{aligned}$$

Definition 4.4. Let $\widetilde{A} = \langle A, \mu\rangle$ and $\widetilde{B} = \langle B, \delta\rangle$ be two intuitionistic fuzzy sets of U and V, respectively. The cartesian product $\widetilde{A} \times \widetilde{B} = \langle U \times V, A \times B, \mu \times \delta\rangle$ is defined by

$$\begin{aligned}(M_A \times M_B)(a, b) =& \text{rmin}\{M_A(a), M_B(b)\}\\ (N_A \times N_B)(a, b) =& \text{rmax}\{N_A(a), N_B(b)\}\\ (\lambda_\mu \times \lambda_\delta)(a, b) =& \max\{\lambda_\mu(a), \lambda_\delta(b)\}\\ (\nu_\mu \times \nu_\delta)(a, b) =& \min\{\nu_\mu(a), \nu_\delta(b)\}\end{aligned}$$

where $A \times B : U \times V \to I[0, 1] \times I[0, 1]$ and $\mu \times \delta : U \times V \to [0, 1] \times [0, 1]$ for all $(a, b) \in U \times V$.

Definition 4.5. A cubic intuitionistic fuzzy subset $\widetilde{A} \times \widetilde{B}$ is called a *cubic intuitionistic fuzzy subalgebra* of $U \times V$ if it satisfies the following conditions: for all $(x_1, y_1), (x_2, y_2) \in U \times V$,

(P1) $(M_A \times M_B)((x_1, y_1) * (x_2, y_2)) \gg \text{rmin}\{(M_A \times M_B)(x_1, y_1), (M_A \times M_B)(x_2, y_2)\}$,
(P2) $(N_A \times N_B)((x_1, y_1) * (x_2, y_2)) \ll \text{rmax}\{(N_A \times N_B)(x_1, y_1), (N_A \times N_B)(x_2, y_2)\}$,
(P3) $(\lambda_\mu \times \lambda_\delta)((x_1, y_1) * (x_2, y_2)) \leq \max\{(\lambda_\mu \times \lambda_\delta)(x_1, y_1), (\lambda_\mu \times \lambda_\delta)(x_2, y_2)\}$,
(P4) $(\nu_\mu \times \nu_\delta)((x_1, y_1) * (x_2, y_2)) \geq \min\{(\nu_\mu \times \nu_\delta)(x_1, y_1), (\nu_\mu \times \nu_\delta)(x_2, y_2)\}$.

Theorem 4.6. *Let $\widetilde{A} = \langle A, \mu\rangle$ and $\widetilde{B} = \langle B, \delta\rangle$ be cubic intuitionistic fuzzy subalgebras of U and V, respectively. Then $\widetilde{A} \times \widetilde{B}$ is a cubic intuitionistic fuzzy subalgebra of $U \times V$.*

Proof. Let (x_1, y_1) and $(x_2, y_2) \in U \times V$. Then we have

$$\begin{aligned}(M_A \times M_B)((x_1, y_1)*(x_2, y_2)) &= (M_A \times M_B)(x_1 * x_2, y_1 * y_2)\\ &= \mathrm{rmin}\{M_A(x_1 * x_2), M_B(y_1 * y_2)\}\\ &\gg \mathrm{rmin}\{\mathrm{rmin}\{M_A(x_1), M_A(x_2)\}, \mathrm{rmin}\{M_B(y_1), M_B(y_2)\}\}\\ &= \mathrm{rmin}\{\mathrm{rmin}\{M_A(x_1), M_B(y_1)\}, \mathrm{rmin}\{M_A(x_2), M_B(y_2)\}\}\\ &= \mathrm{rmin}\{(M_A \times M_B)(x_1, y_1), (M_A \times M_B)(x_2, y_2)\},\end{aligned}$$

