

THE OHM-RUSH CONTENT FUNCTION III: COMPLETION, GLOBALIZATION, AND POWER-CONTENT ALGEBRAS

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ABSTRACT. One says that a ring homomorphism $R \rightarrow S$ is *Ohm-Rush* if extension commutes with arbitrary intersection of ideals, or equivalently if for any element $f \in S$, there is a unique smallest ideal of R whose extension to S contains f , called the *content* of f . For Noetherian local rings, we analyze whether the completion map is Ohm-Rush. We show that the answer is typically ‘yes’ in dimension one, but ‘no’ in higher dimension, and in any case it coincides with the content map having good algebraic properties. We then analyze the question of when the Ohm-Rush property globalizes in faithfully flat modules and algebras over a 1-dimensional Noetherian domain, culminating both in a positive result and a counterexample. Finally, we introduce a notion that we show is strictly between the Ohm-Rush property and the weak content algebra property.

1. Introduction

One of the most useful and important methods of constructing a new commutative ring from an existing one is via polynomial extension. The polynomial algebra $R[x]$ inherits many of the properties of R . There are several important keys in the study of polynomial extensions and their relation to the base ring. Such algebras are of course faithfully flat over R . Moreover there is a natural map from elements of $R[x]$ to finitely generated ideals of R via the well known content map, which sends an element f to the ideal generated by the coefficients of f , denoted $c(f)$.

In [15], Ohm and Rush define a function c from elements of an arbitrary R -algebra S to the set of ideals of R . For S faithfully flat over R , they give criteria for when c can serve as an appropriate generalization of the content function in polynomial extensions. Such extensions are called *content algebras*. In a variation of this concept, Rush in [17] examined what he called *weak content algebras*, whose properties are easier to check than those of a content

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algebra. Also essentially at the same time as Ohm and Rush, Eakin and Silver [3] defined and examined some of the same properties as in [15] and applied the results to locally polynomial rings. More recently in a series of three papers [5–7], the authors further examined content and weak content algebras, as well as defined an intermediary notion (*semicontent* algebras). Nasehpour in [13] studied content algebras that satisfied the additional property that $c(fg) = c(f)c(g)$, calling them *Gaussian*. That is, a Gaussian algebra is an Ohm-Rush algebra in which the content function is a homomorphism of multiplicative semigroups.

We introduce the basic definitions and terminologies in Section 2 that will be used throughout. We then give necessary and sufficient conditions (Theorem 2.4) for the completion of a local Noetherian ring R to be content over R (Gaussian even) in terms of the extension of ideals. Combining this with the work of Hassler and Wiegand [9] on the extension of modules, we prove the main result of this section (Theorem 2.9) which characterizes when \hat{R} is Gaussian over a 1-dimensional reduced local Noetherian ring R . From this it follows (Corollary 2.10) that if R is an analytically irreducible Noetherian local integral domain of dimension one, then \hat{R} is Gaussian over R . Moreover Example 2.11 shows that one cannot expect to extend this result to higher dimensions.

In Section 3 we characterize when an R -algebra is a content algebra, where R is a Dedekind domain. We also show that in many typical cases over a 1-dimensional base, the property of being an Ohm-Rush algebra globalizes (Theorems 3.3 and 3.6). However, the Ohm-Rush property does *not* globalize in general, even for a faithfully flat (albeit non-Noetherian) algebra over \mathbb{Z} . As far as we know, we give here the first known counterexample to globalization of the Ohm-Rush property (see Example 3.8). We then apply our results in Example 3.9 to show that a known locally polynomial algebra is in fact Gaussian.

In the final Section 4 we define and examine power-content algebras, a property that lies strictly between Ohm-Rush algebras and weak content algebras.

2. When is the \mathfrak{m} -adic completion Ohm-Rush?

In this section we examine conditions on a local Noetherian ring R so that its \mathfrak{m} -adic completion is a content (in fact Gaussian) algebra over R , but first we begin with some of the basic definitions that will be used throughout the paper.

Definition 2.1 (See [15]; current nomenclature from [5]). Let R be a ring, M an R -module, and $f \in M$. Then the (*Ohm-Rush*) *content* of f is given by

$$c(f) := \bigcap \{I \subseteq R \text{ ideal} \mid f \in IM\}.$$
¹

¹In [5], we use the symbol Ω for this function.

If $f \in c(f)M$ for all $f \in M$, we say that M is an *Ohm-Rush module*; if M is moreover an R -algebra, we say that it is an *Ohm-Rush algebra* over R .

When $f \in M$, we introduce the notation $L_f := \{I \subseteq R \text{ ideal} \mid f \in IM\}$.

Definition 2.2. Let $R \rightarrow S$ be an Ohm-Rush algebra. We say that it is

- (1) a *content algebra* [15] if it is faithfully flat and for any $f, g \in S$, there is some $n \in \mathbb{N}$ with $c(f)^n c(g) = c(f)^{n-1} c(fg)$,
- (2) a *Gaussian algebra* [13] if it is faithfully flat and for any $f, g \in S$, we have $c(fg) = c(f)c(g)$. That is, one may choose $n = 1$ in (1).

We recall some basic facts about the above properties. In an Ohm-Rush algebra $c(fg) \subseteq c(f)c(g)$ for all $f, g \in S$ [17, Proposition 1.1(i)], while an Ohm-Rush algebra is flat if and only if $c(af) = ac(f)$ for all $a \in R$ and $f \in S$ [15, Corollary 1.6]. It is well known that if an R -module is flat, then the extension of ideals distributes over *finite* intersection. On the other hand, an R -module is Ohm-Rush if and only if the extension of ideals distributes over *arbitrary* intersections [3, (2.2)]. Also in a content algebra, prime ideals extend to prime ideals [17, Theorem 1.2]. (In fact by that same result we know that prime ideals extend to prime ideals in a faithfully flat weak content algebra - see Section 4 for a definition.)

