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ON THE ALMOST SHADOWING PROPERTY FOR HOMEOMORPHISMS

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Abstract. In this paper we investigate some properties concerning the set of shadowable points for homeomorphisms. Then we show that the almost shadowing property is preserved by a topological conjugacy between homeomorphisms. Also, we give an example to illustrate our results.

1. Introduction and preliminaries

The notion of the shadowing property plays an important role in the qualitative theory of dynamical systems(see [1,8]). Walters proved that every expansive homeomorphism with the shadowing property is topologically stable(see [9, Theorem 4]). Variant notions of dynamical systems were studied in the pointwise viewpoint(see [2,3,5–7]). Morales [7] introduced the notion of shadowable points and proved that the shadowing property is equivalent to all points to be shadowable in the class of homeomorphisms on a compact metric space. The purpose of our work is to study some properties of the shadowing property of homeomorphisms from pointwise viewpoint.

We recall some basic notions of dynamical systems which is used in the sequel.

Let X be a compact metric space with metric d and $f : X \to X$ be a homeomorphism.

For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in X is said to be a δ-*pseudo orbit* if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Given $\varepsilon > 0$, a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is said to be ε-*traced*(or *shadowed*) by $z \in X$ if $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. A subset B of X is said to be f-*invariant* for a homeomorphism f if $f(B) = B$.

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We say that a sequence $(x_n)_{n\in\mathbb{Z}}$ of X is *through* some subset $K \subset X$ if $x_0 \in K$. We say that a homeomorphism $f : X \to X$ has the shadowing property *through* a subset K of X if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f through K can be ε-shadowed(see [8]). We say that a homeomorphism $f : X \to X$ has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is ε-shadowed by a point in X. The theory of the shadowing property have been widely studied in the class of homeomorphisms on a compact metric space(see [1, 8]).

We recall the notion of shadowable points for homeomorphisms which is defined to be a point such that the shadowing lemma holds for pseudo orbits through the point.

We say that a point $x \in X$ is a *shadowable point* if for every $\epsilon > 0$ there exists $\delta_x > 0$ such that every δ_x-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ passing through $x_0 = x$ can be ϵ-shadowed by some point. We denoted by $\mathrm{Sh}(f)$ the set of shadowable points of f(see [7, Definition 1.1]).

We remark that a homeomorphism of a compact metric space has the *shadowing property through* a compact subset K of X if and only if every point in K is shadowable(see [7, Lemma 2.3]).

We recall that a homeomorphism$f : X \to X$ has the *almost shadowing property* if $\mathrm{Sh}(f)$ is dense in X(see [7]). Let (X, d) and (Y, d') be compact metric spaces. We denote by $H(X)$ and $H(Y)$ the families of homeomorphisms on X and Y, respectively. We say that $f \in H(X)$ and $g \in H(Y)$ are *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, where the homeomorphism h is called a *topological conjugacy* between f and g.

In this paper we investigate the invariance of the set of shadowable points in the class of homeomorphisms. Then we show that the almost shadowing property is preserved by a topological conjugacy in the class of homeomorphisms on a compact metric space. Also, we give an example to illustrate our results. More precisely, we state our main result.

Theorem 1.1. *Let $f \in H(X)$ and $g \in H(Y)$ be topologically conjugate. Then f has the almost shadowing property if and only if so is g.*

2. Proof of Theorem 1.1

In this section we investigate some properties about the set of shadowable points for homeomorphisms. Then we give a proof of Theorem 1.1 and an example concerning our results.

It is well known that the closure of an invariant set is also invariant. From Theorem 1.1 in [7], we obtain the following result.

Lemma 2.1. *Let $f : X \to X$ be a homeomorphism on a compact metric space. Then*

1. $\mathrm{Sh}(f)$ *and* $\overline{\mathrm{Sh}(f)}$ *are invariant sets;*
2. f *has the shadowing property if and only if* $\mathrm{Sh}(f) = X$.

Here $\overline{A}$ *denotes the closure of a set* A.

We see that the shadowing property of continuous maps of a compact metric space is preserved by a topological conjugacy([1, Theorem 2.36]). For the proof of Theorem 1.1, we need the following result.

Lemma 2.2. *Let $f \in H(X)$ and $g \in H(Y)$ be topologically conjugate by a homeomorphism $h : X \to Y$. Then* $\mathrm{Sh}(g) = h(\mathrm{Sh}(f))$.

