

THE STABILITY OF THE PERIODIC STOKES EQUATIONS ON R^n

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ABSTRACT. In this paper, we will consider the periodic stationary Stokes equations on R^n . For the cube of the period, we set $\Omega = \prod_{i=1}^n (0, L_i)$. And we will study the stability of the solutions on various functional spaces, for the Stokes equations on R^n .

1. Introduction

Obłozja [12, 13] studied the Hyers-Ulam stability of the linear differential equations, $y'(x) + g(x)y(x) = r(x)$. And it was based on Ulam's idea for the stability of homomorphism. Thereafter, this subject for different types of differential equations was discussed by many mathematicians. Consider an open interval $I = (a, b)$ of \mathbb{R} with $-\infty \leq a < b \leq +\infty$. And, for an n times continuously differentiable function, $y : I \rightarrow \mathbb{C}$, let the linear differential equation,

$$(1.1) \quad \mathfrak{F}(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0$$

satisfies on I . For any $\varepsilon > 0$, we assume that there exists a solution $y_0 : I \rightarrow \mathbb{C}$ to (1.1) such that

$$(1.2) \quad |y(x) - y_0(x)| \leq K(\varepsilon), \quad \text{for all } x \in I,$$

where $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$, if $y : I \rightarrow \mathbb{C}$ satisfies the differential inequality

$$(1.3) \quad |\mathfrak{F}(y^{(n)}, y^{(n-1)}, \dots, y', y, x)| \leq \varepsilon.$$

Then we say that the differential equation (1.1) satisfies the Hyers-Ulam stability. One can find many interesting results for the linear differential equations from the following references, [3, 4, 5, 8, 9, 11, 14, 17]. Also, for the partial differential equations, one can refer to [1, 2, 6, 7, 10, 15].

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In this paper, we will study the stability of the solutions for the stationary Stokes equations on R^n , with the periodic boundary condition. For the stability of the periodic stationary Stokes equations, Roh-Jung[10] investigated for R^2 . They used $\Omega = (0, L) \times (0, L)$ for the cube of the period. In this paper, we will use more general cube $\Omega = \prod_{i=1}^n (0, L_i)$ of the period.

We will consider the stationary Stokes problem with the following form: For a given \mathbf{f} , we want to find \mathbf{u} and p such that

$$(1.4) \quad \begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } R^n \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } R^n \end{aligned}$$

$$(1.5) \quad \mathbf{u}(x + L_i \mathbf{e}_i) = \mathbf{u}(x) \quad \text{for all } x \in R^n,$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of R^n , L_i is the period in the i -th direction and $\Omega = \prod_{i=1}^n (0, L_i)$ is the cube of the period.

In next section, we will introduce the functional setting for the solution spaces, with the periodic boundary condition. Finally, we will study the stability of the solutions of the stationary Stokes equations on R^n .

2. Functional spaces for the solution space

For the functional spaces of the solutions of the Stokes equations we will use the Lebesgue space $L^2(R^n)$ with the periodic boundary condition. For the Sobolev space of functions which are in $L^2(\Omega)$, with all their derivatives of order $\leq m$, we set by $H^m(\Omega)$. And with the inner product and the norm,

$$(\mathbf{u}, \mathbf{v})_m = \sum_{|\alpha| \leq m} (D^\alpha \mathbf{u}, D^\alpha \mathbf{v}) \quad \text{and} \quad |\mathbf{u}|_m = [(\mathbf{u}, \mathbf{u})_m]^{1/2},$$

$H^m(\Omega)$ is a Hilbert space. We also consider the subset $H_p^m(\Omega)$ of $H^m(\Omega)$, the space of periodic functions with the cube Ω of period :

$$(2.1) \quad \mathbf{u}(\mathbf{x} + L_i \mathbf{e}_i) = \mathbf{u}(\mathbf{x}) \quad \text{for all } i = 1, \dots, n.$$

For $m = 0$, $H_p^0(\Omega)$ means $L^2(\Omega)$.

