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ON AUTOMORPHISMS IN PRIME RINGS WITH APPLICATIONS

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Abstract. The notions of skew-commuting/commuting/semi-commuting/skew-centralizing/semi-centralizing mappings play an important role in ring theory. $\mathscr{C}^*$-algebras with these properties have been studied considerably less and the existing results are motivating the researchers. This article elaborates the structure of prime rings and $\mathscr{C}^*$-algebras satisfying certain functional identities involving automorphisms.

1. Motivation

The theme of this manuscript is to investigate the n-skew-commuting automorphisms on prime rings and $\mathscr{C}^*$-algebras. The fact that the structure of prime rings and $\mathscr{C}^*$-algebras are describable, has been one of the main motivations for this research. In 1993, Brešar [11] set up the concept of skew-commuting mappings on rings. Specifically, he established that "*If $\mathscr{R}$ is a prime ring of characteristic not* 2*, and $\xi : \mathscr{R} \to \mathscr{R}$ is an additive mapping which is skew-commuting on an ideal $\mathscr{I}$ of $\mathscr{R}$, i.e., $\xi(a)a+a\xi(a) = 0$ for all $a \in \mathscr{I}$, then* $\xi(\mathscr{I}) = (0)$". Also he proved that "*Zero is the only additive mapping which is skew-commuting on a* 2*-torsion free semiprime ring*". On the other hand, in this direction, Brešar [12] characterized commuting additive mappings in this heading by demonstrating that "*Every commuting additive mapping $\xi : \mathscr{R} \to \mathscr{R}$ (where $\mathscr{R}$ is a prime ring, even it is also true for semiprime ring) is of the form $\xi(a) = \lambda a + \gamma(a)$, where $\lambda \in \mathscr{Z}(\mathscr{R})$ and $\gamma : \mathscr{R} \to \mathscr{Z}(\mathscr{R})$ is an additive mapping*". Additional documentation on skew-commuting/commuting mappings and its implementations can be found in [6, 9, 18, 30, 35–38].

Bell and Lucier [8] amplified the idea of skew-commuting/skew-centralizing mappings and they expressed as, "For $n \in \mathbb{Z}^+$, a mapping ξ of a ring $\mathscr{R}$ into itself is called n-skew-commuting (resp. n-skew-centralizing) on S if $\xi(a)a^n + a^n\xi(a) = 0$ (resp. $\xi(a)a^n+a^n\xi(a) \in \mathscr{Z}(\mathscr{R})$) for every $a \in S$". In particular, "for $n = 1, 2$ described as 1-skew-commuting and 2-skew-commuting, respectively".

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In the case of 2-skew-commuting mappings, Fošner [20] analyzed it on 2-torsion free semiprime rings. Specifically, she proved that "*Let $\mathscr{R}$ be a 2-torsion free semiprime ring. Suppose that an additive mapping $\xi : \mathscr{R} \to \mathscr{R}$ satisfies the relation $\xi(a)a^2 + a^2\xi(a) = 0$ for all $a \in \mathscr{R}$. In this case $\xi = 0$*". In [33] Nadeem *et al.* demonstrated that "*If $\mathscr{R}$ is a prime ring with $\mathrm{char}(\mathscr{R}) \neq 2, 3$, $\mathscr{I}$ is an ideal of $\mathscr{R}$ and $\xi : \mathscr{R} \to \mathscr{R}$ an additive mapping such that $\xi(a)a^2 + a^2\xi(a) = 0$ for every $a \in \mathscr{I}$, then $\xi = 0$ on $\mathscr{I}$*". Readers have referred to much more detail about skew-commuting mappings and their extension such as skew-centralizing/semi-centralizing/semi-commuting mappings to [4,17,24,25,29,34]. In [20] one could evaluate the conjecture herein.

Conjecture 1 ([20], Conjecture 6). *Let $n \in \mathbb{Z}^+$ and let $\mathscr{R}$ be a semiprime ring with suitable torsion restrictions. Suppose that an additive mapping $\xi : \mathscr{R} \to \mathscr{R}$ satisfies the relation*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for all $x \in \mathscr{R}$. What can be say about ξ or $\mathscr{R}$?

In [2], Ali set out the following theorem concerning Conjecture 1.

Theorem 1.1 ([2], Theorem 1.1). *Let $n \in \mathbb{Z}^+$ and let $\mathscr{R}$ be a prime ring with $\mathrm{char}(\mathscr{R}) = 0$ or $\mathrm{char}(\mathscr{R}) \geq n$. Suppose that an additive mapping $\xi : \mathscr{R} \to \mathscr{R}$ satisfies the relation*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for every $x \in \mathscr{R}$. In this case $\xi = 0$.

