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VALUE DISTRIBUTIONS OF L-FUNCTIONS CONCERNING POLYNOMIAL SHARING

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Abstract. We mainly study the value distributions of L-functions in the extended selberg class. Concerning weighted sharing, we prove an uniqueness theorem when certain differential monomial of a meromorphic function share a polynomial with certain differential monomial of an L-function which improve and generalize some recent results due to Liu, Li and Yi [11], Hao and Chen [3] and Mandal and Datta [12].

1. Introduction

In today's world the most important open problem in the pure mathematics is the Riemann hypothesis and its extension to the general classes of L-functions. L-functions play very important role in the modern number theory. We may regard the famous Riemann zeta-function, $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z = \prod_p \left(1 - 1/p^z\right)^{-1}$ where $z = a + ib$, $a > 1$ and p denotes prime number and the product is taken over all prime numbers, as the prototype. For a long time a lot of attention have been given by many scholars on the Riemann hypothesis.

Throughout the paper by an L-function L we mean an L-function L in the Selberg class. Such an L-function is defined by $L(z) = \sum_{n=1}^{\infty} a(n)/n^z$ satisfying the following hypothesis.

(i) $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$;
(ii) There exists an integer $k \geq 0$ such that $(z-1)^k L(z)$ is a finite order entire function;
(iii) Every L-function satisfies the functional equation

$$\lambda_L(z) = \omega \overline{\lambda_L(1-\overline{z})},$$

where

$$\lambda_L(z) = L(z) Q^z \prod_{i=1}^{k} \Gamma(\lambda_i z + \nu_i)$$

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with positive real numbers Q, λ_i and complex numbers ν_i, ω with $\mathrm{Re}\nu_i \geq 0$ and $|\omega| = 1$.

(iv) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = \exp(\sum_{k=1}^{\infty} b(p^k)/p^{kz})$ with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ and p denotes prime number.

If L satisfies the hypothesis (i)-(iii), then L is said to be an L-function in the extended Selberg class.

In this paper, concerning weighted sharing we study the uniqueness problems using Nevanlinna's value distribution theory. Here we use the standard notations and definitions of the value distribution theory [4].

2. Preliminaries

Let f and g be two nonconstant meromorphic functions in the open complex plane $\mathbf{C}$. We denote by $S(r, f)$ any function satisfying $S(r, f) = o(T(r, f)$ as $r \to \infty$, outside a possible exceptional set of finite linear measure.

If f and g have the same set of b points with the same multiplicities, we say that f and g share b CM (counting multiplicities) and we say that f and g share b IM (ignoring multiplicities) if we do not consider the multiplicities where $b \in \mathbf{C} \cup \{\infty\}$.

Definition 2.1 ([9]). Let f be a meromorphic function defined in the complex plane. Also let n be a positive integer and $\alpha \in \mathbf{C} \cup \{\infty\}$. By $N(r, \alpha; f \mid\leq n)$ we denote the counting function of the α points of f with multiplicity $\leq n$ and by $\overline{N}(r, \alpha; f \mid\leq n)$ the reduced counting function. Also by $N(r, \alpha; f \mid\geq n)$ we denote the counting function of the α points of f with multiplicity $\geq n$ and by $\overline{N}(r, \alpha; f \mid\geq n)$ the reduced counting function. We define

$$N_n(r, \alpha; f) = \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; f \mid\geq 2) + \cdots + \overline{N}(r, \alpha; f \mid\geq n).$$

Definition 2.2 ([9]). Let f be a meromorphic function defined in the complex plane and $P(Z)$ be a polynomial. Then we denote by $N(r, P; f \mid\leq m)$, $\overline{N}(r, P; f \mid\leq m)$, $N(r, P; f \mid\geq m)$, $\overline{N}(r, P; f \mid\geq m)$, $N_m(r, P; f)$ etc. the counting functions $N(r, 0; f - P \mid\leq m)$, $\overline{N}(r, 0; f - P \mid\leq m)$, $N(r, 0; f - P \mid\geq m)$, $\overline{N}(r, 0; f - P \mid\geq m)$, $N_m(r, 0; f - P)$ etc., respectively.

Definition 2.3 ([5,6]). Let f be a meromorphic function defined in the complex plane and n be an integer (≥ 0) or infinity. For $\alpha \in \mathbf{C} \cup \{\infty\}$ we denote by $E_{n)}(\alpha; f)$ the set of all zeros of $f - \alpha$ with multiplicities not exceeding n, where a zero is counted according to its multiplicity. Also we denote by $\overline{E}_{n)}(\alpha; f)$ the set of all zeros of $f - \alpha$ with multiplicities not exceeding n, where a zero is counted ignoring multiplicity.