$$\begin{aligned}(N_A \times N_B)((x_1, y_1)*(x_2, y_2)) &= (N_A \times N_B)(x_1 * x_2, y_1 * y_2)\\ &= \mathrm{rmax}\{N_A(x_1 * x_2), N_B(y_1 * y_2)\}\\ &\ll \mathrm{rmax}\{\mathrm{rmax}\{N_A(x_1), N_A(x_2)\}, \mathrm{rmax}\{N_B(y_1), N_B(y_2)\}\}\\ &= \mathrm{rmax}\{\mathrm{rmax}\{N_A(x_1), N_B(y_1)\}, \mathrm{rmax}\{N_A(x_2), N_B(y_2)\}\}\\ &= \mathrm{rmax}\{(N_A \times N_B)(x_1, y_1), (N_A \times N_B)(x_2, y_2)\},\end{aligned}$$

$$\begin{aligned}(\lambda_\mu \times \lambda_\delta)((x_1, y_1)*(x_2, y_2)) &= (\lambda_\mu \times \lambda_\delta)(x_1 * x_2, y_1 * y_2)\\ &= \max\{\lambda_\mu(x_1 * x_2), \lambda_\delta(y_1 * y_2)\}\\ &\leq \max\{\max\{\lambda_\mu(x_1), \lambda_\mu(x_2)\}, \max\{\lambda_\delta(y_1), \lambda_\delta(y_2)\}\}\\ &= \max\{\max\{\lambda_\mu(x_1), \lambda_\delta(y_1)\}, \max\{\lambda_\mu(x_2), \lambda_\delta(y_2)\}\}\\ &= \max\{(\lambda_\mu \times \lambda_\delta)(x_1, y_1), (\lambda_\mu \times \lambda_\delta)(x_2, y_2)\} \quad \text{and}\end{aligned}$$

$$\begin{aligned}(\nu_\mu \times \nu_\delta)((x_1, y_1)*(x_2, y_2)) &= (\nu_\mu \times \nu_\delta)(x_1 * x_2, y_1 * y_2)\\ &= \min\{\nu_\mu(x_1 * x_2), \nu_\delta(y_1 * y_2)\}\\ &\geq \min\{\min\{\nu_\mu(x_1), \nu_\mu(x_2)\}, \min\{\nu_\delta(y_1), \nu_\delta(y_2)\}\}\\ &= \min\{\min\{\nu_\mu(x_1), \nu_\delta(y_1)\}, \min\{\nu_\mu(x_2), \nu_\delta(y_2)\}\}\\ &= \min\{(\nu_\mu \times \nu_\delta)(x_1, y_1), (\nu_\mu \times \nu_\delta)(x_2, y_2)\}.\end{aligned}$$

Therefore $\widetilde{A} \times \widetilde{B}$ is a cubic intuitionistic fuzzy subalgebra of $U \times V$. □

Definition 4.7. A cubic intuitionistic fuzzy subset $\widetilde{A} \times \widetilde{B}$ is called a *cubic intuitionistic fuzzy implicative filter* of $U \times V$ if it satisfies the following conditions: for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in U \times V$,

(P5) $(M_A \times M_B)(1, 1) \gg (M_A \times M_B)(x_1, x_2)$ and $(N_A \times N_B)(1, 1) \ll (N_A \times N_B)(x_1, x_2)$,

(P6) $(\lambda_\mu \times \lambda_\delta)(1, 1) \leq (\lambda_\mu \times \lambda_\delta)(x_1, x_2)$ and $(\nu_\mu \times \nu_\delta)(1, 1) \geq (\nu_\mu \times \nu_\delta)((x_1, x_2)$, (P7) $(M_A\times M_B)((x_1,x_2)*(z_1,z_2)) \gg \text{rmin}\{(M_A\times M_B)((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),(M_A\times M_B)((x_1,x_2)*(y_1,y_2))\}$,

(P8) $(N_A\times N_B)((x_1,x_2)*(z_1,z_2)) \ll \text{rmax}\{(N_A\times N_B)((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),(N_A\times N_B)((x_1,x_2)*(y_1,y_2))\}$,