In a recent paper Fošner *et al.* [21] answered in the favour of Conjecture 1 and proved the following:

Theorem 1.2 ([21], Theorem 4). *Let $\mathscr{R}$ be an n-torsion free semiprime ring. Suppose that an additive mapping $\xi : \mathscr{R} \to \mathscr{R}$ satisfies the relation*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for every $x \in \mathscr{R}$. In this case $\xi = 0$.

In 2018, Ali *et al.* [3] discussed n-skew-commuting automorphisms on Lie ideals of prime rings. More exactly they proved that:

Theorem 1.3 ([3], Theorem 1.3). *Let $n \in \mathbb{Z}^+$ and $\mathscr{R}$ be a 2-torsion free semiprime ring. If $\mathscr{R}$ admits an automorphism ξ such that*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for every $x \in \mathscr{R}$, then $\mathscr{R}$ contains a non-zero central ideal.

Given the above-mentioned Conjecture 1 and Theorems 1.1, 1.2, 1.3, it is reasonable to ask what happens if in these results we take ξ as an automorphism/anti-automorphism?

Formulating many results in this setting is achievable by considering suitable conditions on the subset $\mathscr{B}(f, \mathscr{S}) = \{f(x)x^n + x^n f(x) : x \in \mathscr{S}\}$, where $\mathscr{S}$ is an appropriate subset of $\mathscr{R}$, while f is an additive mapping of $\mathscr{R}$ and $n \in \mathbb{Z}^+$. Forasmuch as, numerous researchers have studied $\mathscr{B}(f, \mathscr{S}) = 0$ on well-suited subset of rings involving additive mappings/derivations/automorphisms of $\mathscr{R}$. It would seem natural to interrogate what come into existence if $\mathscr{B}(\xi, \mathscr{S})$ is m-commuting, i.e., $[\mathscr{B}(\xi, \mathscr{S}), x^m] = 0$, where $\mathscr{S} = \mathscr{G}$, a non-central Lie ideal of $\mathscr{R}$ and ξ is an automorphism/anti-automorphism of $\mathscr{R}$. The primary aim of this paper is really to address the above-mentioned question. Especially, for $[\mathscr{B}(\xi, \mathscr{S}), x^m] = 0$, when the prime ring having an automorphism on a non-central Lie ideal of $\mathscr{R}$. Furthermore, the structure of $\mathscr{C}^*$-algebras as an application is identified. The main results of the empirical article are as follows.

Theorem 1.4. *Let $\mathscr{R}$ be a prime ring of characteristic different from 2, ξ be a non-identity automorphism of $\mathscr{R}$, $\mathscr{G}$ be a non-central Lie ideal of $\mathscr{R}$ and $n, m \in \mathbb{Z}^+$. If $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ holds for each $x \in \mathscr{G}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

Theorem 1.5. *Let $\mathscr{A}$ be a primitive $\mathscr{C}^*$-algebra and $m, n \in \mathbb{Z}^+$. If $\xi : \mathscr{A} \to \mathscr{A}$ is an anti-automorphism of $\mathscr{A}$ such that*

$$[\xi(x)x^{*n} + x^{*n}\xi(x), x^{*m}] = 0$$

for every $x \in \mathscr{A}$, then $\mathscr{A}$ is a $\mathscr{C}^* - \mathscr{W}_4$-algebra.

The following example indicates that the presupposition of primeness is necessary in Theorem 1.4.

Example 1.1. Let $\mathscr{R}_1$ be any prime ring not satisfying s_4, the standard identity in four variables and $\mathscr{R}_2 = \mathscr{M}_2(\mathscr{F})$, the ring of 2×2 matrices over a field $\mathscr{F}$. Let $\mathscr{R} = \mathscr{R}_1 \oplus \mathscr{R}_2$ and $\mathscr{G} = \mathscr{M}_2(\mathscr{F}) \oplus 0$. Then $\mathscr{G}$ is a nonzero Lie ideal of $\mathscr{R}$, a semiprime ring. The following is constructed $\zeta : \mathscr{R} \to \mathscr{R}$ such that $\zeta(x_1, x_2) = (x_2, x_1)$. It can be easily seen that ζ is a non-identity automorphism which satisfies Theorem 1.4 but $\mathscr{R}$ does not satisfies s_4.