Definition 2.4 ([5, 6]). Let f and g be two meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity. For $\alpha \in \mathbf{C} \cup \{\infty\}$ we denote by $E_n(\alpha; f)$ the set of all zeros of $f - \alpha$ where a zero of multiplicity k

is counted k times if $k \leq n$ and $n+1$ times if $k > n$. If $E_n(\alpha; f) = E_n(\alpha; g)$, we say that f, g share the value α with weight n.

The definition implies that if f, g share a value α with weight n, then z_1 is an α-point of f with multiplicity $k(\leq n)$ if and only if z_1 is an α-point of g with multiplicity $k(\leq n)$ and z_1 is an α-point of f with multiplicity $k(> n)$ if and only if z_1 is an α-point of g with multiplicity $m(> n)$ where k is not necessarily equal to m.

We write f, g share (α, n) to mean that f, g share the value α with weight n. Clearly if f , g share (α, n), then f, g share (α, m) for all integers $m, 0 \leq m < n$. Also we note that f, g share a value α IM or CM if and only if f, g share $(\alpha, 0)$ or (α, ∞), respectively.

In 2010 Li [10] study the uniqueness problems of meromorphic functions and L-functions and proved the following theorem.

Theorem 2.1 ([10]). *Let f be a nonconstant meromorphic function having finitely many poles and L be a nonconstant L-function. If f and L share (α, ∞) and $(\beta, 0)$, then $L = f$, where α and β are two distinct finite values.*

In 2017, Liu, Li and Yi [11] proved the following uniqueness theorems of L-functions.

Theorem 2.2 ([11]). *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and f be a nonconstant meromorphic function. If $\{f^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share $(1, \infty)$, then $f = \alpha L$ for some nonconstant α satisfying $\alpha^j = 1$.*

Theorem 2.3 ([11]). *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and f be a nonconstant meromorphic function. If $\{f^j\}^{(k)}(z) - (z)$ and $\{L^j\}^{(k)}(z) - (z)$ share $(0, \infty)$, then $f = \alpha L$ for some nonconstant α satisfying $\alpha^j = 1$.*

Definition 2.5 ([12]). Let f be a meromorphic function defined in the complex plane and $P(z)$ be a polynomial. Then we denote by $E_{m)}(P; f)$, $\overline{E}_{m)}(P; f)$ and $E_m(P; f)$ the sets $E_{m)}(0; f - P)$, $\overline{E}_{m)}(0; f - P)$ and $E_m(0; f - P)$, respectively.

Considering weighted sharing in 2018 Hao and Chen [3] proved the following theorem.

Theorem 2.4 ([3]). *Let L be an L-function and f be a meromorphic function defined in the complex plane $\mathbf{C}$ with finitely many poles. Let α_1, $\alpha_2 \in \mathbf{C}$ be distinct and m_1,m_2 be positive integers such that $m_1 m_2 > 1$. If $E_{m_j}(\alpha_j, f) = E_{m_j}(\alpha_j, L)$, $j = 1, 2$, then $L = f$.*

Considering weighted sharing of small functions in 2020 Mandal and Datta [12] proved the following theorem.

Theorem 2.5 ([12]). *Let L be a nonconstant L-function and ρ be a small function of L such that $\rho \neq 0, \infty$. If $\overline{E}_{4)}(\rho; L) = \overline{E}_{4)}(\rho; (L^m)^{(k)})$, $E_{2)}(\rho; L) =$*

$E_{2)}(\rho; (L^m)^{(k)})$ *and*

$$2N_{2+k}(r, 0; L^m) \leq (\sigma + o(1))T(r, L),$$

where $m \geq 1$, $k \geq 1$ *are integers and* $0 < \sigma < 1$, *then* $L = (L^m)^{(k)}$.

Now the following question comes naturally.

Question 2.1. Can we take a polynomial of z in place of z in Theorem 2.3?

Question 2.2. Can we reduce the weight of the sharing of values in Theorems 2.2 and 2.3?

Question 2.3. Is it possible to consider weighted sharing of values of $(f^m)^{(l)}$ and $(L^m)^{(l)}$ in Theorems 2.4 and 2.5?

Definition 2.6 ([5])**.** Let two nonconstant meromorphic functions f and g share a value α IM. We denote by $\overline{N}_*(r, \alpha; f, g)$ the counting function of the α-points of f and g with different multiplicities, where each α-point is counted only once.

Clearly $\overline{N}_*(r, \alpha; f, g) = \overline{N}_*(r, \alpha; g, f)$.

Definition 2.7. Let two nonconstant meromorphic functions f and g share a value α IM. We denote by $\overline{N}(r, \alpha; f| > g)$ the counting function of the α-points of f and g with multiplicities with respect to f is greater than the multiplicities with respect to g, where each α-point is counted once only.