(P9) $(\lambda_\mu\times\lambda_\delta)((x_1,x_2)*(z_1,z_2)) \le \max\{(\lambda_\mu\times\lambda_\delta)((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),(\lambda_\mu\times\lambda_\delta)((x_1,x_2)*(y_1,y_2))\}$,

(P10) $(\nu_\mu\times\nu_\delta)((x_1,x_2)*(z_1,z_2)) \ge \min\{(\nu_\mu\times\nu_\delta)((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),(\nu_\mu\times\nu_\delta)((x_1,x_2)*(y_1,y_2))\}$.

Theorem 4.8. *Let $\widetilde{A}=\langle A,\mu\rangle$ and $\widetilde{B}=\langle B,\delta\rangle$ be cubic intuitionistic fuzzy implicative filters of U and V, respectively. Then $\widetilde{A}\times\widetilde{B}$ is a cubic intuitionistic fuzzy implicative filter of $U\times V$.*

Proof. Let $(x_1,x_2),(y_1,y_2),(x_3,y_3)\in U\times V$. Then we have

$$\begin{aligned}
(M_A\times M_B)(1,1) &= \text{rmin}\{M_A(1),M_B(1)\}\\
&\gg \text{rmin}\{M_A(x_1),M_B(x_2)\}=(M_A\times M_B)(x_1,x_2),\\
(N_A\times N_B)(1,1) &= \text{rmax}\{N_A(1),N_B(1)\}\\
&\ll \text{rmax}\{N_A(x_1),N_B(x_2)\}=(N_A\times N_B)(x_1,x_2),\\
(\lambda_\mu\times\lambda_\delta)(1,1) &= \max\{\lambda_\mu(1),\lambda_\delta(1)\}\le\max\{\lambda_\mu(x_1),\lambda_\delta(x_2)\}\\
&= \max(\lambda_\mu\times\lambda_\delta)(x_1,x_2) \text{ and}\\
(\nu_\mu\times\nu_\delta)(1,1) &= \min\{\nu_\mu(1),\nu_\delta(1)\}\ge\min\{\nu_\mu(x_1),\nu_\delta(x_2)\}\\
&= \min(\nu_\mu\times\nu_\delta)(x_1,x_2).
\end{aligned}$$

$$\begin{aligned}
(M_A\times M_B)((x_1,x_2)*(z_1,z_2)) &= (M_A\times M_B)(x_1*z_1,x_2*z_2)\\
&= \text{rmin}\{M_A(x_1*z_1),M_B(x_2*z_2)\}\\
&\gg \text{rmin}\{\text{rmin}\{M_A(x_1*(y_1*z_1)),M_A(x_1*y_1)\},\\
&\quad \text{rmin}\{M_B(x_2*(y_2*z_2)),M_B(x_2*y_2)\}\}\\
&= \text{rmin}\{\text{rmin}\{M_A(x_1*(y_1*z_1)),M_B(x_2*(y_2*z_2))\},\\
&\quad \text{rmin}\{M_A(x_1*y_1),M_B(x_2*y_2)\}\}\\
&= \text{rmin}\{(M_A\times M_B)(x_1*(y_1*z_1),x_2*(y_2*z_2)),\\
&\quad (M_A\times M_B)(x_1*y_1,x_2*y_2)\}\\
&= \text{rmin}\{(M_A\times M_B)((x_1,x_2)*((y_1,y_2)*(z_1,z_2)),\\
&\quad (M_A\times M_B)((x_1,y_1)*(y_1,y_2))\},
\end{aligned}$$