2. Preliminaries

Some conceptual notions are necessarily demonstrated to develop the proof of the key theorems. Roughly, these ideas are well known and written compactly. The standard identity s_4 in four variables is defined as follows:

$$s_4(X_1, X_2, X_3, X_4) = \sum (-1)^\eta X_{\eta(1)} X_{\eta(2)} X_{\eta(3)} X_{\eta(4)},$$

where $(-1)^\eta$ is the sign of a permutation η of the symmetric group of degree 4.

Throughout the manuscript unless otherwise mention, "$\mathscr{R}$ is a prime ring, $\mathscr{Q}$ is the maximal right ring of quotients of $\mathscr{R}$ with center $\mathscr{C}$, and $\mathscr{R}\mathscr{C}$ is the central closure of $\mathscr{R}$. It is known that $\mathscr{Q}$ is also prime and $\mathscr{C}$ is a field which is called extended centroid of $\mathscr{R}$ (see [7, Chapter 2] for further key definitions and related properties)". *"An automorphism ξ is called X-inner if there exists invertible element $g \in \mathscr{Q}$ such that $\xi(s) = gsg^{-1}$ for every $s \in \mathscr{R}$". "An automorphism ξ is called X-outer if it is not X-inner"*. From [5,7,16], let $\mathscr{A}$ be a ring with a simple left module M and set $\mathscr{D} = \mathrm{End}({}_{\mathscr{A}}M)$. By Schur's Lemma, $\mathscr{D}$ is a division ring. Next, let $\mathrm{End}(M)$ be a ring of all endomorphism of an additive group M acting on M from the left and consider the ring $\mathrm{End}(M_{\mathscr{D}})$ as a subring of $\mathrm{End}(M)$. Given $a \in \mathscr{A}$, define a map $L_a : M \to M$ by $L_a x = ax$ for all $x \in M$. *"An automorphism ξ of $\mathscr{A}$ is called M-inner if there exists an invertible element $T \in \mathrm{End}(M)$ such that $TL_aT^{-1} = L_{\xi(a)}$ for all $a \in \mathscr{A}$; otherwise it is called M-outer"* (see [16] for details). A few well known results can be collected which is able be utilized all through the text.

Fact 2.1 ([5, Lemma 7.1])**.** Let $\mathscr{V}_{\mathscr{D}}$ be a vector space over a division ring $\mathscr{D}$ with $\dim \mathscr{V}_{\mathscr{D}} \geq 2$ and $\mathscr{P} \in \mathrm{End}(\mathscr{V})$. If s and $\mathscr{P}s$ are $\mathscr{D}$-dependent for every $s \in \mathscr{V}$, then there exists $\chi \in \mathscr{D}$ such that $\mathscr{P}s = \chi s$ for every $s \in \mathscr{V}$.

Fact 2.2 ([14, Theorem 1])**.** Let $\mathscr{R}$ be a prime ring and $\mathscr{I}$ be a two sided ideal of $\mathscr{R}$. Then $\mathscr{I}, \mathscr{R}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identities (GPIs) with automorphisms.

Fact 2.3 ([28, Proposition])**.** Let $\mathscr{R}$ be a prime algebra over an infinite field k and let $\mathscr{K}$ be a field extension over k. Then $\mathscr{R}$ and $\mathscr{R} \otimes_k \mathscr{K}$ satisfy the same generalized polynomial identities with coefficients in $\mathscr{R}$.

Fact 2.4 ([38, Lemma 2.1])**.** Let $\mathscr{R}$ be a prime ring with extended centroid $\mathscr{C}$. Then the following conditions are equivalent:

(i) $\dim_{\mathscr{C}} \mathscr{R}\mathscr{C} \leq 4$.
(ii) $\mathscr{R}$ satisfies s_4, the standard identity in four variables.
(iii) $\mathscr{R}$ is commutative or $\mathscr{R}$ embeds in $\mathscr{M}_2(\mathscr{F})$ for a field $\mathscr{F}$.
(iv) $\mathscr{R}$ is algebraic of bounded degree 2 over $\mathscr{C}$.
(v) $\mathscr{R}$ satisfies $[[x^2, y], [x, y]]$.

Fact 2.5 ([5, Theorem 3.2])**.** Let $\mathscr{A}$ be a primitive ring with nonzero socle and let M be a faithful simple left $\mathscr{A}$-module. Then every automorphism of $\mathscr{A}$ is M-inner.