Definition 2.8. Let two nonconstant meromorphic functions f and g share a value α IM. We denote by $\overline{N}_E(r, \alpha; f, g| > m)$ the counting function of the α-points of f and g with multiplicities greater than m and the multiplicities with respect to f is equal to the multiplicities with respect to g, where each α-point is counted once only.

Definition 2.9 ([8])**.** We denote by $N_{\otimes}(r, 0; f^{(k)})$ $(\overline{N}_{\otimes}(r, 0; f^{(k)}))$ the counting function (reduced counting function) of those zeros of $f^{(k)}$ which are not the zeros of $f(f-1)$.

Definition 2.10 ([8])**.** We denote by $N_0(r, 0; f^{(k)})$ $(\overline{N}_0(r, 0; f^{(k)}))$ the counting function (reduced counting function) of those zeros of $f^{(k)}$ which are not the zeros of the nonconstant meromorphic function f.

Throughout the paper we mean by f, g two nonconstant meromorphic functions defined in the open complex plane $\mathbf{C}$.

3. Main results

Using weighted sharing we try to solve Questions 2.1, 2.2 and 2.3 and prove the following theorem.

Theorem 3.1. *Let f be a nonconstant meromorphic function and L be a nonconstant L-function. If $(f^m)^{(l)}$, $(L^m)^{(l)}$ share (P,k) and f, L share $(\infty,0)$ where $m \geq 1$, $l \geq 1$, $k \geq 0$ are integers and $P(z)$ is an n-th degree polynomial. Then $f = dL$, for some constant d satisfying $d^m = 1$ if one of the following holds*

(i) $k = 0$ *and* $m > 5l + 7$;
(ii) $k = 1$ *and* $m > \frac{5l+9}{2}$;
(iii) $k \geq 2$ *and* $m > 2l + 2$.

4. Lemmas

In this section we present some necessary lemmas.

Henceforth we denote by Ω the function defined by

$$\Omega = (\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}) - (\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}),$$

where Φ and Ψ are meromorphic functions defined in the complex plane.

Lemma 4.1 ([13])**.** *Let L be an L-function with degree q. Then*

$$T(r,L) = \frac{q}{\pi} r \log r + O(r).$$

Lemma 4.2 ([12])**.** *Let L be an L-function. Then $N(r,\infty;L) = S(r,L) = O(\log r)$.*

Lemma 4.3. *Let f be a nonconstant meromorphic function and L be an L-function. If f and L share $(\infty,0)$, then $\overline{N}(r,\infty;f) = S(r,L) = O(\log r)$.*

Proof. Since f and L share $(\infty,0)$ therefore by Lemma 4.2 we have $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;L) = S(r,L) = O(\log r)$. This completes the proof. □

Lemma 4.4 ([16])**.** *Let $f(z) = \frac{\alpha_0+\alpha_1 z+\cdots+\alpha_n z^n}{\beta_0+\beta_1 z+\cdots+\beta_m z^m}$ be a nonconstant rational function defined in the complex plane $\boldsymbol{C}$, where $\alpha_0, \alpha_1, \ldots, \alpha_n (\neq 0)$ and $\beta_0, \beta_1, \ldots, \beta_m (\neq 0)$ are complex constants. Then*

$$T(r,f) = \max\{m,n\} \log r + O(1).$$

Lemma 4.5 ([16])**.** *Let f be a transcendental meromorphic function defined in the complex plane $\boldsymbol{C}$. Then*

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty.$$

Lemma 4.6 ([15])**.** *Let f be a nonconstant meromorphic function and let l be a positive integer. If ρ be a small function of f, then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + N(r,0;f) + N(r,\rho,f^{(l)}) - N(r,0,(\frac{f^{(l)}}{\rho})') + S(r,f).$$

Lemma 4.7. *Let f be a nonconstant meromorphic function and L be a nonconstant L-function. If $(f^m)^{(l)}$, $(L^m)^{(l)}$ share $(P(z), 0)$, where $m \geq 1$, $l \geq 1$, $k \geq 0$ are integers and $P(z)$ is an n-th degree polynomial and $m > l+1$, then f and L are transcendental meromorphic functions.*

Proof. Let the degree of L be q. Then by Lemma 4.1 we have $T(r, L) = \frac{q}{\pi} r \log r + O(r)$. Hence by Lemmas 4.4 and 4.5 L is a transcendental meromorphic function.