Common examples of content algebras include polynomial algebras [2, 12], monoid algebras where the monoid is torsion-free and cancellative [14], and power series algebras when the base ring is Noetherian [4]. On the other hand, the Gaussian algebra property is much stronger, and for instance only holds for polynomial algebras over an integral domain when the domain is Prüfer [8, 16, 18]. See also [6] for properties of ring elements $f \in S$ for which $c(f)c(g) = c(fg)$ for all $g \in S$.

Recall that a ring extension $R \subseteq S$ is called *cyclically pure* if every ideal of R is contracted from S . That is, for every ideal I of R , we have $IS \cap R = I$. In particular, any faithfully flat extension is cyclically pure.

Proposition 2.3. *Let $R \rightarrow S$ be a cyclically pure (e.g. faithfully flat) ring homomorphism. Let $f \in S$. Suppose there is some $r \in R$ such that $fS = rS$. Then $c(f) = rR$, so $f \in c(f)S$.*

If every $f \in S$ has this property, then S is a Gaussian R -algebra.

Proof. Let $f \in S$ and $r \in R$ with $fS = rS$. Let I be an ideal of R such that $f \in IS$. Then $r \in fS \cap R \subseteq IS \cap R = I$ by cyclic purity. Since this holds for all such ideals I , we have $r \in c(f)$. On the other hand, since $f \in rS$, we have $c(f) \subseteq rR$. Hence $c(f) = rR$. Thus, $f \in rS = c(f)S$.

Now suppose that *every* principal ideal of S is extended from a principal ideal of R . Let $f, g \in S$. There exist $a, b \in R$ such that $fS = aS$ and $gS = bS$. Moreover, we have $fgS = abS$. So by the above, $c(f)c(g) = (aR)(bR) = abR = c(fg)$.

Since we have shown that the map is Ohm-Rush, we can check flatness by showing that $c(af) = ac(f)$ for all $a \in R$, $f \in S$, due to [15, Corollary 1.6].

So let $f \in S$ and $a, r \in R$ with $fS = rS$. Then $afS = arS$, so $c(af) = arR = ac(f)$. Finally, for any maximal ideal \mathfrak{m} of R , we have $\mathfrak{m}S \cap R = \mathfrak{m}$, whence $\mathfrak{m}S \neq S$, finishing the proof of faithful flatness and the Gaussian property. \square

In the case of the completion map, we have the following characterization:

Theorem 2.4. *Let (R, \mathfrak{m}) be a Noetherian local ring, and let (\hat{R}, \mathfrak{n}) be its \mathfrak{m} -adic completion. Then the following are equivalent:*

- (1) *The map $R \rightarrow \hat{R}$ is Ohm-Rush.*
- (2) *The map $R \rightarrow \hat{R}$ is Gaussian.*
- (3) *For any $g \in \hat{R}$, there is some $r \in R$ such that $g\hat{R} = r\hat{R}$. That is, every principal ideal of \hat{R} is extended from a principal ideal of R .*
- (4) *Every ideal of \hat{R} is extended from R .*

Proof. We have (2) \implies (1) by definition, and Proposition 2.3 shows that (3) \implies (2). To see that (4) \implies (3), let $g \in \hat{R}$. Then there is some ideal I of R with $g\hat{R} = I\hat{R}$, which by [6, Remark 3.3] must be a principal ideal of R .

It remains only to show that (1) \implies (4). So suppose the completion map is Ohm-Rush. Let $0 \neq g \in \hat{R}$. For each $t \in \mathbb{N}$, choose $g_t \in R$ such that $g - g_t \in \mathfrak{n}^t$. Set $I_t := g_tR + \mathfrak{m}^t$. Then $I_t\hat{R} = g\hat{R} + \mathfrak{n}^t$. By the Krull intersection theorem applied to the quotient ring $\hat{R}/g\hat{R}$, then we have $\bigcap_t (I_t\hat{R}) = g\hat{R}$. Thus, $c(g) \subseteq \bigcap_t I_t = (\bigcap_t I_t)\hat{R} \cap R = (\bigcap_t I_t\hat{R}) \cap R = g\hat{R} \cap R$. Hence, $g\hat{R} \subseteq c(g)\hat{R} \subseteq (g\hat{R} \cap R)\hat{R} \subseteq g\hat{R}$. It follows that $g\hat{R} = c(g)\hat{R}$. Now let J be any nonzero ideal of \hat{R} . Say $J = (f_1, \dots, f_n)$ with each $f_i \neq 0$. Then by the above, $J = (\sum_{i=1}^n c(f_i))\hat{R}$. \square

This yields the following useful necessary criterion for Ohm-Rushness of the completion map:

Corollary 2.5. *Let (R, \mathfrak{m}) be a Noetherian local ring and (\hat{R}, \mathfrak{n}) its \mathfrak{m} -adic completion. Suppose there is some nonzero $g \in \hat{R}$ such that $g\hat{R} \cap R = 0$. Then the completion map $R \rightarrow \hat{R}$ is not Ohm-Rush.*

Proof. If it is Ohm-Rush, then by Theorem 2.4, $g\hat{R} = r\hat{R}$ for some $r \in R$. In particular, $r \neq 0$. But then $0 = g\hat{R} \cap R = (rR)\hat{R} \cap R = rR \neq 0$, which is absurd. \square

This theorem allows us to use a result of Hassler and Wiegand regarding extended *modules*. First recall the following:

Definition 2.6 ([9]). Let R and S be Noetherian local rings and $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ a flat local homomorphism. Given a finitely generated S -module N , we say N is *extended* (from R) provided there is an R -module M such that $S \otimes_R M$ is isomorphic to N as an S -module.

