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DEFERRED STRONGLY CESÀRO SUMMABLE AND STATISTICALLY CONVERGENT FUNCTIONS

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Abstract. In this paper, firstly we introduce the concepts of deferred Cesàro summable and deferred statistically convergent function, and secondly we introduce the concepts of deferred almost summable and deferred almost statistically convergent functions. Furthermore, we investigate the relations between these concepts.

1. Introduction

A sequence (x_k) is said to be strongly Cesàro summable to the number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0.$$

Nearly all of the transformations used in the theory of summability have undesirable features. For example the Cesàro transformation of any given positive order increases ultimate bounds and oscillations of certain sequences of functions, and does not always preserve uniform convergence, or continuous convergence, of sequences of functions. Deferred Cesàro means have useful properties not possessed by Cesàro's and other well known transformations. R. P. Agnew [1] defined the deferred Cesàro mean $D(p_n, q_n)$ as a generalization of Cesàro mean of real (or complex) valued sequence (x_k) by

$$D_n(x_k) = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \quad n = 1, 2, 3, ...,$$

where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of non-negative integers satisfying $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$.

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In the notation of matrix transformation

$$D_n(x_k) = \sum_{k=0}^{\infty} a_{nk} x_k,$$

where

$$a_{nk} = \begin{cases} \dfrac{1}{q(n) - p(n)} & , \quad p(n) < k \leq q(n) \\ 0 & , \quad \textit{otherwise}. \end{cases}$$

In [1] it is known that $D(p(n), q(n))$ is regular if $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$. Note that $D_n(n-1, n)$ is the identity transformation and $D(0, n)$ is the $(C, 1)$ transformation of (x_k).

A sequence (x_k) is said to be strongly deferred Cesàro convergent to $L \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - L| = 0.$$

The idea of statistical convergence was introduced by Fast [6] and since then has been studied by other authors including [3], [4], [5], [8], [9], [15], [17], [18], [19] and [20]. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. There is a natural relationship between statistical convergence and strong Cesàro summability [4].

A sequence (x_k) is said to be statistically convergent to the number $L \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The concept of deferred statistical convergence was introduced in [10].

A sequence (x_k) is said to be deferred statistically convergent to the number $L \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{p(n) - q(n)} \left| \left\{ p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon \right\} \right| = 0.$$

Let $A = (a_{nk})$ be an infinite matrix and $x = (x_k)$ be a sequence. Let E and F be two non-empty subset of the space w of complex numbers. We write $Ax = (A_n x)$ if $A_n(x) = \sum_{k=1}^{\infty} a_{kn} x_k$ converges for each n. If $x = (x_k) \in E$ implies $Ax \in F$, we say that A defines a matrix transformation from E to F. A matrix A is said to be regular if A transforms every convergent sequence to convergent sequence by preserving the limit.

Following conditions are, by the Silverman-Toeplitz Theorem [13], necessary and sufficient conditions for regularity of $A = (a_{nk})$:

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} a_{nk} = 0$, for each $k \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

Definition 1.1. *[16] Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be Cesàro summable to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{n} \int_1^n f(t)dt = \ell.$$

Definition 1.2. *[2] Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be strongly Cesàro summable to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{n} \int_1^n |f(t) - \ell| dt = 0.$$

Definition 1.3. *[16] Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be statistically convergent to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{n} \big| \big\{ 1 < t \leq n : |f(t) - \ell| \geq \varepsilon \big\} \big| = 0,$$

where the vertical bars indicate the Lebesque measure of the enclosed set. It is denoted by $f(t) \to \ell(S)$.

2. Deferred Strongly Cesàro Summable and Statistically Convergent Functions

Definition 2.1. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be deferred Cesàro summable to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \int_{p(n)}^{q(n)} f(t)dt = \ell.$$

Definition 2.2. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be strongly deferred Cesàro summable to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \int_{p(n)}^{q(n)} |f(t) - \ell| dt = 0.$$

In this case, we write $f(t) \to \ell(D[p, q])$.

It is clear that; if $q(n) = n$ and $p(n) = 0$, then Definition 2.2 coincides with the definition of strongly Cesàro summable function.

Definition 2.3. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1,\infty)$. f is said to be strongly r-deferred Cesàro summable $(0 < r < \infty)$ to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \int_{p(n)}^{q(n)} |f(t)-\ell|^r dt = 0.$$

Definition 2.4. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1,\infty)$. f is said to be deferred statistically convergent to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left|\left\{p(n) < t \le q(n) : |f(t)-\ell| \ge \varepsilon\right\}\right| = 0,$$

where the vertical bars indicate the Lebesque measure of the enclosed set. In this case, we write $f(t) \to \ell(DS[p,q])$.