Let $\Phi = \frac{(f^m)^{(l)}}{P}$ and $\Psi = \frac{(L^m)^{(l)}}{P}$. Since $(f^m)^{(l)}$, $(L^m)^{(l)}$ share $(P(z), 0)$ therefore Φ, Ψ share $(1, l)$ except zeros of P. Let z_0 be a zero of L of multiplicity n_0 but not a zero of P. Then z_0 is a zero of L^m with multiplicity mn_0 and so z_0 is a zero of $\frac{L^m}{P}$ of multiplicity mn_0. Hence z_0 is a zero of $(\frac{(L^m)^{(l)}}{P})'$ with multiplicity at least $mn_0 - l - 1$. Again let z_1 be a zero of $\frac{(L^m)^{(l)}}{P} - 1$ with multiplicity n_1, then z_1 is a zero of $(\frac{(L^m)^{(l)}}{P} - 1)'$ with multiplicity $n_1 - 1$.

Hence By Lemma 4.2, Lemma 4.4 and Lemma 4.6 we have

$$
\begin{aligned}
T(r, L^m) &= mT(r, L) + O(1) \\
&\leq \overline{N}(r, \infty; L^m) + N(r, 0; L^m) + N(r, P, (L^m)^{(l)}) \\
&\quad - N(r, 0, (\frac{(L^m)^{(l)}}{P})') + S(r, L) \\
&\leq (l+1)\overline{N}(r, 0; L) + \overline{N}(r, 0, \frac{(L^m)^{(l)}}{P} - 1) \\
&\quad - N_{\otimes}(r, 0, (\frac{(L^m)^{(l)}}{P})') + O(\log r) \\
&\leq (l+1)\overline{N}(r, 0; L) + \overline{N}(r, 0, \frac{(f^m)^{(l)}}{P} - 1) \\
&\quad - N_{\otimes}(r, 0, (\frac{(L^m)^{(l)}}{P})') + O(\log r) \\
&\leq (l+1)T(r, L) + T(r, (f^m)^{(l)}) + O(\log r). \qquad (4.1)
\end{aligned}
$$

By Lemma 4.4 we have from (4.1)

$$
(m - l - 1)T(r, L) \leq T(r, (f^m)^{(l)}) + O(\log r). \tag{4.2}
$$

Since $m > l + 1$ therefore from (4.2) it follows that f is a transcendental meromorphic function. □

Lemma 4.8 ([17]). *Let f be a nonconstant meromorphic function and k, p are two positive integers. Then*

$$
N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f)
$$

and

$$
N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).
$$

Lemma 4.9 ([14]). *Let two meromorphic functions f and g be such that $(f^m)^{(l)} = (g^m)^{(l)}$ where m, l are positive integer. If $m > l+1$, then $f = dg$ for some constant d such that $d^m = 1$.*

Lemma 4.10. *Let f, L share $(\infty, 0)$, where f is a nonconstant meromorphic function and L is a nonconstant L-function. Let $\Phi = \frac{(f^m)^{(l)}}{P}$ and $\Psi = \frac{(L^m)^{(l)}}{P}$, where m, l are positive integers such that $m > 2l+2$ and $P(z)$ is an nth degree polynomial. If $\Omega = 0$, then either $(f^m)^{(l)}(L^m)^{(l)} = P^2$ or $f = dL$ for some constant d such that $d^m = 1$.*

Proof. Let $\Omega = (\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi-1}) - (\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi-1}) = 0$.

Integrating we have

$$\Phi - 1 = \frac{\Psi - 1}{C - D(\Psi - 1)}, \tag{4.3}$$

where $C(\neq 0)$ and D are constants.

Now we have to consider the following two cases.

Case 1. Let $D = 0$. Then from (4.3) we have

$$\Phi - 1 = \frac{(\Psi - 1)}{C}. \tag{4.4}$$

If $C \neq 1$, then from (4.4)

$$\overline{N}(r, 0; \Phi) = \overline{N}(r, 1 - C; \Psi). \tag{4.5}$$

Since f, L share $(\infty, 0)$ therefore by Lemma 4.2, Lemma 4.8 and the second fundamental theorem we have

$$\begin{aligned} T(r, L^m) &\leq T(r, \Psi) + N_{l+1}(r, 0; L^m) - \overline{N}(r, 0; \Psi) + S(r, L) \\ &\leq \overline{N}(r, \infty; \Psi) + \overline{N}(r, 0; \Psi) + \overline{N}(r, 1 - C; \Psi) + N_{l+1}(r, 0; L^m) \\ &\quad - \overline{N}(r, 0; \Psi) + S(r, L) \\ &\leq \overline{N}(r, \infty; L) + \overline{N}(r, 0; \Phi) + N_{l+1}(r, 0; L^m) + O(\log r) \\ &\leq l\overline{N}(r, \infty; f) + (l+1)\overline{N}(r, 0; f) + N_{l+1}(r, 0; L^m) + O(\log r) \\ &\leq l\overline{N}(r, \infty; L) + (l+1)\overline{N}(r, 0; f) + (l+1)\overline{N}(r, 0; L) + O(\log r) \\ &\leq (l+1)T(r, L) + (l+1)T(r, f) + O(\log r). \end{aligned} \tag{4.6}$$