Remark 2.7. Note that the R -module M is forced to be finitely generated, by [1, Ch. 1, §3(6), Proposition 11]. Moreover, if $\mathfrak{m}S = \mathfrak{n}$, then $\mu_R(M) = \mu_S(N)$

(where we use the symbol μ to denote the minimal number of generators of a module). To see this, recall the well-known formula that for a finite-length R -module L , if $S/\mathfrak{m}S$ has finite vector-space dimension over R , then we have $\lambda_S(S \otimes_R L) = \lambda_S(S/\mathfrak{m}S) \cdot \lambda_R(L)$. Thus,

$$\begin{aligned} \mu_S(S \otimes_R M) &= \lambda_S((S \otimes_R M)/\mathfrak{n}(S \otimes_R M)) = \lambda_S((S \otimes_R M)/\mathfrak{m}(S \otimes_R M)) \\ &= \lambda_S(S \otimes_R M/\mathfrak{m}M) = \lambda_S(S/\mathfrak{m}S) \cdot \lambda_R(M/\mathfrak{m}M) = 1 \cdot \mu_R(M). \end{aligned}$$

In particular, N is a cyclic S -module if and only if M is a cyclic R -module.

It follows that J is an extended ideal from R (in the usual sense) $\iff S/J$ is an extended S -module from R in the Hassler-Wiegand sense. For one direction, if J is an extended ideal, then $J = IS$ for some ideal I of R , whence $S/J = S/IS \cong S \otimes_R R/I$. For the other direction, if S/J is extended from R , say $S/J \cong S \otimes_R M$, then by the above, we have $1 = \mu_S(S/J) = \mu_R(M)$, whence M is cyclic. In particular, $M \cong R/I$, where $I = \text{ann}_R(M)$. Thus, $S/J \cong S \otimes_R R/I = S/IS$ as S -modules, whence $J = IS$.

Recall the following useful result on extended modules:

Proposition 2.8 ([9, Corollary 4.5]). *Let (R, \mathfrak{m}) and (S, \mathfrak{n}) be one-dimensional Noetherian local rings, and let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism such that $\mathfrak{n} = \mathfrak{m}S$ and the induced map $R/\mathfrak{m} \rightarrow S/\mathfrak{m}S$ of residue fields is an isomorphism. Let $K(S)$ be the quotient ring of S obtained by inverting the complement of the union of the height zero primes of S . The following are equivalent:*

- (1) *For any finitely generated S -module N such that $K(S) \otimes_S N$ is projective as a $K(S)$ -module, N is extended from R .*
- (2) *The natural map $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is bijective.*

Next is the main theorem of this section.

Theorem 2.9. *Let (R, \mathfrak{m}) be a 1-dimensional reduced Noetherian local ring. Then the following are equivalent:*

- (1) *Every finitely generated \hat{R} -module is extended from R .*
- (2) *Every ideal of \hat{R} is extended from R .*
- (3) *Every prime ideal of \hat{R} is extended from R .*
- (4) *\hat{R} is reduced and the natural map $\text{Spec}(\hat{R}) \rightarrow \text{Spec}(R)$ is a bijection.*
- (5) *The completion map $R \rightarrow \hat{R}$ is Ohm-Rush.*
- (6) *The completion map $R \rightarrow \hat{R}$ is Gaussian.*

Proof. If \hat{R} is reduced and the Spec map is bijective, then $K(\hat{R})$ is a finite product of fields, so all modules over it are projective. Hence by Proposition 2.8, all finite \hat{R} -modules are extended. Thus (4) \implies (1). The fact that (1) \implies (2) follows from Remark 2.7. The implication (2) \implies (3) is trivial. The equivalence of the three conditions (2), (5) and (6) follows from Theorem 2.4.

For the implication (3) \implies (4), suppose that all prime ideals of \hat{R} are extended from R . If P is a prime ideal of \hat{R} , let J be an ideal of R with

$P = J\hat{R}$, then $P \cap R = J\hat{R} \cap R = J$ by purity of the map $R \rightarrow \hat{R}$. Hence, $P = J\hat{R} = (P \cap R)\hat{R}$. Now let $P, P' \in \text{Spec } \hat{R}$ with $P \cap R = P' \cap R$. Then $P = (P \cap R)\hat{R} = (P' \cap R)\hat{R} = P'$. Hence, the Spec map is injective. For surjectivity, we need to show that all prime ideals of R are contracted from prime ideals of \hat{R} , so let \mathfrak{p} be a prime ideal of R . If $\mathfrak{p} = \mathfrak{m}$, then $\mathfrak{p} = \mathfrak{m}\hat{R} \cap R = \mathfrak{m}$ is contracted from the maximal ideal of \hat{R} . If \mathfrak{p} is a minimal prime of R , then by the Going-Down property (associated to the inclusion $\mathfrak{p} \subset \mathfrak{m}$ and the contraction $\mathfrak{m} = \hat{\mathfrak{m}} \cap R$) there is some prime P of \hat{R} lying over \mathfrak{p} . Since prime ideals of \hat{R} are extended, we have $P = (P \cap R)\hat{R} = \mathfrak{p}\hat{R}$. Thus, $\mathfrak{p}\hat{R}$ is prime and $\mathfrak{p}\hat{R} \cap R = \mathfrak{p}$, so the Spec map is surjective. Thus, it is bijective, and we have moreover shown that every prime ideal of \hat{R} is of the form $\mathfrak{p}\hat{R}$, where \mathfrak{p} is a prime ideal of R .

Finally, we want to prove that \hat{R} is reduced. For this, let P_1, \dots, P_n be the minimal primes of \hat{R} . Then we have $P_j = \mathfrak{p}_j\hat{R}$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal primes of R . So the nilradical of \hat{R} is $\bigcap_{i=1}^n P_i = \bigcap_i (\mathfrak{p}_i\hat{R}) = (\bigcap_i \mathfrak{p}_i)\hat{R}$ since the completion map is flat. But $\bigcap_{i=1}^n \mathfrak{p}_i$ is the nilradical of R , hence 0 since R is reduced. Thus, the nilradical of \hat{R} is also zero. \square

In particular, for a 1-dimensional Noetherian local ring whose completion is a domain, the completion map is Gaussian:

Corollary 2.10. *Let (R, \mathfrak{m}) be a Noetherian local integral domain of Krull dimension 1. Then the map $R \rightarrow \hat{R}$ is Gaussian if and only if it is Ohm-Rush if and only if R is analytically irreducible.*