Proof. We claim that $\mathrm{Sh}(g) = h(\mathrm{Sh}(f))$. Let $y \in h(\mathrm{Sh}(f))$ and $\varepsilon > 0$. Then there exists $x \in \mathrm{Sh}(f)$ such that $h(x) = y$. It follows from uniform continuity of h that there is $\varepsilon_1 > 0$ such that $d(x, x') < \varepsilon_1$ implies $d'(h(x), h(x')) < \varepsilon$. Since x is shadowable in X, there exists $\delta_1 > 0$ such that each δ_1-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ of f passing through $x_0 = x$ is ε_1-shadowed by some point. Then it is enough to show that each δ-pseudo orbit $\{y_i\}_{i\in\mathbb{Z}}$ of g passing through $y_0 = y$ is ε-shadowed by some point in Y. Putting $x_i = h^{-1}(y_i)$ for each $i \in \mathbb{Z}$, since $d'(g(y_i), y_i) < \delta$ for all $i \in \mathbb{Z}$, we have

$$\begin{aligned} d(f(x_i), x_{i+1}) &= d(f \circ h^{-1}(y_i), h^{-1}(y_{i+1})) \\ &= d(h^{-1} \circ g(y_i), h^{-1}(y_{i+1})) < \delta_1 \end{aligned}$$

for each $i \in \mathbb{Z}$ and $x_0 = x = h^{-1}(y)$. Thus $\{x_i\}_{i\in\mathbb{Z}}$ is a δ_1-pseudo orbit of f passing through $x_0 = x$ and so it is ε_1-shadowed by some $z' \in X$.

Hence we have

$$d'(h \circ f^i(z'), h(x_i)) = d'(g^i \circ h(z'), y_i) < \varepsilon$$

for each $i \in \mathbb{Z}$. Thus the δ-pseudo orbit $\{y_i\}_{i\in\mathbb{Z}}$ of g passing through $y_0 = y$ is ε-shadowed by some $z = h(z') \in Y$. Hence $y \in \mathrm{Sh}(g)$. This proves that $h(\mathrm{Sh}(f)) \subset \mathrm{Sh}(g)$. By applying the similar argument for h^{-1}, we can prove that the converse inclusion is true. Hence we have $\mathrm{Sh}(g) = h(\mathrm{Sh}(f))$. This completes the proof. □

We say that a homeomorphism $f : X \to X$ has the *shadowing property on* $K \subset X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that every

δ-pseudo-orbit of f in K can be ε-shadowed(see [8]). We see that this definition is weaker than the shadowing property through K.

LEMMA 2.3. *[8, Lemma 1.1.1] If a homeomorphism $f : X \to X$ on a compact metric space X has the finite shadowing property on $A \subset X$, then f has the shadowing property on A.*

From Lemma 2.3, we obtain the following result.

LEMMA 2.4. *[4, Lemma 3.1] Let X be a compact metric space and let A be a dense invariant subspace of X. Then a homeomorphism $f : X \to X$ has the shadowing property if and only if the restricted homeomorphism $f|_A : A \to A$ has the shadowing property.*

From Lemma 2.4, we obtain the following result.

PROPOSITION 2.5. *Let $f : X \to X$ be a homeomorphism of a compact metric space X and $\mathrm{Sh}(f)$ be a dense subset of X. If the restricted homeomorphism $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ has the shadowing property on $\mathrm{Sh}(f)$, then f also has the shadowing property.*

REMARK 2.6. From our results, we see that if $\mathrm{Sh}(f)$ is closed in X, then $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ has the shadowing property on $\mathrm{Sh}(f)$.

Now, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that f has the almost shadowing property. Then $\mathrm{Sh}(f)$ is a dense subset of X. By the topological property of the density and Lemma 2.2, we see that $h(\mathrm{Sh}(f))$ is a dense subset of Y and $\mathrm{Sh}(g) = h(\mathrm{Sh}(f))$. Thus there exists a dense subset $\mathrm{Sh}(g)$ of Y. Hence g has the almost shadowing property. Similarly, we can prove that the converse is true. This completes the proof. □

We give an example to illustrate main results. In fact, there are a non-compact subset $\mathrm{Sh}(f)$ of a compact metric space X and a homeomorphism $f : X \to X$ such that $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ does not have the shadowing property on $\mathrm{Sh}(f)$.

EXAMPLE 2.7. *[7, Example 2.1] Let $X = C \cup [1, 2]$ be the compact metric subspace of $\mathbb{R}$, where C be the ternary Cantor set of $[0, 1]$. Take $f : X \to X$ as the identity of X. Then we see that $\mathrm{Sh}(f) = C \setminus \{1\}$ is a non-compact subset of X and $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ does not have the shadowing property on $\mathrm{Sh}(f)$. Furthermore, $f|_C : C \to C$ has the shadowing property on C. Here $C = \overline{\mathrm{Sh}(f)}$ is a compact invariant subspace of X.*

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