Without loss of generality we may assume that $T(r, f) \leq T(r, L)$ for $r \in I$ where I is a set of infinite measure. Hence for $r \in I$ we have from (4.6)

$$mT(r, L) \leq (2l+2)T(r, L) + O(\log r),$$

which is a contradiction since $m > 2l + 2$.

Hence $C = 1$ and from (4.4) we have $\Phi = \Psi$.

Therefore by Lemma 4.9 we have $f = dL$, for some constant d such that $d^m = 1$.

Case 2. Let $D \neq 0$.

In this case we have to consider the following two subcases.

Subcase 2.1. Let $C = -D$.

If $D = 1$, then from (4.3) we have $\Phi\Psi = 1$. Hence $(f^m)^{(l)}(L^m)^{(l)} = P^2$.

If $D \neq 1$, then from (4.3) we have $\frac{1}{\Phi} = \frac{-D\Psi}{(1-D)\Psi-1}$.

Hence $\overline{N}(r, 0; \Phi) = \overline{N}(r, \frac{1}{1-D}; \Psi)$.

Now proceeding as in the Case 1 we arrive at a contradiction.

Subcase 2.2. Let $C \neq -D$.

If $D = 1$, then from (4.3) we have

$$\Phi = \frac{-C}{\Psi - C - 1}. \tag{4.7}$$

Since f, L share $(\infty, 0)$ therefore by Lemma 4.2 and Lemma 4.4 we have from (4.7)

$$\begin{aligned}\overline{N}(r, C+1; \Psi) &= \overline{N}(r, \infty; \Phi)\\ &= \overline{N}(r, \infty; f) + O(\log r)\\ &= O(\log r).\end{aligned}$$

Now proceeding as in the Case 1 we arrive at a contradiction.

If $D \neq 1$, then from (4.3) we have

$$\Phi - (1 - \frac{1}{D}) = \frac{-C}{D^2(\Psi - \frac{C+D}{D})}.$$

Therefore

$$\begin{aligned}\overline{N}(r, \frac{C+D}{D}; \Psi) &= \overline{N}(r, \infty; \Phi)\\ &= \overline{N}(r, \infty; f) + O(\log r)\\ &= O(\log r).\end{aligned}$$

Hence proceeding as in the Case 1 we arrive at a contradiction.

This completes the proof of the lemma. $\square$

Lemma 4.11 ([1])**.** *Let Φ and Ψ be two nonconstant meromorphic functions sharing* $(1, 0)$*. If $\Omega \neq 0$, then*

$$\begin{aligned}T(r, \Phi) \leq\ & N_2(r, 0; \Phi) + N_2(r, \infty; \Phi) + N_2(r, 0; \Psi) + N_2(r, \infty; \Psi)\\ &+ 2\overline{N}(r, 0; \Phi) + 2\overline{N}(r, \infty; \Phi) + \overline{N}(r, 0; \Psi) + \overline{N}(r, \infty; \Psi)\\ &+ S(r, \Phi) + S(r, \Psi).\end{aligned}$$

Lemma 4.12 ([1])**.** *Let Φ and Ψ be two nonconstant meromorphic functions sharing* $(1, 1)$*. If $\Omega \neq 0$, then*

$$\begin{aligned}T(r, \Phi) \leq\ & N_2(r, 0; \Phi) + N_2(r, \infty; \Phi) + N_2(r, 0; \Psi) + N_2(r, \infty; \Psi)\\ &+ \frac{1}{2}\overline{N}(r, 0; \Phi) + \frac{1}{2}\overline{N}(r, \infty; \Phi) + S(r, \Phi) + S(r, \Psi).\end{aligned}$$

Lemma 4.13 ([7])**.** *Let f be a nonconstant meromorphic function. Then*

$$N_0(r, 0; f^{(l)}) \leq l\overline{N}(r, \infty; f) + N(r, 0; f| < l) + l\overline{N}(r, 0, f| \geq l) + S(r, f).$$

Lemma 4.14 ([2])**.** *Let f and g be two nonconstant meromorphic functions sharing $(1,l)$, where $2 \leq l \leq \infty$. Then*

$$\begin{aligned}
&N(r,1;g) - \overline{N}(r,1;g)\\
\geq\ &\overline{N}(r,1;f| = 2) + 2\overline{N}(r,1;f| = 3) + \cdots + (l-1)\overline{N}(r,1;f| = l)\\
&+ l\overline{N}(r,1;f| > g) + (l+1)\overline{N}(r,1;g| > f) + l\overline{N}_E(r,1;f,g| > l+1).
\end{aligned}$$