However, the above results do not extend to higher dimension, even in the case of regular local rings essentially of finite type over a field, as the following example demonstrates:

Example 2.11. Let $R = k[x, y, z_1, \dots, z_n]_{(x, y, z_1, \dots, z_n)}$ where k is a field with $\text{char}(k) \neq 2$. Then $\hat{R} = k[[x, y, z_1, \dots, z_n]]$. Let $f := x^2 - y^2(y + 1)$. By the Eisenstein irreducibility criterion applied to the prime element $y + 1$ of $k[[y]]$ in the polynomial extension $R = k[[y]][x, z_1, \dots, z_n]$, we have that f is a prime element of R . However, the coefficients in the Maclaurin expansion of the square root of $y + 1$ can be written with denominators that only involve powers of 2. Hence, since $1/2 \in R \subseteq \hat{R}$, $y + 1$ has a square root in \hat{R} . Therefore, $f = x^2 - y^2(y + 1) = (x + y\sqrt{y + 1})(x - y\sqrt{y + 1})$ presents $f \in \hat{R}$ as a nontrivial product of elements of the maximal ideal of \hat{R} . It follows that $f\hat{R}$ is neither prime nor the unit ideal. Hence, since the extension $f\hat{R}$ of the prime ideal fR of R is not prime in \hat{R} , $R \rightarrow \hat{R}$ is not a content algebra. But then by Theorem 2.4, it cannot even be Ohm-Rush.

We also note that if $\text{char}(k) = 2$, then $f := x^3 - y^3(y + 1)$ is also irreducible in R by the Eisenstein criterion. We leave it to the reader to apply (the generalized) Hensel's Lemma to see that $y + 1$ has a cube root in $k[[y]]$, from

which it follows that f is not irreducible in \hat{R} . Thus in this case as well \hat{R} is not a content R -algebra.

3. Globalization of the Ohm-Rush property over a one-dimensional Noetherian domain

In this section, we analyze the question of whether and when the Ohm-Rush property globalizes over a 1-dimensional Noetherian domain base. That is, when N is a faithfully flat R -module such that $N_{\mathfrak{m}}$ is an Ohm-Rush $R_{\mathfrak{m}}$ -module for every maximal ideal \mathfrak{m} of R , does it follow that N is an Ohm-Rush R -module? We will see that the answer is ‘yes’ if

- (1) R is a Dedekind domain and N is a Noetherian module over some R -algebra (See Theorem 3.4), or
- (2) R is a 1-dimensional integral domain and N is a Noetherian R -algebra that is an integral domain (See Theorem 3.6).

We end the section with two examples, one of which is a faithfully flat \mathbb{Z} -algebra that is Ohm-Rush locally but not globally.

We start with a criterion to detect when a faithfully flat algebra over a DVR is a content algebra.

Proposition 3.1. *Let (R, \mathfrak{m}) be a DVR, and let S be a faithfully flat R -algebra. Then S is a content R -algebra (equivalently Gaussian) if and only if $\mathfrak{m}S \in \text{Spec } S$ and $\bigcap_n \mathfrak{m}^n S = 0$.*

Proof. By [10, Exercise 1.1(5)], $J = \bigcap_n \mathfrak{m}^n S$ is a prime ideal of S . If $R \rightarrow S$ is a content algebra, then we know that $\mathfrak{m}S$ and (0) are prime ideals of S , and $\text{ht}(\mathfrak{m}S) = \text{ht}(\mathfrak{m}) = 1$ by [7, Theorem 5.4]. So the intersection must be 0.

For the converse, first note by [15, Proposition 2.1] that S is an Ohm-Rush R -algebra. Then since $\mathfrak{m}S$ is prime, and \mathfrak{m} is the only maximal ideal of R , an appeal to [5, Theorem 4.7] finishes the proof. \square

For Dedekind domain bases, the criteria for a faithfully flat map being Ohm-Rush are a bit more subtle. We begin with a lemma about detecting the Ohm-Rush property over a 1-dimensional Noetherian domain:

Lemma 3.2. *Let R be a 1-dimensional Noetherian domain. Let M be a flat R -module (or at least an R -module where extension of ideals commutes with finite intersection), and let $0 \neq f \in M$. Then $f \in c(f)M$ if and only if L_f satisfies the descending chain condition.*

Proof. If $f \in c(f)M$, then $c(f) \neq 0$, and $c(f)$ is the unique minimal element of L_f . Hence, the ideals of L_f are in one-to-one order-preserving correspondence with the ideals of the Artinian ring $R/c(f)$. Hence L_f satisfies DCC.

Conversely, suppose L_f satisfies DCC. Since it is nonempty (e.g. $R \in L_f$), it contains a minimal element. Moreover, suppose that $I, I' \in L_f$, with I minimal. Then we have $f \in IM \cap I'M = (I \cap I')M$, so $I \cap I' \in L_f$. By

minimality of I , it follows that $I = I \cap I'$, whence $I \subseteq I'$. Thus, every element of L_f contains I , so $I = c(f) \in L_f$, whence $f \in c(f)M$. \square

We next present a criterion for a flat module over a Dedekind domain to be Ohm-Rush.

Theorem 3.3. *Let R be a Dedekind domain, and let M be a flat (i.e., torsion-free) R -module. Let $0 \neq f \in M$. Then $f \in c(f)M$ if and only if the following two conditions hold:*

- (1) $L_f \cap \text{Spec } R$ is finite, and
- (2) There is some $n \in \mathbb{N}$ such that for all $\mathfrak{p} \in \text{Spec } R$, $f \notin \mathfrak{p}^n M$.

Hence M is an Ohm-Rush module if and only if (1) and (2) hold for every nonzero $f \in M$.