Fact 2.6. Let $\mathscr{R}$ be a prime ring and $\mathscr{L}$ be a non-central Lie ideal of $\mathscr{R}$. If $\mathrm{char}(\mathscr{R}) \neq 2$, by [10, Lemma 1] there exists a nonzero ideal $\mathscr{I}$ of $\mathscr{R}$ such that $0 \neq [\mathscr{I}, \mathscr{R}] \subseteq \mathscr{L}$. If $\mathrm{char}(\mathscr{R}) = 2$ and $\dim_{\mathscr{C}} \mathscr{R}\mathscr{C} > 4$, i.e., $\mathrm{char}(\mathscr{R}) = 2$ and $\mathscr{R}$ does not satisfy s_4, then by [27, Theorem 13] there exists a nonzero ideal $\mathscr{I}$ of $\mathscr{R}$ such that $0 \neq [\mathscr{I}, \mathscr{R}] \subseteq \mathscr{L}$. Thus if either $\mathrm{char}(\mathscr{R}) \neq 2$ or $\mathscr{R}$ does not satisfy s_4, then we may conclude that there exists a nonzero ideal $\mathscr{I}$ of $\mathscr{R}$ such that $[\mathscr{I}, \mathscr{I}] \subseteq \mathscr{L}$.

3. Results on prime rings

Beginning with the following propositions which are needed for the key results to develop the proofs.

Proposition 3.1. *Let $\mathscr{R}$ be a dense subring of $\mathrm{End}(\mathscr{V}_{\mathscr{D}})$, ξ be an automorphism of $\mathscr{R}$ and $n, m \in \mathbb{Z}^+$. If $\mathscr{R}$ satisfies $[\xi([x, y])[x, y]^n + [x, y]^n\xi([x, y]), [x, y]^m] = 0$*

for every $x, y \in \mathscr{R}$*, then either* $\dim(\mathscr{V}_{\mathscr{D}}) \leq 2$ *or* ξ *is an identity map on* $\mathrm{End}(\mathscr{V}_{\mathscr{D}})$*.*

Proof. Since $\mathscr{R}$ is a primitive ring with nonzero socle. By Jacobson [22, Isomorphism Theorem, p. 79], there exists a semi-linear automorphism $\mathscr{P} \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$ such that $\xi(s) = \mathscr{P}s\mathscr{P}^{-1}$ for every $s \in R$. Moreover, for every $v \in \mathscr{V}$, $\lambda \in \mathscr{D}$, $\mathscr{P}(v\lambda) = (\mathscr{P}v)\eta(\lambda)$, where η is an automorphism of $\mathscr{D}$. Using the hypotheses, the following can be obtained

$$\begin{aligned} 0 &= [\xi([x,y])[x,y]^n + [x,y]^n\xi([x,y]), [x,y]^m] \\ &= [\mathscr{P}[x,y]\mathscr{P}^{-1}[x,y]^n + [x,y]^n\mathscr{P}[x,y]\mathscr{P}^{-1}, [x,y]^m] \end{aligned}$$

for all $x, y \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$. Expect to begin with that v and $\mathscr{P}^{-1}v$ are $\mathscr{D}$-dependent for each $v \in \mathscr{V}$. Chuang *et al.,* [15, Lemma 1] provided $\mathscr{P}^{-1}v = v\theta$, where $\theta \in \mathscr{D}$ and $v \in \mathscr{V}$. Hence for each $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$, $\mathscr{P}^{-1}(sv) = sv\theta$ and consequently $sv = \mathscr{P}(sv\theta) = \mathscr{P}(s(v\theta)) = \mathscr{P}s\mathscr{P}^{-1}(v) = \xi(s)v$ for all $v \in \mathscr{V}$, $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$. In addition $(\xi(s) - s)\mathscr{V} = (0)$ for each $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$. In this way in all, $\xi(s) = s$ for each $s \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$. This appears that ξ is an identity map on $\mathrm{End}(\mathscr{V}_{\mathscr{D}})$, as required.

In this way, there exists $v \in \mathscr{V}$ such that v and $\mathscr{P}^{-1}v$ are straightly $\mathscr{D}$-independent. Firstly, one can accept that $\dim(\mathscr{V}_{\mathscr{D}}) \geq 4$. At that point take $u, w \in \mathscr{V}$ such that $\{u, v, w, \mathscr{P}^{-1}v\}$ is $\mathscr{D}$-independent. For this, let $x, y \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$ such that

$$\begin{aligned} &xv = 0, \quad x\mathscr{P}^{-1}v = 0, \quad xu = \mathscr{P}^{-1}w, \quad xw = u; \\ &yv = 0, \quad y\mathscr{P}^{-1}v = u, \quad yu = -w, \qquad yw = 0. \end{aligned}$$

It is taken note that $[x,y]\mathscr{P}^{-1}v = \mathscr{P}^{-1}w$, $[x,y]w = w$, $[x,y]v = 0$ and subsequently, the presumption yields

$$\begin{aligned} 0 &= ([\mathscr{P}[x,y]\mathscr{P}^{-1}[x,y]^n + [x,y]^n\mathscr{P}[x,y]\mathscr{P}^{-1}, [x,y]^m])v \\ &= -w, \end{aligned}$$

a contradiction, implying that $\dim(\mathscr{V}_{\mathscr{D}}) \leq 3$.