5. Proof of the main result

Proof of Theorem 3.1. Let $\Phi = \frac{(f^m)^{(l)}}{P}$ and $\Psi = \frac{(L^m)^{(l)}}{P}$. Since $(f^m)^{(l)}$, $(L^m)^{(l)}$ share (P,k) and f, L share $(\infty,0)$ therefore Φ, Ψ share $(1,k)$ except the zeros of P and $(\infty,0)$.

Now we have to consider the following two cases.

Case 1. Let $\Omega = 0$. By Lemma 4.10 we have either $(f^m)^{(l)}(L^m)^{(l)} = P^2$ or $f = dL$ for some constant d such that $d^m = 1$.

If $(f^m)^{(l)}(L^m)^{(l)} = P^2$, then

$$\Phi\Psi = 1. \tag{5.1}$$

Since f and L share $(\infty,0)$ and $m > 2l+2$ therefore from (5.1) it is clear that if z_0 is not a zero of P, then z_0 is neither a zero of f nor a zero of L. Again since f and L share $(\infty,0)$ and $m > l+2$ therefore from (5.1) it is clear that ∞ is a Picard exceptional value of f and L. Hence

$$T(r,L) = \frac{|A|r}{\pi}(1 + O(1)) + O(\log r), \tag{5.2}$$

where A is a nonzero constant.

Therefore from Lemma 4.1 and (5.2) we arrive at a contradiction.

Hence $f = dL$ for some constant d such that $d^m = 1$.

Case 2. Let $\Omega \neq 0$.

In this case we have to consider the following three subcases.

Subcase 2.1. Let $k = 0$. From Lemma 4.3, Lemma 4.4 and Lemma 4.8 we have

$$\begin{aligned}
N_2(r,0;\Phi) &\leq N_2(r,0;(f^m)^{(l)}) + O(\log r)\\
&\leq T(r,(f^m)^{(l)}) - mT(r,f) + N_{l+2}(r,0;f^m) + O(\log r) + S(r,f)\\
&\leq T(r,\Phi) - mT(r,f) + N_{l+2}(r,0;f^m) + O(\log r) + S(r,f). \qquad (5.3)
\end{aligned}$$

From (5.3) we have

$$mT(r,f) \leq T(r,\Phi) - N_2(r,0;\Phi) + N_{l+2}(r,0;f^m) + O(\log r) + S(r,f). \tag{5.4}$$

By Lemma 4.2, Lemma 4.3, Lemma 4.8, Lemma 4.11 and (5.4) we have

$$\begin{aligned}
mT(r,f) &\leq 4\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;L) + 2\overline{N}(r,0;(f^m)^{(l)}) + N_{l+2}(r,0;f^m)\\
&\quad + N_2(r,0;(L^m)^{(l)}) + \overline{N}(r,0;(L^m)^{(l)}) + O(\log r) + S(r,f)\\
&\leq 2N_{l+1}(r,0;f^m) + 2l\overline{N}(r,\infty;f) + (l+2)\overline{N}(r,0;f)
\end{aligned}$$

$$
\begin{aligned}
&+ N_{l+2}(r,0;L^m) + l\overline{N}(r,\infty;L) + N_{l+1}(r,0;L^m)\\
&+ l\overline{N}(r,\infty;L) + O(\log r) + S(r,f)\\
&\le (3l+4)\overline{N}(r,0;f) + (2l+3)\overline{N}(r,0;L) + O(\log r) + S(r,f)\\
&\le (5l+7)T(r) + O(\log r) + S(r,f), \qquad (5.5)
\end{aligned}
$$

where $T(r) = \max\{T(r,f), T(r,L)\}$.

Similarly we have

$$mT(r,L) \le (5l+7)T(r) + S(r,f) + O(\log r). \tag{5.6}$$

From (5.5) and (5.6) we arrive at a contradiction since $m > 5l+7$.

Subcase 2.2. Let $k = 1$.