Proof. First suppose $f \in c(f)M$. Since R is a Dedekind domain and $c(f) \neq 0$, there is a unique (up to ordering) prime decomposition of $c(f)$. Say $c(f) = \prod_{j=1}^t P_j^{n_j}$, where the P_j s are distinct maximal ideals of R , $t \geq 0$, and each $n_j \geq 1$. If $P \in L_f \cap \text{Spec } R$, then $f \in PM$, whence $c(f) \subseteq P$. Thus some $P_j \subseteq P$, so $P_j = P$. That is, $L_f \cap \text{Spec } R = \{P_j \mid 1 \leq j \leq t\}$ is finite. Now set $n := \max\{n_j \mid 1 \leq j \leq t\} + 1$. If $f \in \mathfrak{p}^n M$ for some \mathfrak{p} , then $\mathfrak{p} = P_j$ for some $1 \leq j \leq t$. Then $f \in P_j^n M$, whence $c(f) \subseteq P_j^n$. It follows that $P_j^{n_j} \subseteq P_j^n$, whence $n_j \geq n$, which is a contradiction.

Conversely, suppose $f \notin c(f)M$. Then by Lemma 3.2, L_f admits an infinite descending chain $J_0 \supsetneq J_1 \supsetneq J_2 \supsetneq \dots$. Suppose that (1) holds. Say $L_f \cap \text{Spec } R = \{P_1, \dots, P_t\}$. Then these are the only primes in the prime decompositions of the J_i . By the proper containment condition, for each $j \in \mathbb{N}$, there is some $a_j \in \{1, \dots, t\}$ such that $J_{j+1} \subseteq J_j P_{a_j}$. Let $n \in \mathbb{N}$. By the pigeonhole principle, $J_{(n-1)t+1} \in P_i^n$ for some $1 \leq i \leq t$. Since n was arbitrary, (2) fails. \square

The above then allows us to show that the property of being a faithfully flat Ohm-Rush algebra over a Dedekind domain globalizes, at least when the target ring is Noetherian. We will later see (cf. Example 3.8) a counterexample over \mathbb{Z} when the target ring is not Noetherian.

Theorem 3.4. *Let R be a Dedekind domain. Let $R \rightarrow S$ be a ring homomorphism. Let M be a Noetherian S -module, considered as an R -module via restriction of scalars. Suppose that for each maximal ideal \mathfrak{m} of R , the $R_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$ is faithfully flat and Ohm-Rush. Then M is a faithfully flat Ohm-Rush R -module.*

Proof. Since faithful flatness globalizes, we have that M is faithfully flat (and hence also torsion-free) over R . Let $0 \neq f \in M$. We want to show that it satisfies conditions (1) and (2) of Theorem 3.3.

First note that since M is torsion-free over R , for any maximal ideal \mathfrak{m} of R we have $f/1 \neq 0$ in $M_{\mathfrak{m}}$. Then by Theorem 3.3 (applied to the flat Ohm-Rush $R_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$), there is some positive integer n (dependent on \mathfrak{m}) with $f/1 \notin \mathfrak{m}^n M_{\mathfrak{m}}$. It follows that $f \notin \mathfrak{m}^n M$.

Next we prove (1). Accordingly, let $0 \neq f \in M$. Suppose $f \in \mathfrak{p}M$ for infinitely many $\mathfrak{p} \in \text{Spec } R$. Enumerate a countable set of such \mathfrak{p}_i , $i \in \mathbb{N}$, and for each i find the unique n_i such that $f \in \mathfrak{p}_i^{n_i} M \setminus \mathfrak{p}_i^{n_i+1} M$, whose existence is guaranteed by the previous paragraph. Let $Q_0 := Sf$, and inductively for each $i \geq 1$, set $Q_i := (Q_{i-1} :_M \mathfrak{p}_i^{n_i})$. This is an ascending chain of S -submodules of M (i.e., $Q_{i-1} \subseteq Q_i$ for all $i \geq 1$). Since M is a Noetherian S -module, we just need to prove the chain is strict to get a contradiction.

Claim 1. For all $t \geq 1$, $Q_{t-1} \subseteq \mathfrak{p}_t^{n_t} M$.

Proof of Claim 1. Let $x \in Q_{t-1}$. Then by an easy induction, we have

$$\left(\prod_{i=1}^{t-1} \mathfrak{p}_i^{n_i} \right) x \subseteq Q_0 = Sf \subseteq \left(\prod_{i=1}^t \mathfrak{p}_i^{n_i} \right) M.$$

To see the last containment, we have

$$\begin{aligned} f \in \bigcap_{i=1}^t (\mathfrak{p}_i^{n_i} M) &= \left(\bigcap_{i=1}^t \mathfrak{p}_i^{n_i} \right) M && \text{by flatness} \\ &= \left(\prod_{i=1}^t \mathfrak{p}_i^{n_i} \right) M && \text{since } R \text{ is a Dedekind domain.} \end{aligned}$$

Thus, we have

$$\begin{aligned} x &\in \left(\left(\prod_{i=1}^t \mathfrak{p}_i^{n_i} \right) M :_M \left(\prod_{i=1}^{t-1} \mathfrak{p}_i^{n_i} \right) \right) \\ &= \left(\prod_{i=1}^t \mathfrak{p}_i^{n_i} :_R \prod_{i=1}^{t-1} \mathfrak{p}_i^{n_i} \right) M && \text{again by flatness} \\ &= \mathfrak{p}_t^{n_t} M && \text{again since } R \text{ is Dedekind. } \quad \square \end{aligned}$$

Claim 2. For any $t \geq 0$, we have $Q_t \not\subseteq \mathfrak{p}_t M$.

Proof of Claim 2. Let W be the complement of the set $\bigcup_{i=1}^t \mathfrak{p}_i$ in R . Write $A = W^{-1}R$, $B = W^{-1}S$, and $N = W^{-1}M$. It is enough to show that $W^{-1}Q_t$ is not contained in $\mathfrak{p}_t N$. Since localization commutes with colon and finite products, we may assume (by replacing R with A) that R is a *semilocal* Dedekind domain, hence a PID. After which, we have that each \mathfrak{p}_i is principal, say with generator p_i . Thus, $f = p_1^{n_1} \cdots p_t^{n_t} g$ for some $g \in M$. Therefore, $Q_t = (Sf :_M (\prod_{i=1}^t p_i^{n_i}))$, whence $g \in Q_t$. So if the claim is false, $g \in \mathfrak{p}_t M$, which implies that $f \in \mathfrak{p}_t^{n_t+1} M$, a contradiction to our construction of n_t . This finishes the proof of the claim. \square

Since $Q_{t-1} \subsetneq Q_t$ for all $t \geq 1$, we have a strict ascending chain of S -submodules of M , contradicting the fact that M is Noetherian as an S -module.