$$
\begin{aligned}
(N_A \times N_B)((x_1,x_2)*(z_1,z_2)) &= (N_A \times N_B)(x_1*z_1, x_2*z_2)\\
&= \mathrm{rmax}\{N_A(x_1*z_1), N_B(x_2*z_2)\}\\
&\ll \mathrm{rmax}\{\mathrm{rmax}\{N_A(x_1*(y_1*z_1)), N_A(x_1*y_1)\},\\
&\quad \mathrm{rmax}\{N_B(x_2*(y_2*z_2)), N_B(x_2*y_2)\}\}\\
&= \mathrm{rmax}\{\mathrm{rmax}\{N_A(x_1*(y_1*z_1)), N_B(x_2*(y_2*z_2))\},\\
&\quad \mathrm{rmax}\{N_A(x_1*y_1), N_B(x_2*y_2)\}\}\\
&= \mathrm{rmax}\{(N_A \times N_B)(x_1*(y_1*z_1), x_2*(y_2*z_2)),\\
&\quad (N_A \times N_B)(x_1*y_1, x_2*y_2)\}\\
&= \mathrm{rmax}\{(N_A \times N_B)((x_1,x_2)*((y_1,y_2)*(z_1,z_2)),\\
&\quad (N_A \times N_B)((x_1,y_1)*(y_1,y_2))\},
\end{aligned}
$$

$$
\begin{aligned}
(\lambda_\mu \times \lambda_\delta)((x_1,x_2)*(z_1,z_2)) &= (\lambda_\mu \times \lambda_\delta)(x_1*z_1, x_2*z_2)\\
&= \max\{\lambda_\mu(x_1*z_1), \lambda_\delta(x_2*z_2)\}\\
&\le \max\{\max\{\lambda_\mu(x_1*(y_1*z_1)), \lambda_\mu(x_1*y_1)\},\\
&\quad \max\{\lambda_\delta(x_2*(y_2*z_2)), \lambda_\delta(x_2*y_2)\}\}\\
&= \max\{\max\{\lambda_\mu(x_1*(y_1*z_1)), \lambda_\delta(x_2*(y_2*z_2))\},\\
&\quad \max\{\lambda_\mu(x_1*y_1), \lambda_\delta(x_2*y_2)\}\}\\
&= \max\{(\lambda_\mu \times \lambda_\delta)(x_1*(y_1*z_1), x_2*(y_2*z_2)),\\
&\quad (\lambda_\mu \times \lambda_\delta)(x_1*y_1, x_2*y_2)\}\\
&= \max\{(\lambda_\mu \times \lambda_\delta)((x_1,x_2)*((y_1,y_2)*(z_1,z_2)),\\
&\quad (\lambda_\mu \times \lambda_\delta)((x_1,y_1)*(y_1,y_2))\} \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
(\nu_\mu \times \nu_\delta)((x_1,x_2)*(z_1,z_2)) &= (\nu_\mu \times \nu_\delta)(x_1*z_1, x_2*z_2)\\
&= \min\{\nu_\mu(x_1*z_1), \nu_\delta(x_2*z_2)\}\\
&\ge \min\{\min\{\nu_\mu(x_1*(y_1*z_1)), \nu_\mu(x_1*y_1)\},\\
&\quad \min\{\nu_\delta(x_2*(y_2*z_2)), \nu_\delta(x_2*y_2)\}\}\\
&= \min\{min\{\nu_\mu(x_1*(y_1*z_1)), \nu_\delta(x_2*(y_2*z_2))\},\\
&\quad \min\{\nu_\mu(x_1*y_1), \nu_\delta(x_2*y_2)\}\}\\
&= \min\{(\nu_\mu \times \nu_\delta)(x_1*(y_1*z_1), x_2*(y_2*z_2)),\\
&\quad (\nu_\mu \times \nu_\delta)(x_1*y_1, x_2*y_2)\}\\
&= \min\{(\nu_\mu \times \nu_\delta)((x_1,x_2)*((y_1,y_2)*(z_1,z_2)),\\
&\quad (\nu_\mu \times \nu_\delta)((x_1,y_1)*(y_1,y_2))\}.
\end{aligned}
$$

Therefore $\widetilde{A} \times \widetilde{B}$ is a cubic intuitionistic fuzzy implicative filter of $U \times V$. □