Using Lemma 4.2, Lemma 4.3 and Lemma 4.8 we have

$$
\begin{aligned}
N_2(r,0;\Phi) &\le N_2(r,0;(f^m)^{(l)}) + O(\log r)\\
&\le T(r,(f^m)^{(l)}) - mT(r,f) + N_{l+2}(r,0;f^m) + O(\log r) + S(r,f)\\
&\le T(r,\Phi) - mT(r,f) + N_{l+2}(r,0;f^m) + O(\log r) + S(r,f). \qquad (5.7)
\end{aligned}
$$

From (5.7) we have

$$mT(r,f) \le T(r,\Phi) - N_2(r,0;\Phi) + N_{l+2}(r,0;f^m) + O(\log r) + S(r,f). \tag{5.8}$$

By Lemma 4.2, Lemma 4.3, Lemma 4.8, Lemma 4.12 and (5.8) we have

$$
\begin{aligned}
mT(r,f) &\le \frac{5}{4}\overline{N}(r,\infty;f) + \frac{1}{2}\overline{N}(r,0;(f^m)^{(l)}) + N_{l+2}(r,0;f^m)\\
&\quad + N_2(r,0;(L^m)^{(l)}) + O(\log r) + S(r,f)\\
&\le \frac{1}{2}\{lN_{l+1}(r,0;f) + l\overline{N}(r,\infty;f)\} + N_{l+2}(r,0;f^m)\\
&\quad + N_2(r,0;(L^m)^{(l)}) + O(\log r) + S(r,f)\\
&\le \frac{3l+5}{2}\overline{N}(r,0;f) + N_{l+2}(r,0;L^m) + O(\log r) + S(r,f)\\
&\le \frac{5l+9}{2}T(r) + O(\log r) + S(r,f), \qquad (5.9)
\end{aligned}
$$

where $T(r) = \max\{T(r,f), T(r,L)\}$.

Similarly we have

$$mT(r,L) \le \frac{5l+9}{2}T(r) + O(\log r) + S(r,f). \tag{5.10}$$

From (5.9) and (5.10) we arrive at a contradiction since $m > \frac{5l+9}{2}$.

Subcase 2.3. Let $k \ge 2$. Clearly Ω has only simple poles and the possible poles of Ω occur at

(i) poles of Φ and Ψ with different multiplicities,

(ii) multiple zeros of Φ and Ψ,

(iii) 1 points of Φ and Ψ with different multiplicities,

(iv) zeros of Φ' which are not zeros of $\Phi(\Phi-1)$,

(v) zeros of Ψ' which are not zeros of $\Psi(\Psi-1)$.

Hence

$$N(r,\infty;\Omega) \le \overline{N}_*(r,\infty;\Phi,\Psi) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}(r,0;\Phi|\ge 2) + \overline{N}(r,0;\Psi|\ge 2) + \overline{N}_\otimes(r,0;\Phi') + \overline{N}_\otimes(r,0;\Psi'). \tag{5.11}$$

Now let z_1 be a simple zero of $\Phi-1$ but not a zero of P. Then z_1 is a simple zero of $\Psi-1$ and a zero of Ω.

Hence by Lemma 4.2 we get

$$N(r,1;\Phi\le 1) \le N(r,0;\Omega) \le N(r,\infty;\Omega) + S(r,f) + O(\log r). \tag{5.12}$$

From (5.11) and (5.12) we have

$$\begin{aligned}\overline{N}(r,1;\Phi) &\le \overline{N}(r,1;\Phi|\ge 2) + N(r,1;\Phi\le 1)\\ &\le \overline{N}_*(r,\infty;\Phi,\Psi) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}(r,0;\Phi|\ge 2)\\ &\quad + \overline{N}(r,0;\Psi|\ge 2) + \overline{N}_\otimes(r,0;\Phi') + \overline{N}_\otimes(r,0;\Psi')\\ &\quad + \overline{N}(r,1;\Phi|\ge 2) + S(r,f) + O(\log r).\end{aligned} \tag{5.13}$$