Finally, to prove (2), let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the elements of $L_f \cap \text{Spec } R$. By the second paragraph of the proof, for each $1 \leq i \leq k$ there is some positive integer n_i with $f \notin \mathfrak{p}_i^{n_i} M$. Set $n := \max\{n_i : 1 \leq i \leq k\}$. Then for each i , we have $f \notin \mathfrak{p}_i^n M$. But for any other maximal ideal \mathfrak{m} of R , we have $f \notin \mathfrak{m}M$, whence $f \notin \mathfrak{m}^n M$, completing the proof of (2) from Theorem 3.3, and thus the proof that M is an Ohm-Rush R -module. \square

Corollary 3.5. *Let R be a Dedekind domain. Let $R \rightarrow S$ be a Noetherian R -algebra such that for every maximal ideal \mathfrak{m} of R , $S_{\mathfrak{m}}$ is a faithfully flat Ohm-Rush $R_{\mathfrak{m}}$ -algebra. Then S is a faithfully flat Ohm-Rush R -algebra.*

Proof. Substitute $M = S$ in Theorem 3.4. \square

One could also obtain a corollary to Theorem 3.4 by substituting $R = S$, but this is unnecessary since any flat finitely generated module is projective [11, Corollary to Theorem 7.12], and any projective module is Ohm-Rush [15, Corollary 1.4].

By a quite different proof, we next present a theorem that weakens the condition on R (to being merely a 1-dimensional Noetherian domain) but strengthens the condition on M and S (requiring S to also be a domain and $M = S$).

Theorem 3.6. *Let R be a 1-dimensional integral domain. Let $R \rightarrow S$ be a faithfully flat map, where S is a Noetherian integral domain. Assume that for each maximal ideal \mathfrak{m} of R , we have that $R_{\mathfrak{m}} \rightarrow S_{\mathfrak{m}}$ is Ohm-Rush (where in both cases, we are inverting the multiplicative set $R \setminus \mathfrak{m}$). Then $R \rightarrow S$ is Ohm-Rush.*

Proof. First we prove the result under the apparently stronger assumption that for any finite set $X = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$ of maximal ideals of R , we have that the map $W^{-1}R \rightarrow W^{-1}S$ is Ohm-Rush, where $W := R \setminus \bigcup_{i=1}^t \mathfrak{m}_i$. We introduce the notation $R_X := W^{-1}R$ and $S_X := W^{-1}S$ for this.

Accordingly, let $0 \neq f \in S$. If $c(f) = (1)$, then $f \in c(f)S$. Otherwise let P_1, P_2, \dots, P_n be the primes of S minimal over fS . Let $X := \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$, where $\mathfrak{p}_i = P_i \cap R$. Claim: $L_f \cap \text{Spec } R \subseteq X$. To see this let $\mathfrak{p} \in L_f \cap \text{Spec } R$. Note that $0R \notin L_f$, since $f \neq 0$. Hence $\text{ht } \mathfrak{p} = 1$. Let $P \in \text{Spec } S$ be minimal over $\mathfrak{p}S$. Then since S is Noetherian and faithfully flat over R , R is also Noetherian by [11, Exercise 7.9], and so $\text{ht } P = 1$ by [11, Theorem 15.1]. But fS is not in any height zero prime of S (since, again, $f \neq 0$ and S is a domain), so P is minimal over fS . Thus, $P = P_i$ for some i , whence $\mathfrak{p} = P \cap R$ (by faithful flatness) $= P_i \cap R = \mathfrak{p}_i \in X$.

Now let $I \in L_f$. Then $IR_X \in L_{f/1}$ where $f/1$ is the image of f in S_X . By our assumption $R_X \rightarrow S_X$ is Ohm-Rush, whence $f/1 \in c(f/1)S_X$, so that in particular $c(f/1) \neq 0$ since $f/1 \neq 0$ by torsion-freeness of S over R . Let $J := c(f/1) \cap R$, which is thus also nonzero. We have $c(f/1) \subseteq IR_X$

by definition. But also we have $IR_X \cap R = I$. To see this, first note that $V(I) \subseteq L_f \cap \text{Spec } R \subseteq X$ by the previous paragraph. In particular, all the associated primes of I are in X . So let $a \in R \setminus I$. Then $(I : a) \subseteq \mathfrak{p}$ for some associated prime \mathfrak{p} of R/I (since R is Noetherian), hence for some $\mathfrak{p} \in X$. Thus, $a/1 \notin IR_{\mathfrak{p}}$. Since $IR_{\mathfrak{p}} \supseteq IR_X$, we conclude that $a \notin IR_X \cap R$. Thus, $J = c(f/1) \cap R \subseteq IR_X \cap R \subseteq I$. But since $I \in L_f$ was arbitrary, we have that L_f is in one-to-one correspondence with some set of ideals in the Artinian ring R/J . Hence L_f satisfies the descending chain condition. Thus by Lemma 3.2, $f \in c(f)S$.