Secondly, one could assume that $\dim(\mathscr{V}_{\mathscr{D}}) = 3$. Take $w \in \mathscr{V}$ such that $\{v, w, \mathscr{P}^{-1}v\}$ is $\mathscr{D}$-independent and then $\{v, w, \mathscr{P}^{-1}v\}$ forms a $\mathscr{D}$-basis of $\mathscr{V}$. If $\mathscr{P}(v + \mathscr{P}^{-1}v + w) \in v\mathscr{D}$ and $\mathscr{P}(\mathscr{P}^{-1}v + w) \in v\mathscr{D}$, then $\mathscr{P}v, \mathscr{P}(\mathscr{P}^{-1}v + w) \in v\mathscr{D}$ and then $v, \mathscr{P}^{-1}v + w \in \mathscr{P}^{-1}(v\mathscr{D}) = \mathscr{P}^{-1}(v)\xi^{-1}(\mathscr{D}) = \mathscr{P}^{-1}v\mathscr{D}$, contradicting the fact that $\{v, \mathscr{P}^{-1}v + w\}$ is $\mathscr{D}$-independent. Therefore, one can pick $\rho \in \{0, 1\}$ such that $u = \rho v + \mathscr{P}^{-1}v + w$ and $\mathscr{P}u \notin v\mathscr{D}$. Write $\mathscr{P}u = v\alpha + \mathscr{P}^{-1}v\beta + w\gamma$, where $\alpha, \beta, \gamma \in \mathscr{D}$ and β, γ both are not zero. By density of theorem, there exist $x, y \in \mathrm{End}(\mathscr{V}_{\mathscr{D}})$ such that

$$\begin{aligned} &xv = 0, \quad x\mathscr{P}^{-1}v = w, \quad xw = 0; \\ &yv = 0, \quad y\mathscr{P}^{-1}v = 0, \quad yw = -u. \end{aligned}$$

That is $xu = w$, $yu = -u$, $x\mathscr{P}u = w\beta$ and $y\mathscr{P}u = -u\gamma$. Therefore, it can be seen that $[x,y]v = 0$, $[x,y]\mathscr{P}^{-1}v = u$, $[x,y]w = -w$. Also, $[x,y]u = u - w$,

$[x,y]\mathscr{P}u = u\beta - w\gamma$ and $[x,y]^2\mathscr{P}u = (u-w)\beta + w\gamma$. If $m+n$ is even, then $[x,y]^{m+n}\mathscr{P}u = (u-w)\beta + w\gamma$ and $[x,y]^{m+n}\mathscr{P}u = (u\beta - w\gamma)$, when $m+n$ is odd. As β,γ are not both zero and u,w are $\mathscr{D}$-dependent, so it is easy to see that $[x,y]^{m+n}\mathscr{P}u \neq 0$. Thus in all, it is that

$$\begin{aligned} 0 &= ([\mathscr{P}[x,y]\mathscr{P}^{-1}[x,y]^n + [x,y]^n\mathscr{P}[x,y]\mathscr{P}^{-1}, [x,y]^m])v \\ &= -[x,y]^2\mathscr{P}u, \end{aligned}$$

a contradiction, implying that $\dim(\mathscr{V}_{\mathscr{D}}) \leq 2$. □

Theorem 3.1. *Let $\mathscr{R}$ be a non-commutative prime ring, ξ be a non-identity automorphism of $\mathscr{R}$ and $n,m \in \mathbb{Z}^+$. If $[\xi([x,y])[x,y]^n + [x,y]^n\xi([x,y]), [x,y]^m] = 0$ holds for each $x,y \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