5. Conclusion

Jun [9] defined the notion of a cubic intuitionistic fuzzy set, which is extending the concept of a cubic set. He investigated some related properties of it. Senapati et al. [15, 16] applied to this structure to ideals of BCI-algebras and B-algebras. In this paper, we defined the notion of cubic intuitionistic fuzzy filters and cubic intuitionistic fuzzy implicative filters of BE-algebras and studied some related properties of them. We investigated relations between cubic intuitionistic fuzzy filters with cubic intuitionistic fuzzy implicative filters of BE-algebras. These definitions and results can be similarity extended to some algebraic systems such as Lie algebras, Hyper algebras and lattices.

Conflict of interest

All authors declare no conflicts of interest in this paper.

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable suggestions.

References

[1] S. S. Ahn and K. K. So, *On ideals and upper sets in BE-algebras*, Sci. Math. Japon. **68** (2008), 279–285.
[2] S. S. Ahn and K. K. So, *On genrealized upper sets in BE-algebras*, Bull. Korean Math. Soc. **46** (2009), 281–287.
[3] K. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems **31** (1989), no. 3, 343–349.
[4] M. Gulistan, N. Yaqoob, T. Vougiouklis, and H. Abdul Wahab, *Extensions of cubic ideals in weak left almost semihypergroups*, J. Intell. Fuzzy Syst. **34** (2018), no. 6, 4161–4172.
[5] M. Gulistan and A. M. Al-roqi, *Cubic soft sets with applications in BCK/BCI-algebras*, Ann. Fuzzy Math. Inf. **8** (2014), 291–304.
[6] Y. B. Jun and S. S. Ahn, *Fuzzy implicative filters with degree in the interval* $(0, 1]$, J. Comput. Anal. Appl. **15** (2013), 1456–1466.
[7] Y. B. Jun, C. S. Kim, and K. O. Yang, *Cubic sets*, Ann. Fuzzy Math. Inf. **4** (2012), 83–98.
[8] Y. B. Jun, G. Muhiuddin, M. A. Ozturk, and E. H. Roh, *Cubic soft ideals in BCK/BCI-algebras*, J. Comput. Anal. Appl. **22** (2017), 929–940.

[9] Y. B. Jun, *A novel extension of cubic sets and its applications in BCK/BCI-algebras*, Ann. Fuzzy Math. Inf. **14** (2017), no. 5, 475–486.

[10] M. Khan, Y. B. Jun, M. Gulistan, and N. Yaqoob, *The generalized version of Jun's cubic sets in semigroups*, J. Intell. Fuzzy Syst. **28** (2015), no. 2, 947–960.

[11] H. S. Kim and Y. H. Kim, *On BE-algebras*, Sci. Math. Japon. **66** (2007), 113–116.

[12] G. Muhiuddin, F. Feng, and Y. B. Jun, *Subalgebras of BCK/BCI-algebras based on cubic soft sets*, Sci. World J. **2014** (2014), no. 9, Aricle ID 458638.

[13] T. Senapati, C. S. Kim, M. Bhowmik, and M. Pal, *Cubic subalgebras and cubic closed ideals of B-algebras*, Fuzzy Inf. Eng. **7** (2015), no. 2, 129–149.

[14] T. Senapati and K. P. Shum, *Cubic implicative ideals of BCK-algebras*, Missouri J. Math. Sci. **29** (2017), no. 2, 125–138.

[15] T. Senapati, Y. B. Jun, G. Muhiuddin, and K. P. Shum, *Cubic intuitionistic structures applied to ideals of BCI-algebras*, An. St. Univ. Ovidius Constanta **25** (2019), no. 2, 213–232.

[16] T. Senapati, Y. B. Jun, and K. P. Shum, *Cubic intuitionistic subalgebras and closed cubic intuitionistic ideals of B-algebras*, J. Intell. Fuzzy Syst. **36** (2019), no. 2, 1563–1571.

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