To complete the proof, all we need is the following result. □

Proposition 3.7. *Let R be a Noetherian, 1-dimensional, semilocal domain. Let S be a Noetherian R -algebra. If $S_{\mathfrak{m}}$ is Ohm-Rush over $R_{\mathfrak{m}}$ for all maximal ideals \mathfrak{m} of R , then S is Ohm-Rush over R .*

Proof. Let $0 \neq f \in S$ with $c(f) \neq (1)$. Let $I \in L_f$. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the set of maximal ideals of R . For each $1 \leq i \leq n$, let c_i be the content function associated to $R_{\mathfrak{m}_i} \rightarrow S_{\mathfrak{m}_i}$. Since $f \in IS$, we have $f/1 \in IS_{\mathfrak{m}_i} = (IR_{\mathfrak{m}_i})S_{\mathfrak{m}_i}$, whence $IR_{\mathfrak{m}_i} \in L_{f/1}$ for each $i = 1, \dots, n$. Thus $0 \neq c_i(f/1) \subseteq IR_{\mathfrak{m}_i}$ for each i . Hence $c_i(f/1) \cap R$ is not zero and is contained in $IR_{\mathfrak{m}_i} \cap R$. Furthermore $I = \bigcap_i (IR_{\mathfrak{m}_i} \cap R)$. Hence $\bigcap_i (c_i(f/1) \cap R) \subseteq I$.

Now let $J = \bigcap_i (c_i(f/1) \cap R)$. Since J is a finite intersection of nonzero ideals in the integral domain R , we must have that $J \neq 0$. But J is contained in every element of L_f . Since R/J is Artinian, L_f thus satisfies the descending chain condition. So $f \in c(f)S$ by Lemma 3.2. □

It is reasonable to ask whether the Noetherian condition on the S -module M in Theorem 3.4 and on the ring S in Theorem 3.6 can be dropped. The following example shows that these conditions are necessary.

Example 3.8. At the beginning of [3], Eakin and Silver give the following example. Let $R = \mathbb{Z}$, let x be an indeterminate, and let S be the \mathbb{Z} -subalgebra of $\mathbb{Q}[x]$ generated over \mathbb{Z} by the elements x/p , taken over all positive prime numbers p . As they note, this ring is locally a polynomial ring over \mathbb{Z} , for if we take any particular prime number p , localizing $\mathbb{Z} \rightarrow S$ at the multiplicative set $\mathbb{Z} \setminus p\mathbb{Z}$ yields the ring map $\mathbb{Z}_{(p)} \rightarrow \mathbb{Z}_{(p)}[x/p]$. Therefore, $S_{p\mathbb{Z}} \cong \mathbb{Z}_{(p)}[x/p]$ satisfies the much weaker condition of being an Ohm-Rush $\mathbb{Z}_{(p)}$ -algebra. However, $\mathbb{Z} \rightarrow S$ is not Ohm-Rush, since for any prime number p , we have $x = p(x/p) \in pS$, whence $c(x) \subseteq \bigcap_p p\mathbb{Z} = 0$, even though $x \neq 0$, so that $x \notin c(x)S$.

Example 3.9. From the same article, we can recover a Gaussian algebra that is locally polynomial but not finitely generated. Namely, in [3, Example 3.15], they construct a \mathbb{Z} -algebra S that is a Noetherian UFD that is not finitely generated over \mathbb{Z} , such that for every positive prime number p , we have $(\mathbb{Z} \setminus p\mathbb{Z})^{-1}S \cong \mathbb{Z}_{(p)}[x]$. Then either Theorem 3.6 or Theorem 3.4 is enough to show that S is an Ohm-Rush R -algebra. After this, an appeal to the fact that the

Gaussian property globalizes [7, Proposition 3.3] proves that S is a Gaussian \mathbb{Z} -algebra, since a polynomial extension in one variable of the Prüfer domain $\mathbb{Z}_{(p)}$ is always Gaussian by Gauss's lemma.

4. Power-content algebras

Rush in [17] defined a *weak content* algebra over R as an Ohm-Rush algebra S such that $\sqrt{c(fg)} = \sqrt{c(f)c(g)}$ for all $f, g \in S$. As indicated by the terminology, any content algebra is a weak content algebra. In this final section, we explore a property strictly between Ohm-Rush and weak content algebra.

First, we recall the notion of the content of an *ideal*:

Definition 4.1 ([5, just prior to Lemma 3.8]). Let $R \rightarrow S$ be an Ohm-Rush algebra and J an ideal of S . Then $c(J) := \bigcap \{I \subseteq R \text{ ideal} : J \subseteq IS\}$. Equivalently, $c(J) = \sum_{g \in S} c(g)$. Hence, $J \subseteq c(J)S$.

Definition 4.2. Let $R \rightarrow S$ be an Ohm-Rush algebra. We say it is a *power-content* algebra if for any ideal J of S , we have $c(\sqrt{J}) \subseteq \sqrt{c(J)}$.

Lemma 4.3. *Let $R \rightarrow S$ be an Ohm-Rush algebra. Then it is a power-content algebra if and only if for any radical ideal I of R , we have that IS is a radical ideal of S .*

Proof. Suppose we have a power-content algebra, and let I be a radical ideal of R . Let $f \in \sqrt{IS}$. That is, there is some n with $f^n \in IS$. Then $c(f^n) \subseteq I$. Since $f \in \sqrt{(f^n)}$, we have $c(f) \subseteq c(\sqrt{(f^n)}) \subseteq \sqrt{c(f^n)} \subseteq \sqrt{I} = I$. Thus, $f \in IS$, whence IS is radical.

Conversely, suppose radical ideals extend to radical ideals. Let J be an ideal of S . Then $\sqrt{c(J)S}$ is a radical ideal of S , and $J \subseteq c(J)S \subseteq \sqrt{c(J)S}$, whence we have $\sqrt{J} \subseteq \sqrt{c(J)S}$. Thus, $c(\sqrt{J}) \subseteq \sqrt{c(J)}$. \square

We immediately see two distinctions among content-defined classes of R -algebras.

Example 4.4. Not all faithfully flat Ohm-Rush algebras are power-content. For instance, consider the ring homomorphism $R \rightarrow R[x]/(x^2) =: S$, where R is any commutative ring and x is an indeterminate over R . Then it is faithfully flat and Ohm-Rush, since S is a free R -module of rank 2. But it is not power-content, since $\sqrt{c(0)} = \sqrt{0}$, but $c(\sqrt{0}) \supseteq c(x) = R$, whereas the nilradical of a nonzero ring is always a proper ideal.