Proof. Firstly, the assumption can be made that ξ is X-inner, i.e., $s^{\xi} = psp^{-1}$ for every $s \in \mathscr{R}$. As ξ is the non-identity map, so $p \notin \mathscr{C}$. Then $[p[x,y]p^{-1}[x,y]^n + [x,y]^n p[x,y]p^{-1}, [x,y]^m] = 0$ is a non-trivial generalized polynomial identity (GPI) of $\mathscr{R}$. By Fact 2.2, $[p[x,y]p^{-1}[x,y]^n + [x,y]^n p[x,y]p^{-1}, [x,y]^m] = 0$ is also a generalized polynomial identity (GPI) of $\mathscr{Q}$. Let $\mathscr{F}$ be the algebraic closure of $\mathscr{C}$ if $\mathscr{C}$ is infinite and set $\mathscr{F} = \mathscr{C}$ for $\mathscr{C}$ finite. By Fact 2.3, $[p[x,y]p^{-1}[x,y]^n + [x,y]^n p[x,y]p^{-1}, [x,y]^m] = 0$ is also a generalized polynomial identity (GPI) of $\mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F}$. In addition, $\mathscr{Q} \otimes \mathscr{C}\mathscr{F}$ is a prime ring with $\mathscr{F}$ as its extended centroid, in view of [19, Theorem 3.5]. Then $\mathscr{Q} \otimes \mathscr{C}\mathscr{F}$ is a prime ring that satisfies a non-trivial generalized polynomial identity (GPI), and its extended centroid $\mathscr{F}$ is either a closed algebraic field or a finite field. By Martindale's theorem [5, Corollary 6.1.7], $\mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F}$ may be a primitive ring having nonzero socle with the field $\mathscr{F}$ as its related division ring. By Jacobson theorem [22, p. 75], $\mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F}$ is isomorphic to a dense subring of the ring of linear transformations on a vector space $\mathscr{V}$ over $\mathscr{F}$, containing nonzero linear transformations of finite rank. By suspicion, $\mathscr{R}$ is non-commutative so $\mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F}$ is non-commutative. By Proposition 3.1, $\dim \mathscr{V}_{\mathscr{F}} \leq 2$. This implies that $\mathscr{R} \subseteq \mathscr{Q} \cong \mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F} \subseteq \mathscr{Q} \otimes_{\mathscr{C}} \mathscr{F} \cong \mathscr{M}_2(\mathscr{F})$. We get the appropriate conclusion in view of Fact 2.4.

Secondly, suppose that ξ is X-outer. Therefore

$$[\xi([x,y])[x,y]^n + [x,y]^n\xi([x,y]), [x,y]^m] = 0$$

is a nontrivial differential identity with automorphism ξ. Otherwise, if it is trivial we get a contradiction, as ξ is the non-identity map. It follows $\mathscr{R}$ is a GPI ring from Main Theorem in [13]. By exploiting the techniques presented above, one can easily seen that $\mathscr{R}$ could be a primitive ring with a faithful simple (left) $\mathscr{R}$-module $\mathscr{V}$ and related division ring $\mathscr{C}$ such that $H = soc(\mathscr{R}) \neq 0$. At that point, by the Jacobson density theorem [23], $\mathscr{R}$ is isomorphic to a dense ring of linear transformations of vector space $\mathscr{V}$ over $\mathscr{C}$. Take note that $\dim \mathscr{V}_{\mathscr{C}} \geq 2$ (something else, $\mathscr{R}$ is commutative). On the off chance that $\dim \mathscr{V}_{\mathscr{C}} = 2$, at that point $\mathscr{R} \cong \mathscr{M}_2(\mathscr{C})$, i.e., $\mathscr{R}$ fulfills s_4, as required. In addition, ξ is $\mathscr{V}$-inner in see of Fact 2.5. Subsequently, there exists an invertible element $T \in \mathrm{End}(\mathscr{V})$

such that $TL_rT^{-1} = L_{\xi(r)}$ for all $r \in \mathscr{R}$. In the event that the ring $\mathscr{R}$ is primitive and $\mathscr{V}$ could be a faithful $\mathscr{R}$-module, able to consider $\mathscr{R}$ as a subring of $\mathrm{End}(\mathscr{V}_{\mathscr{C}})$ by means of the embedding $r \to L_r$ (see [16] for points of interest). Hence in see of Proposition 3.1, we get the specified conclusion. □

Hence, having all the necessary frames to execute the proof of the first fundamental result of this paper.