Using Lemma 4.13 and Lemma 4.14 we have

$$\begin{aligned}&\overline{N}_\otimes(r,0;\Psi') + \overline{N}(r,1;\Phi|\ge 2) + \overline{N}_*(r,1;\Phi,\Psi)\\ \le\ &\overline{N}_\otimes(r,0;\Psi') + \overline{N}(r,1;\Phi|=2)\\ &+ \overline{N}(r,1;\Phi|=3) + \cdots + \overline{N}(r,1;\Phi|=k) + \overline{N}_E(r,1;\Phi,\Psi|>k+1)\\ &+ \overline{N}(r,1;\Phi|>\Psi) + \overline{N}(r,0;\Phi|>\Psi) + \overline{N}_*(r,1;\Phi,\Psi)\\ \le\ &\overline{N}_\otimes(r,0;\Psi') - \overline{N}(r,1;\Phi|=3) - \cdots - (k-2)\overline{N}(r,1;\Phi|=k)\\ &- (k-1)\overline{N}_E(r,1;\Phi,\Psi|>k+1) - (k-1)\overline{N}(r,1;\Phi|>\Psi)\\ &- k\overline{N}(r,0;\Psi|>\Phi) + N(r,1;\Psi) - \overline{N}(r,1;\Psi) + \overline{N}_*(r,1;\Phi,\Psi)\\ \le\ &\overline{N}_\otimes(r,0;\Psi') + N(r,1;\Psi) - \overline{N}(r,1;\Psi)\\ &- (k-2)\overline{N}(r,1;\Phi|>\Psi) - (k-1)\overline{N}(r,0;\Psi|>\Phi)\\ \le\ &N_0(r,0;\Psi') - (k-2)\overline{N}(r,1;\Phi|>\Psi) - (k-1)\overline{N}(r,0;\Psi|>\Phi)\\ \le\ &\overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) - (k-2)\overline{N}(r,1;\Phi|>\Psi)\\ &- (k-1)\overline{N}(r,0;\Psi|>\Phi) + S(r,L)\\ \le\ &\overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) - (k-2)\overline{N}_*(r,1;\Phi,\Psi)\\ &- \overline{N}(r,0;\Psi|>\Phi) + S(r,L).\end{aligned} \tag{5.14}$$

Hence by the second fundamental theorem we get from (5.13) and (5.14), Lemma 4.2, Lemma 4.3 and Lemma 4.8

$$\begin{aligned}mT(r,f) &\le T(r,\Phi) + N_{l+2}(r,0;f^m) - N_2(r,0;\Phi) + S(r,f) + O(\log r)\\ &\le \overline{N}(r,0;\Phi) + \overline{N}(r,\infty;\Phi) + \overline{N}(r,1;\Phi) + N_{l+2}(r,0;f^m)\end{aligned}$$

$$
\begin{aligned}
&- N_2(r,0;\Phi) - \overline{N}_{\otimes}(r,0;\Phi') + S(r,f) + O(\log r)\\
&\le \overline{N}(r,0;\Phi) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;L)\\
&\quad + \overline{N}(r,0;\Phi| \ge 2) + \overline{N}(r,0;\Psi| \ge 2) + \overline{N}(r,1;\Phi| \ge 2)\\
&\quad + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}_{\otimes}(r,0;\Psi') + \overline{N}_{\otimes}(r,0;\Phi') + N_{l+2}(r,0;f^m)\\
&\quad - N_2(r,0;\Phi) - \overline{N}_{\otimes}(r,0;\Phi') + S(r,f) + O(\log r)\\
&\le \overline{N}(r,0;\Phi) + 2\overline{N}(r,\infty;L) + \overline{N}(r,0;\Phi| \ge 2)\\
&\quad + \overline{N}(r,0;\Psi| \ge 2) + \overline{N}(r,1;\Phi| \ge 2) + \overline{N}_*(r,1;\Phi,\Psi)\\
&\quad + \overline{N}_{\otimes}(r,0;\Psi') + N_{l+2}(r,0;f^m) - N_2(r,0;\Phi) + S(r,f) + O(\log r)\\
&\le \overline{N}(r,0;\Psi| \ge 2) + \overline{N}(r,0;\Psi) + N_{l+2}(r,0;f^m) + \overline{N}(r,\infty;\Psi)\\
&\quad - (k-2)\overline{N}_*(r,1;\Phi,\Psi) - \overline{N}(r,0;\Psi| > \Phi) + S(r,f) + O(\log r)\\
&\le N_2(r,0;\Psi) + N_{l+2}(r,0;f^m) + \overline{N}(r,\infty;L) - (k-2)\overline{N}_*(r,1;\Phi,\Psi)\\
&\quad - \overline{N}(r,0;\Psi| > \Phi) + S(r,f) + O(\log r)\\
&\le N_2(r,0;\Psi) + N_{l+2}(r,0;f^m) - (k-2)\overline{N}_*(r,1;\Phi,\Psi)\\
&\quad - \overline{N}(r,0;\Psi| > \Phi) + S(r,f) + O(\log r)\\
&\le N_2(r,0;(L^m)^{(l)}) + N_{l+2}(r,0;f^m) + S(r,f) + O(\log r)\\
&\le (l+2)\overline{N}(r,0;L) + (l+2)\overline{N}(r,0;f) + S(r,f) + O(\log r)\\
&\le (l+2)\{T(r,f) + T(r,L)\} + S(r,f) + O(\log r). \qquad (5.15)
\end{aligned}
$$

Similarly we have

$$ mT(r,L) \le (l+2)\{T(r,f) + T(r,L)\} + S(r,f) + S(r,L) + O(\log r). \qquad (5.16)$$

From (5.15) and (5.16) we arrive at a contradiction since $m > 2l+2$. This completes the proof. □

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