Example 4.5. Not all faithfully flat power-content algebras are weak content algebras. For instance, let R be any commutative ring, let x, y be any indeterminates over R , and consider the algebra $R \rightarrow R[x, y]/(xy) =: S$. Again it is a faithfully flat Ohm-Rush algebra because it is free as an R -module. Let \mathfrak{p} be a prime ideal of R . Then $xy = 0 \in \mathfrak{p}S$, whereas $x \notin \mathfrak{p}S$ and $y \notin \mathfrak{p}S$. Hence this is not a weak content algebra. But if I is a radical ideal of R ,

then $S/IS \cong (R/I)[x, y]/(xy)$ is reduced, as is easy to show. Hence this is a power-content algebra.

Proposition 4.6. *Let $R \rightarrow S$ be a ring homomorphism. Then it is a weak content algebra if and only if it is a power-content algebra such that for all ideals I, J of S , we have $c(I) \cap c(J) \subseteq \sqrt{c(I \cap J)}$.*

Proof. First suppose it is a weak content algebra. Let I be a radical ideal of R . Say $I = \bigcap_{\mathfrak{p} \in U} \mathfrak{p}$ for some subset $U \subseteq \text{Spec } R$. Then $IS = (\bigcap_{\mathfrak{p} \in U} \mathfrak{p})S = \bigcap_{\mathfrak{p} \in U} (\mathfrak{p}S)$ by the Ohm-Rush property. But since $R \rightarrow S$ is a weak content algebra, each $\mathfrak{p}S$ is either prime or the unit ideal, so their intersection must be a radical ideal. Hence by Lemma 4.3, it is a power-content algebra. Now let I, J be ideals of S . Let \mathfrak{p} be a prime ideal of R that contains $c(I \cap J)$. Then $I \cap J \subseteq \mathfrak{p}S$, so that since $\mathfrak{p}S$ is either the unit ideal or prime, either $I \subseteq \mathfrak{p}S$ or $J \subseteq \mathfrak{p}S$. Thus, either $c(I) \subseteq \mathfrak{p}$ or $c(J) \subseteq \mathfrak{p}$, and in either case we have $c(I) \cap c(J) \subseteq \mathfrak{p}$. We have shown that any prime ideal that contains $c(I \cap J)$ contains $c(I) \cap c(J)$, whence $c(I) \cap c(J) \subseteq \sqrt{c(I \cap J)}$.

Conversely, suppose $R \rightarrow S$ is a power-content algebra such that $c(I) \cap c(J) \subseteq \sqrt{c(I \cap J)}$ for all ideals I, J of S . Let $f, g \in S$. Then

$$\begin{aligned} c(f)c(g) &\subseteq c(f) \cap c(g) \subseteq \sqrt{c((f) \cap (g))} \subseteq \sqrt{c(\sqrt{(f) \cap (g)})} \\ &= \sqrt{c(\sqrt{(fg)})} \subseteq \sqrt{\sqrt{c(fg)}} = \sqrt{c(fg)}. \end{aligned} \quad \square$$

Finally we show that the power-content property is transitive and, in the presence of the Ohm-Rush property, globalizes. These results are analogous to our results of this type for Gaussian, weak content, and semicontent algebras [7, Propositions 3.1-3.3].

Theorem 4.7. *Let $\varphi : R \rightarrow S$ be a flat Ohm-Rush algebra. The following are equivalent:*

- (a) $R \rightarrow S$ is a power-content algebra.
- (b) For every multiplicative subset W of R , $W^{-1}R \rightarrow W^{-1}S$ is a power-content algebra.
- (c) For every maximal ideal \mathfrak{m} of R , $R_{\mathfrak{m}} \rightarrow S_{\mathfrak{m}}$ is a power-content algebra, where $S_{\mathfrak{m}}$ is the localization of S at the multiplicative set $\varphi(R \setminus \mathfrak{m})$.

Proof. First we prove that (a) \implies (b). Recall [15, Theorem 3.1] that $W^{-1}S$ is an Ohm-Rush $W^{-1}R$ -algebra. Now let I be a radical ideal of $W^{-1}R$. Let J be the contraction of I to R . Then J is also radical, whence JS is radical by Lemma 4.3, whence $W^{-1}(JS) = I(W^{-1}S)$ is a radical ideal. Then by Lemma 4.3 again, $W^{-1}R \rightarrow W^{-1}S$ is a power-content algebra.

Since it is obvious that (b) \implies (c), it remains only to prove that (c) \implies (a). Accordingly, let I be a radical ideal of R . Let $f \in S$ such that $f^n \in IS$. Let \mathfrak{m} be a maximal ideal of R . By assumption, $R_{\mathfrak{m}} \rightarrow S_{\mathfrak{m}}$ is a power-content

algebra. Thus, $IS_{\mathfrak{m}} = (IR_{\mathfrak{m}})S_{\mathfrak{m}}$ is a radical ideal of $S_{\mathfrak{m}}$ by Lemma 4.3. Since $(f/1)^n \in IS_{\mathfrak{m}}$, it follows that $f/1 \in IS_{\mathfrak{m}}$. Thus $(IS :_R f) \not\subseteq \mathfrak{m}$. Since \mathfrak{m} was an arbitrary maximal ideal of R , we have $(IS :_R f) = R$, whence $f \in IS$. Thus, IS is radical, whence by Lemma 4.3, $R \rightarrow S$ is power-content. \square

Theorem 4.8. *Let $R \rightarrow S$ and $S \rightarrow T$ be power-content algebras. Then $R \rightarrow T$ is a power-content algebra. That is, the property is transitive.*

Proof. We have that $R \rightarrow T$ is Ohm-Rush by repeated use of [15, 1.2(ii)]. Now let I be a radical ideal of R . Then by Lemma 4.3, IS is a radical ideal of S , whence $IT = (IS)T$ is a radical ideal of T . Hence by Lemma 4.3 again, $R \rightarrow T$ is power-content. \square

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