It is clear that; if $q(n) = n$ and $p(n) = 0$, then Definition 2.4 coincides with the definition of statistical convergence of a function.

3. Inclusion Relations

Theorem 3.1. *Let $\{p(n)\}, \{q(n)\}, \{p^{'}(n)\}$ and $\{q^{'}(n)\}$ be sequences of non-negative integers satisfying $p(n) \le p^{'}(n) < q^{'}(n) \le q(n)$ and*

$$\left\{ \frac{q(n)-p(n)}{q^{'}(n)-p^{'}(n)} \right\}$$

is bounded for all $n \in \mathbb{N}$, then

$$f(t) \to \ell(DS[p,q]) \quad \textit{implies} \quad f(t) \to \ell(DS[p^{'},q^{'}]).$$

Proof. From the inclusion

$$\left\{p'(n) < t \le q'(n) : |f(t)-\ell| \ge \varepsilon\right\} \subset \left\{p(n) < t \le q(n) : |f(t)-\ell| \ge \varepsilon\right\},$$

we can write the inequality

$$\frac{1}{q^{'}(n)-p^{'}(n)} \left|\left\{p'(n) < t \le q'(n) : |f(t)-\ell| \ge \varepsilon\right\}\right|$$

$$\le \frac{q(n)-p(n)}{q^{'}(n)-p^{'}(n)} \frac{1}{q(n)-p(n)} \left|\left\{p(n) < t \le q(n) : |f(t)-\ell| \ge \varepsilon\right\}\right|.$$

After taking limit when $n \to \infty$, desired result is obtained. □

Theorem 3.2. *Let $\{p(n)\}, \{q(n)\}, \{p^{'}(n)\}$ and $\{q^{'}(n)\}$ be sequences of non-negative integers satisfying $p(n) \le p^{'}(n) < q^{'}(n) \le q(n)$ and*

$$\left\{ \frac{q(n)-p(n)}{q^{'}(n)-p^{'}(n)} \right\}$$

is bounded for all $n \in \mathbb{N}$, *then*

$$f(t) \to \ell(D[p,q]) \ \ \textit{implies} \ \ f(t) \to \ell(D[p^{'},q^{'}]).$$

Proof. Since the proof similar to the proof of Theorem 3.1, we omit it. □

Theorem 3.3. *If* $f(t) \to \ell(D[p,q])$, *then* $f(t) \to \ell(DS[p,q])$.

Proof. Let $f(t) \to \ell(D[p,q])$. For an arbitrary $\varepsilon > 0$, we get

$$\begin{aligned}
&\frac{1}{q(n)-p(n)} \int_{p(n)}^{q(n)} |f(t)-\ell| dt \\
&\quad = \Bigg(\frac{1}{q(n)-p(n)} \int_{\substack{p(n) \\ |f(t)-\ell| \geq \varepsilon}}^{q(n)} |f(t)-\ell| dt \\
&\qquad\qquad + \frac{1}{q(n)-p(n)} \int_{\substack{p(n) \\ |f(t)-\ell| < \varepsilon}}^{q(n)} |f(t)-\ell| dt \Bigg) \\
&\quad \geq \frac{1}{q(n)-p(n)} \int_{\substack{p(n) \\ |f(t)-\ell| \geq \varepsilon}}^{q(n)} |f(t)-\ell| dt \\
&\quad \geq \frac{\varepsilon}{q(n)-p(n)} \big| \big\{ p(n) < t \leq q(n) : |f(t)-\ell| \geq \varepsilon \big\} \big|.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \big| \big\{ p(n) < t \leq q(n) : |f(t)-\ell| \geq \varepsilon \big\} \big| = 0,$$

that is, $f(t) \to \ell(DS[p,q])$. □

Theorem 3.4. *If* $f(t)$ *bounded and* $f(t) \to \ell(DS[p,q])$, *then* $f(t) \to \ell(D[p,q])$.