Proof of Theorem 1.4. The presumption gives $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ for all $x \in \mathscr{G}$. In see of Fact 2.6, there exists an ideal $\mathscr{I}$ of $\mathscr{R}$ such that $(0) \neq [\mathscr{I}, \mathscr{R}] \subseteq \mathscr{G}$ and $[\mathscr{G}, \mathscr{G}] \neq (0)$. Moreover, $\mathscr{R}$ is non-commutative as $\mathscr{G}$ is a non-central Lie ideal of $\mathscr{R}$. In this way by the given speculations, $\mathscr{I}$ as well as $\mathscr{R}$ (Fact 2.2) satisfies $[\xi([x,y])[x,y]^n + [x,y]^n\xi([x,y]), [x,y]^m] = 0$. Hence Theorem 3.1 delivers the specified conclusion. □

The taking after corollary could be a special case of Theorems 3.1 and 1.4, which is of autonomous intrigued.

Corollary 3.1. *Let $\mathscr{R}$ be a prime ring of characteristic different from 2, ξ be a non-identity automorphism of $\mathscr{R}$ and $n, m \in \mathbb{Z}^+$. If $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ holds for each $x \in \mathscr{R}$, then $\mathscr{R}$ satisfies s_4, the standard identity in four variables.*

4. Results on $\mathscr{C}^*$-algebras

This section addresses the applications of strict ring theoretical framework on $\mathscr{C}^*$-algebras. Let's introduce some well known facts and elementary definitions for the purpose of fullness. *"A Banach algebra is a linear associate algebra which, as a vector space, is a Banach space with norm $\|\cdot\|$ satisfying the multiplicative inequality; $\|xy\| \leq \|x\|\|y\|$ for all x and y in $\mathscr{A}$".*

A Banach algebra $\mathscr{A}$ is a PI-algebra if and only if there exist $n \in \mathbb{N}$ and a polynomial $q \in \mathscr{W}_n$, $q \neq 0$, such that $q(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$, where $\mathscr{W}_n$ is the set of all complex polynomials in n non-commuting variables. *"An involution on an algebra $\mathscr{A}$ is a map $x \mapsto x^*$ of $\mathscr{A}$ onto such that the following conditions are hold*: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, *and* (iii) $(x + \lambda y)^* = x^* + \overline{\lambda}y^*$ *for all $x, y \in \mathscr{A}$ and $\lambda \in \mathbb{C}$ the field of complex number, where $\overline{\lambda}$ is the conjugate of λ".* Of course the prototypical case of an involution on a Banach algebra is the adjoint operation on $\mathscr{B}(\mathscr{H})$, the set of bounded linear operators on Hilbert space $\mathscr{H}$. Another important example is a complex conjugation on $\mathbb{C}(\mathbb{X})$, the set of all continuous complex valued functions on $\mathbb{X}$, a compact Hausdroff space defined as $f^*(x) := \overline{f(x)}$. *"An algebra equipped with an involution is called a $*$-algebra or algebra with involution".* A Banach $*$-algebra is a Banach algebra $\mathscr{A}$ together with an isometric involution $\|x^*\| = \|x\|$ for all $x \in \mathscr{A}$. A Banach $*$-algebra is called a $\mathscr{C}^*$-algebra $\mathscr{A}$ if $\|x^*x\| = \|x\|^2$ for all $x \in \mathscr{A}$. "A $\mathscr{C}^*$-algebra $\mathscr{A}$ is primitive if its zero ideal is primitive, i.e., if $\mathscr{A}$ has a faithful non-zero irreducible representation".

Let $\mathscr{W}_n$ denote the standard polynomial of degree n in n non-commuting variables, $\mathscr{W}_n(a_1, a_2, \ldots, a_n) = \Sigma_{\sigma \in S_n}$ sign $(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}$, where S_n is the set of all permutations of $\{1, 2, 3, \ldots, n\}$ and sign$(\sigma) = \pm 1$ for σ even (odd) (see [26], [31] and references therein). An algebra $\mathscr{A}$ is said to be a $\mathscr{C}^*$-$\mathscr{W}_n$-algebra if $\mathscr{W}_n(a_1, a_2, \ldots, a_n) = 0$ for each choice of elements $a_1, a_2, \ldots, a_n \in \mathscr{A}$. In particular, an algebra is a $\mathscr{C}^* - \mathscr{W}_4$-algebra if it satisfies the standard identity $\mathscr{W}_4(a_1, a_2, a_3, a_4) = 0$ for all $a_1, a_2, a_3, a_4 \in \mathscr{A}$. Moreover, an algebra is a $\mathscr{C}^* - \mathscr{W}_2$-algebra if and only if it is commutative, i.e., a $\mathscr{C}^* - \mathscr{W}_2$-algebra is commutative if it satisfies the standard identity $\mathscr{W}_2(a_1, a_2) = 0$ for all $a_1, a_2 \in \mathscr{A}$. Many researchers discussed Gelfand's theory for Banach algebra and $\mathscr{C}^*$-algebra namely, Banach-$\mathscr{W}_{2n}$-algebra and $\mathscr{C}^* - \mathscr{W}_{2n}$-algebra.