Then, $H_p^m(\Omega)$ is also a Hilbert space and the functions in $H_p^m(\Omega)$ are characterized by their Fourier series expansion

$$(2.2) \quad H_p^m(\Omega) = \left\{ \mathbf{u} : \mathbf{u} = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}} e^{2i\pi \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \bar{c}_{\mathbf{k}} = c_{-\mathbf{k}}, \sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \frac{\mathbf{k}}{L} \right|^{2m} |c_{\mathbf{k}}|^2 < \infty \right\},$$

where $\frac{\mathbf{x}}{L} = (\frac{x_1}{L_1}, \dots, \frac{x_n}{L_n})$ and $\frac{\mathbf{k}}{L} = (\frac{k_1}{L_1}, \dots, \frac{k_n}{L_n})$. We also denote

$$(2.3) \quad \dot{H}_p^m(\Omega) = \{ \mathbf{u} \in H_p^m(\Omega) \text{ of type (2.2)} : c_0 = 0 \}.$$

Now, one note that $\dot{H}_p^m(\Omega)$ is a Hilbert space with the norm $[\sum_{\mathbf{k} \in \mathbb{Z}^n} |\frac{\mathbf{k}}{L}|^{2m} |c_{\mathbf{k}}|^2]^{1/2}$, and $\dot{H}_p^{-m}(\Omega)$ is the dual space of $\dot{H}_p^m(\Omega)$ for all $m \in \mathbb{N}$. Now, we introduce an

new Hilbert space, $\mathbf{H}_p^m(\Omega) = \{H_p^m(\Omega)\}^n$ with the inner product and the norm,

$$((\mathbf{u}, \mathbf{v})) = \sum_{i=1}^n \left(\frac{\partial \mathbf{u}}{\partial x_i}, \frac{\partial \mathbf{v}}{\partial x_i} \right), \quad \|\mathbf{u}\| = \{((\mathbf{u}, \mathbf{u}))\}^{1/2}.$$

And for the solution spaces, we will use the following functional spaces,

$$(2.4) \quad \mathbf{V} = \{\mathbf{u} \in \mathbf{H}_p^1(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } R^n\},$$

$$(2.5) \quad \mathbf{H} = \{\mathbf{u} \in \mathbf{H}_p^0(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } R^n\},$$

where \mathbf{V} is also a Hilbert space with this norm. More about the functional spaces for the solutions with the periodic boundary condition can be found in [16].

The stationary Stokes problem (1.4) with the periodic boundary condition (1.5) have the following; Given $\mathbf{f} \in \dot{\mathbf{H}}_p^0(\Omega)$ or $\dot{\mathbf{H}}_p^{-1}(\Omega)$, find $\mathbf{u} \in \dot{\mathbf{H}}_p^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$(2.6) \quad -\Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega.$$

For the Fourier expansions of \mathbf{u} , p and \mathbf{f} , we set as the following :

$$\mathbf{u} = \sum_{\mathbf{k} \in \mathbb{Z}^n} u_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \quad p = \sum_{\mathbf{k} \in \mathbb{Z}^n} p_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \quad \mathbf{f} = \sum_{\mathbf{k} \in \mathbb{Z}^n} f_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}},$$

where $\frac{\mathbf{x}}{L} = (\frac{x_1}{L_1}, \dots, \frac{x_n}{L_n})$.

Now, by (2.6), for every $\mathbf{k} \neq 0$ one obtain

$$(2.7) \quad 4\pi^2 \left| \frac{\mathbf{k}}{L} \right|^2 u_{\mathbf{k}} + 2\pi i \frac{\mathbf{k}}{L} p_{\mathbf{k}} = f_{\mathbf{k}}$$

and

$$(2.8) \quad \frac{\mathbf{k}}{L} \cdot u_{\mathbf{k}} = 0,$$

where $\frac{\mathbf{k}}{L} = (\frac{k_1}{L_1}, \dots, \frac{k_n}{L_n})$.