Proof. Suppose that $f(t) \rightarrow \ell(DS[p,q])$ and $f(t)$ is bounded, say $|f(t)-\ell| \le M$ for all $t \in (1,\infty)$. Given $\varepsilon > 0$, we get

$$\begin{aligned}
&\frac{1}{q(n)-p(n)} \int_{p(n)}^{q(n)} |f(t)-\ell|dt \\
&\qquad = \frac{1}{q(n)-p(n)} \left(\underset{|f(t)-\ell|\ge\varepsilon}{\int_{p(n)}^{q(n)}} |f(t)-\ell|dt + \underset{|f(t)-\ell|<\varepsilon}{\int_{p(n)}^{q(n)}} |f(t)-\ell|dt \right) \\
&\qquad \le \frac{1}{q(n)-p(n)} \left(M \underset{|f(t)-\ell|\ge\varepsilon}{\int_{p(n)}^{q(n)}} dt + \varepsilon \underset{|f(t)-\ell|<\varepsilon}{\int_{p(n)}^{q(n)}} dt \right) \\
&\qquad \le M \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < t \le q(n) : |f(t)-\ell| \ge \varepsilon \right\} \right| \\
&\qquad\quad + \varepsilon \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < t \le q(n) : |f(t)-\ell| < \varepsilon \right\} \right|
\end{aligned}$$

and so,

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \int_{p(n)}^{q(n)} |f(t)-\ell|dt = 0.$$

□

Theorem 3.5. *If the sequence* $\left\{ \frac{p(n)}{q(n)-p(n)} \right\}_{n\in\mathbb{N}}$ *is bounded, then* $f(t) \rightarrow \ell(S)$ *implies* $f(t) \rightarrow \ell(DS[p,q])$.

Proof. Let $f(t) \rightarrow \ell(S)$. Then, for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ t \le n : |f(t)-\ell| \ge \varepsilon \right\} \right| = 0.$$

Hence, for every $\varepsilon > 0$, we can write

$$\lim_{n\to\infty} \frac{1}{q(n)} \left| \left\{ t \le q(n) : |f(t)-\ell| \ge \varepsilon \right\} \right| = 0.$$

From the inclusion

$$\left\{ p(n) < t \le q(n) : |f(t)-\ell| \ge \varepsilon \right\} \subseteq \left\{ t \le q(n) : |f(t)-\ell| \ge \varepsilon \right\}$$

and the inequality

$$\left| \left\{ p(n) < t \le q(n) : |f(t)-\ell| \ge \varepsilon \right\} \right| \le \left| \left\{ t \le q(n) : |f(t)-\ell| \ge \varepsilon \right\} \right|,$$

we have

$$\frac{1}{q(n)-p(n)}\left|\left\{p(n) < t \leq q(n) : |f(t)-\ell| \geq \varepsilon\right\}\right|$$
$$\leq \left(1+\frac{p(n)}{q(n)-p(n)}\right)\frac{1}{q(n)}\left|\left\{t \leq q(n) : |f(t)-\ell| \geq \varepsilon\right\}\right|$$

and so, we obtain $f(t) \to \ell(DS[p,q])$. □

Theorem 3.6. *Let $q(n) = n$ for all $n \in \mathbb{N}$. Then, $f(t) \to \ell(DS[p,n])$ if and only if $f(t) \to \ell(S)$.*

Proof. Assume that $f(t) \to \ell(DS[p,n])$. Then for each $n \in \mathbb{N}$, letting $p(n) > n^{(1)} > p(n^{(1)}) = n^{(2)} > p(n^{(2)}) = n^{(3)} > \ldots$, we may write

$$\begin{aligned}\left\{t \leq n : |f(t)-\ell| \geq \varepsilon\right\} &= \left\{t \leq n^{(1)} : |f(t)-\ell| \geq \varepsilon\right\}\\ &\quad \cup\left\{n^{(1)} < t \leq n : |f(t)-\ell| \geq \varepsilon\right\},\end{aligned}$$

$$\begin{aligned}\left\{t \leq n^{(1)} : |f(t)-\ell| \geq \varepsilon\right\} &= \left\{t \leq n^{(2)} : |f(t)-\ell| \geq \varepsilon\right\}\\ &\quad \cup\left\{n^{(1)} < t \leq n^{(2)} : |f(t)-\ell| \geq \varepsilon\right\}\end{aligned}$$

and

$$\begin{aligned}\left\{t \leq n^{(2)} : |f(t)-\ell| \geq \varepsilon\right\} &= \left\{t \leq n^{(3)} : |f(t)-\ell| \geq \varepsilon\right\}\\ &\quad \cup\left\{n^{(2)} < t \leq n^{(3)} : |f(t)-\ell| \geq \varepsilon\right\}.\end{aligned}$$

This process may be continued until for some positive integer h depending on n, we obtain

$$\begin{aligned}\left\{t \leq n^{(h-1)} : |f(t)-\ell| \geq \varepsilon\right\} &= \left\{t \leq n^{(h)} : |f(t)-\ell| \geq \varepsilon\right\}\\ &\quad \cup\left\{n^{(h-1)} < t \leq n^{(h)} : |f(t)-\ell| \geq \varepsilon\right\},\end{aligned}$$

where $n^{(h)} \geq 1$ and $n^{(h+1)} = 0$. Therefore, we can write

$$\frac{1}{n}\left|\left\{1 < t \leq n : |f(t)-\ell| \geq \varepsilon\right\}\right| = \sum_{r=0}^{h} \frac{n^{(r)}-n^{(r+1)}}{n} t_r$$

for every n, where

$$t_r = \frac{1}{n^{(r)}-n^{(r+1)}}\left\{n^{(r)} < t \leq n^{(r+1)} : |f(t)-\ell| \geq \varepsilon\right\}.$$

If we consider a matrix $A = (a_{nr})$ as

$$a_{nr} = \begin{cases} \dfrac{n^{(r)}-n^{(r+1)}}{n} & , \quad r = 0,1,2,\ldots,h \\ 0 & , \quad \textit{otherwise} \end{cases}$$

where $n^{(0)} = n$, then the sequence

$$\left\{\frac{1}{n}\left|\left\{1 < t \le n : |f(t) - \ell| \ge \varepsilon\right\}\right|\right\}$$

is the (a_{nr}) transformation of the sequence (t_r). Since the matrix (a_{nr}) is a regular and the sequence

$$\left\{\frac{1}{n^{(r)} - n^{(r+1)}}\left|\left\{n^{(r)} < t \le n^{(r+1)} : |f(t) - \ell| \ge \varepsilon\right\}\right|\right\}$$

is convergent to zero, we have

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{1 < t \le n : |f(t) - \ell| \ge \varepsilon\right\}\right| = 0.$$

Conversely, since $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ is bounded for $q(n) = n$, by Theorem 3.5, we have $f(t) \to \ell(S)$ implies $f(t) \to \ell(DS[p, q])$. □

4. Deferred Almost Summability

A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be a Banach limit if

(i) $\phi(x) \ge 0$ when the sequence (x_k) has $x_k \ge 0$ for all k,

(ii) $\phi(e) = 1$ where $e = (1, 1, 1, ...)$ and

(iii) $\phi(x_{k+1}) = \phi(x_k)$, for all $(x_k) \in l_\infty$.

A sequence $(x_k) \in l_\infty$ is said to be almost convergent to the value L if all of its Banach limits equal to L. Lorentz [12] has given the following characterization.

A bounded sequence (x_k) is almost convergent to the number L if and only if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_{m+k} = L,$$

uniformly in m.

Maddox [14] has defined strongly almost convergent sequence as follows:

A bounded sequence (x_k) is said to be strongly almost convergent to the number L if and only if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_{m+k} - L| = 0,$$

uniformly in m.

By a lacunary sequence [7], we mean an increasing integer sequence $\theta = \{k_n\}$ such that $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$ as $n \to \infty$. The intervals determined by θ will be denoted by $I_n = (k_{n-1}, k_n]$.

Let $\theta = \{k_n\}$ be a lacunary sequence and f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be strongly lacunary almost summable to ℓ if

$$\lim_{n\to\infty} \frac{1}{h_n} \int_{k_{n-1}+m+1}^{k_n+m} |f(t) - \ell| dt = 0,$$

uniformly in m.

Let $\theta = \{k_n\}$ be a lacunary sequence and f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be lacunary almost statistically convergent to $\ell = \ell_f$ if

$$\lim_{n\to\infty} \frac{1}{h_n} \big|\big\{k_{n-1} + m + 1 \le t \le k_n + m : |f(t) - \ell| \ge \varepsilon\big\}\big| = 0,$$

uniformly in m.

Let (λ_n) be a non-decreasing sequence of positive numbers tending to ∞, and $\lambda_{n+1} - \lambda_n \le 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean [11] is defined by

$$\frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} x_k.$$

Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be strongly λ-almost summable to $\ell = \ell_f$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int_{k=n-\lambda_n+m+1}^{n+m} |f(t) - \ell| dt = 0,$$

uniformly in m.

Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be λ-statistically almost convergent to $\ell = \ell_f$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \big|\big\{n - \lambda_n + m + 1 \le t \le n + m : |f(t) - \ell| \ge \varepsilon\big\}\big| = 0,$$

uniformly in m.

Definition 4.1. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be deferred almost summable to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \int_{p(n)+m+1}^{q(n)+m} f(t) dt = \ell,$$

uniformly in m.