Taking the scalar product of (2.7) with $\frac{\mathbf{k}}{L}$ and using (2.8) we find

$$(2.9) \quad p_{\mathbf{k}} = \frac{\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}}{2\pi i \left| \frac{\mathbf{k}}{L} \right|^2} \text{ for } \mathbf{k} \in \mathbb{Z}^n, \mathbf{k} \neq 0$$

and by (2.7) we have

$$(2.10) \quad u_{\mathbf{k}} = \frac{1}{4\pi^2 \left| \frac{\mathbf{k}}{L} \right|^2} \left(f_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}) \frac{\mathbf{k}}{L}}{\left| \frac{\mathbf{k}}{L} \right|^2} \right) \text{ for } \mathbf{k} \in \mathbb{Z}^n, \mathbf{k} \neq 0.$$

3. The stability of the solutions

Now, we will study the stability of the solutions for the n -dimensional stationary Stokes equations with the periodic boundary condition. The existence of the solutions for the n -dimensional stationary Stokes equations with the periodic boundary condition is very well known. With assumption of the existence of the solution, we will study the various stability of the solutions.

THEOREM 3.1. *Let the function $\mathbf{v} \in \dot{\mathbf{H}}_p^2(\Omega)$ and $q \in \dot{\mathbf{H}}_p^1(\Omega)$ satisfy the equations*

$$(3.1) \quad -\Delta \mathbf{v} + \nabla q - \mathbf{f} = \mathbf{g} \quad \text{in } \Omega, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega,$$

where $\|\mathbf{g}\|_{L^2} \leq \varepsilon$ and $\mathbf{f}, \mathbf{g} \in \dot{\mathbf{H}}_p^0(\Omega)$. Then there exist $\mathbf{u} \in \dot{\mathbf{H}}_p^2(\Omega)$ and $p \in \dot{\mathbf{H}}_p^1(\Omega)$ satisfying

$$(3.2) \quad -\Delta \mathbf{u} + \nabla p - \mathbf{f} = 0 \quad \text{in } \Omega, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega$$

such that

$$(3.3) \quad \|\mathbf{u} - \mathbf{v}\|_{H^i} \leq K_i \|\mathbf{g}\|_{L^2} \leq K_i \varepsilon \quad \text{for } i = 0, 1, 2,$$

$$(3.4) \quad \|p - q\|_{H^i} \leq M_i \|\mathbf{g}\|_{L^2} \leq M_i \varepsilon \quad \text{for } i = 0, 1$$

for some constants K_i and M_i depending on L_{max} where $L_{max} = \max\{L_1, \dots, L_n\}$.

Proof. For existence of the solution $\mathbf{u} \in \dot{\mathbf{H}}_p^2(Q)$ and $p \in \dot{\mathbf{H}}_p^1(Q)$, one can have from (2.9) and (2.10). Next, to obtain (3.3) and (3.4) we denote the Fourier expansions of \mathbf{v} , q and \mathbf{g} as the followings;

$$\mathbf{v} = \sum_{\mathbf{k} \in \mathbb{Z}^2} v_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \quad q = \sum_{\mathbf{k} \in \mathbb{Z}^2} q_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \quad \mathbf{g} = \sum_{\mathbf{k} \in \mathbb{Z}^2} g_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}},$$

where $\frac{\mathbf{x}}{L} = (\frac{x_1}{L_1}, \dots, \frac{x_n}{L_n})$. By (2.9), (2.10) and (3.1) we have

$$(3.5) \quad q_{\mathbf{k}} = \frac{\frac{\mathbf{k}}{L} \cdot (f_{\mathbf{k}} + g_{\mathbf{k}})}{2\pi i |\frac{\mathbf{k}}{L}|^2} \quad \text{for } \mathbf{k} \in \mathbb{Z}^n, \quad \mathbf{k} \neq 0$$