Definition 4.2. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1, \infty)$. f is said to be strongly deferred almost summable to $\ell = \ell_f$ if*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \int_{p(n)+m+1}^{q(n)+m} |f(t) - \ell| dt = 0,$$

uniformly in m. In this case, we write $f(t) \to \ell(\hat{D}[p, q])$.

Definition 4.3. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1,\infty)$. f is said to be strongly r-deferred almost summable $(0<r<\infty)$ to $\ell=\ell_f$ if*

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\int_{p(n)+m+1}^{q(n)+m}|f(t)-\ell|^r=0,$$

uniformly in m. In this case, we write $f(t)\to\ell(\hat{D}^r[p,q])$.

It is clear that;

- If $q(n)=n$ and $p(n)=0$, then Definition 4.2 coincides with the definition of strong almost summability of f.
- Let $\theta=\{k_n\}$ be a lacunary sequence. If we consider $q(n)=k_n$ and $p(n)=k_{n-1}$, then Definition 4.2 coincides with the lacunary strong almost summabilitiy of f.
- If $q(n)=n$ and $p(n)=n-\lambda_n$, then Definition 4.2 coincides with the strong λ-almost summability of f.

Definition 4.4. *Let f be a real valued function, measurable (in the Lebesgue sense) in the interval $(1,\infty)$. f is said to be deferred almost statistically convergent to $\ell=\ell_f$ if for every $\varepsilon>0$,*

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\left|\left\{p(n)+m<t\leq q(n)+m:|f(t)-\ell|\geq\varepsilon\right\}\right|=0,$$

uniformly in m. In this case, we write $f(t)\to\ell(\hat{DS}[p,q])$.

It is clear that;

- If $q(n)=n$ and $p(n)=0$, then Definition 4.4 coincides with the definition of almost statistical convergence of f.
- Let $\theta=\{k_n\}$ be a lacunary sequence. If we consider $q(n)=k_n$ and $p(n)=k_{n-1}$, then Definition 4.4 coincides with the lacunary almost statistical convergence of f.
- If $q(n)=n$ and $p(n)=n-\lambda_n$, then Definition 4.4 coincides with the almost λ-statistical convergence of f.

Since the proofs of the following theorems are similar to the proofs of the theorems given in the Section 3, we give the theorems without proof in order not to repeat them.

Theorem 4.5. *Let $\{p(n)\},\{q(n)\},\{p^{'}(n)\}$ and $\{q^{'}(n)\}$ be sequences of non-negative integers satisfying $p(n)\leq p^{'}(n)<q^{'}(n)\leq q(n)$ for all $n\in\mathbb{N}$ and*

$$\limsup_{n\to\infty}\frac{q(n)-p(n)}{q^{'}(n)-p^{'}(n)}<\infty,$$

then $f(t)\to\ell(\hat{DS}[p,q])$ implies $f(t)\to\ell(\hat{DS}[p^{'},q^{'}])$.

Theorem 4.6. *Let $\{p(n)\}, \{q(n)\}, \{p^{'}(n)\}$ and $\{q^{'}(n)\}$ be sequences of non-negative integers satisfying $p(n) \leq p^{'}(n) < q^{'}(n) \leq q(n)$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty} \frac{q(n)-p(n)}{q^{'}(n)-p^{'}(n)} < \infty,$$

then $f(t) \to \ell(\hat{D}[p,q])$ implies $f(t) \to \ell(\hat{D}[p^{'},q^{'}])$.

Theorem 4.7. *If $f(t)$ is strongly deferred almost convergent to ℓ, then $f(t)$ is deferred almost statistically convergent to ℓ, that is, if $f(t) \to \ell(\hat{D}[p,q])$, then $f(t) \to \ell(\hat{DS}[p,q])$.*

Theorem 4.8. *If $f(t)$ is bounded and deferred almost statistically convergent to ℓ, then $f(t)$ is strongly deferred almost convergent to ℓ, that is, if f is bounded and $f(t) \to \ell(\hat{DS}[p,q])$, then $f(t) \to \ell(\hat{D}[p,q])$.*

Theorem 4.9. *If the sequence $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ is bounded, then $f(t) \to \ell(\hat{S})$ implies $f(t) \to \ell(\hat{DS}[p,q])$.*

Theorem 4.10. *Let $q(n) = n$ for all $n \in \mathbb{N}$. Then, $f(t) \to \ell(\hat{DS}[p,n])$ if and only if $f(t) \to \ell(\hat{S})$.*

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