and

$$(3.6) \quad v_{\mathbf{k}} = \frac{1}{4\pi^2 |\frac{\mathbf{k}}{L}|^2} \left((f_{\mathbf{k}} + g_{\mathbf{k}}) - \frac{(\frac{\mathbf{k}}{L} \cdot [f_{\mathbf{k}} + g_{\mathbf{k}}]) \frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} \right) \quad \text{for } \mathbf{k} \in \mathbb{Z}^n, \quad \mathbf{k} \neq 0,$$

where $\frac{\mathbf{k}}{L} = (\frac{k_1}{L_1}, \dots, \frac{k_n}{L_n})$. And, due to (2.9), (2.10) and (3.2) we can get

$$(3.7) \quad p_{\mathbf{k}} = \frac{\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}}{2\pi i |\frac{\mathbf{k}}{L}|^2} \quad \text{for } \mathbf{k} \in \mathbb{Z}^n, \quad \mathbf{k} \neq 0$$

and

$$(3.8) \quad u_{\mathbf{k}} = \frac{1}{4\pi^2 |\frac{\mathbf{k}}{L}|^2} \left(f_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}) \frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} \right) \quad \text{for } \mathbf{k} \in \mathbb{Z}^n, \quad \mathbf{k} \neq 0,$$

where $\frac{\mathbf{k}}{L} = (\frac{\mathbf{k}_1}{L_1}, \dots, \frac{\mathbf{k}_n}{L_n})$. Therefore, from (3.5) - (3.8), we obtain the Fourier expansions of $\mathbf{u} - \mathbf{v}$ and $p - q$ as the following ;

$$(3.9) \quad u_{\mathbf{k}} - v_{\mathbf{k}} = -\frac{1}{4\pi^2 |\frac{\mathbf{k}}{L}|^2} \left(g_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot g_{\mathbf{k}}) \frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} \right), \quad p_{\mathbf{k}} - q_{\mathbf{k}} = -\frac{\frac{\mathbf{k}}{L} \cdot g_{\mathbf{k}}}{2\pi i |\frac{\mathbf{k}}{L}|^2},$$

where $\frac{\mathbf{k}}{L} = (\frac{\mathbf{k}_1}{L_1}, \dots, \frac{\mathbf{k}_n}{L_n})$. Then, for $|\mathbf{k}| > 1$, from (3.9) we have

$$|u_{\mathbf{k}} - v_{\mathbf{k}}| \leq \frac{L_{max}^2}{4\pi^2} |g_{\mathbf{k}}|,$$

where $L_{max} = \max\{L_1, \dots, L_n\}$. Next, to prove for $|\mathbf{k}| = 1$, we assume $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_n)$ as $\mathbf{k}_i = 0$ for all $i \neq j$ and $\mathbf{k}_j = 1$ or -1 . Then for $g_{\mathbf{k}} = (g_{\mathbf{k}}^1, \dots, g_{\mathbf{k}}^n)$, we have

$$g_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot g_{\mathbf{k}}) \frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} = w_{\mathbf{k}},$$

where $w_{\mathbf{k}}^i = g_{\mathbf{k}}^i$ for all $i \neq j$, and $w_{\mathbf{k}}^j = 0$. Therefore, we have

$$(3.10) \quad |u_{\mathbf{k}} - v_{\mathbf{k}}| \leq \frac{L_{max}^2}{4\pi^2} |g_{\mathbf{k}}|.$$

Hence, we have

$$(3.11) \quad \|\mathbf{u} - \mathbf{v}\|_{L^2} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |u_{\mathbf{k}} - v_{\mathbf{k}}|^2 \right]^{1/2} \leq \frac{L_{max}^2}{4\pi^2} \|\mathbf{g}\|_{L^2} \leq K_1 \varepsilon$$

and similarly we obtain

$$(3.12) \quad \|p - q\|_{L^2} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |p_{\mathbf{k}} - q_{\mathbf{k}}|^2 \right]^{1/2} \leq \frac{L_{max}}{2\pi} \|\mathbf{g}\|_{L^2} \leq M_1 \varepsilon.$$

Also, for H^1 -norm, by (3.9) we get

$$(3.13) \quad \|\mathbf{u} - \mathbf{v}\|_{H^1} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \frac{\mathbf{k}}{L} \right|^2 |u_{\mathbf{k}} - v_{\mathbf{k}}|^2 \right]^{1/2} \leq \frac{L_{max}}{2\pi^2} \|\mathbf{g}\|_{L^2} \leq K_2 \varepsilon$$

and

$$(3.14) \quad \|p - q\|_{H^1} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \frac{\mathbf{k}}{L} \right|^2 |p_{\mathbf{k}} - q_{\mathbf{k}}|^2 \right]^{1/2} \leq \frac{1}{2\pi} \|\mathbf{g}\|_{L^2} \leq M_2 \varepsilon.$$

Similarly, for H^2 -norm, we have

$$(3.15) \quad \|\mathbf{u} - \mathbf{v}\|_{H^2} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \frac{\mathbf{k}}{L} \right|^4 |u_{\mathbf{k}} - v_{\mathbf{k}}|^2 \right]^{1/2} \leq \frac{1}{4\pi^2} \|\mathbf{g}\|_{L^2} \leq K_3 \varepsilon.$$

Hence, by (3.11) - (3.15), we complete the proof. \square

REMARK 3.2. We will see that the constant in (3.10) is optimal. Assume that $L_{max} = \max\{L_1, \dots, L_n\} = L_j$. Consider the function g as $g_{\mathbf{k}} = 0$ if $\mathbf{k} \neq (0, \dots, 0, 1, 0, \dots, 0)$, and

$$g_{\mathbf{k}} = \left(\frac{\epsilon}{\sqrt{n-1}}, \dots, \frac{\epsilon}{\sqrt{n-1}}, 0, \frac{\epsilon}{\sqrt{n-1}}, \dots, \frac{\epsilon}{\sqrt{n-1}} \right)$$

if $\mathbf{k} = (0, \dots, 0, 1, 0, \dots, 0)$ which means that every component equal to zero except j -th component. Then the inequality (3.10) becomes to the equality. Therefore, our Hyers-Ulam constant is **optimal**.

COROLLARY 3.3. Assume that the function $\mathbf{v} \in \dot{\mathbf{H}}_p^2(\Omega)$ and $q \in \dot{\mathbf{H}}_p^1(\Omega)$ satisfy the equations

$$(3.16) \quad -\Delta \mathbf{v} + \nabla q - \mathbf{f} = \mathbf{g} \text{ in } \Omega, \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,$$

where $\mathbf{f} \in \dot{\mathbf{H}}_p^0(\Omega)$ and $\mathbf{g} \in \mathbf{H}$ with $\|\mathbf{g}\|_{L^2} \leq \epsilon$. Then there exist $\mathbf{u} \in \dot{\mathbf{H}}_p^2(\Omega)$ and $p \in \dot{\mathbf{H}}_p^1(\Omega)$ satisfying

$$(3.17) \quad -\Delta \mathbf{u} + \nabla p - \mathbf{f} = 0 \text{ in } \Omega, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega.$$

such that

$$(3.18) \quad \|\mathbf{u} - \mathbf{v}\|_{H^i} \leq K_i \|\mathbf{g}\|_{L^2} \leq K_i \epsilon \quad \text{for } i = 0, 1, 2,$$

$$(3.19) \quad \|p - q\|_{H^i} = 0 \quad \text{for } i = 0, 1,$$

for some constants K_i depending on L_{max} where $L_{max} = \max\{L_1, \dots, L_n\}$.

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