Theorem 4.1. *Let $\mathscr{A}$ be a primitive $\mathscr{C}^*$-algebra and $m, n \in \mathbb{Z}^+$. If $\xi : \mathscr{A} \to \mathscr{A}$ is an automorphism of $\mathscr{A}$ such that*

$$[\xi(x)x^n + x^n\xi(x), x^m] = 0$$

for all $x \in \mathscr{A}$, then $\mathscr{A}$ is a $\mathscr{C}^* - \mathscr{W}_4$-algebra.

Proof. In the light of hypothesis given, $\xi : \mathscr{A} \to \mathscr{A}$ is satisfying $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ for all $x \in \mathscr{A}$. Within the light of [32, Theorem 5.4.5], $\mathscr{A}$ is a prime $\mathscr{C}^*$-algebra, as it is primitive and hence $\mathscr{A}$ is a prime ring [1]. Therefore, implementation of Corollary 3.1 heeds the needed result, thereby proving the theorem. □

Proof of Theorem 1.5. The assumption gives $[\xi(x)x^{*m} + x^{*m}\xi(x), x^{*n}] = 0$ for all $x \in \mathscr{A}$. In the last expression, exchanging x^* for x, gives $[\xi(x^*)x^n + x^n\xi(x^*), x^m] = 0$ for every $x \in \mathscr{A}$. Let us set a map $\psi : \mathscr{A} \to \mathscr{A}$ such that $\psi(x) = \xi(x^*)$ for all $x \in \mathscr{A}$. Then, understandably $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in \mathscr{A}$, i.e., ψ is an automorphism of $\mathscr{A}$ thereby getting for all $x \in \mathscr{A}$, $[\psi(x)x^n + x^n\psi(x), x^m] = 0$. The conclusion can be drawn from Theorem 4.1. This finishes out theorem's proof. □

References

[1] G. Abrams and M. Tomforde, *A class of C^*-algebras that are prime but not primitive*, Münster J. Math. **7** (2014), no. 2, 489–514. https://doi.org/10.17879/58269761901

[2] S. Ali, *On n-skew-commuting mappings of prime rings with applications to C^*-algebras*, Proc. Indian Acad. Sci. (Math. Sci), Accepted, 2017.

[3] S. Ali, M. Ashraf, M. A. Raza, and A. N. Khan, *N-commuting mappings on (semi)-prime rings with applications*, Comm. Algebra **47** (2019), no. 5, 2262–2270. https://doi.org/10.1080/00927872.2018.1536203

[4] M. Ashraf, M. A. Raza, and S. A. Pary, *Commutators having idempotent values with automorphisms in semi-prime rings*, Math. Rep. (Bucur.) **20(70)** (2018), no. 1, 51–57.

[5] K. I. Beidar and M. Brešar, *Extended Jacobson density theorem for rings with derivations and automorphisms*, Israel J. Math. **122** (2001), 317–346. https://doi.org/10.1007/BF02809906

[6] K. I. Beidar, Y. Fong, P. Lee, and T. Wong, *On additive maps of prime rings satisfying the Engel condition*, Comm. Algebra **25** (1997), no. 12, 3889–3902. https://doi.org/10.1080/00927879708826093

[7] K. I. Beidar, W. S. Martindale, III, and A. V. Mikhalev, *Rings with generalized identities*, Monographs and Textbooks in Pure and Applied Mathematics, 196, Marcel Dekker, Inc., New York, 1996.
[8] H. E. Bell and J. Lucier, *On additive maps and commutativity in rings*, Results Math. **36** (1999), no. 1-2, 1–8. https://doi.org/10.1007/BF03322096
[9] H. E. Bell and W. S. Martindale, III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987), no. 1, 92–101. https://doi.org/10.4153/CMB-1987-014-x
[10] J. Bergen, I. N. Herstein, and J. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra **71** (1981), no. 1, 259–267. https://doi.org/10.1016/0021-8693(81)90120-4
[11] M. Brešar, *On skew-commuting mapping of rings*, Bull. Austral. Math. Soc. **41** (1993), 291–296.
[12] ______, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), no. 2, 385–394. https://doi.org/10.1006/jabr.1993.1080
[13] C.-L. Chuang, *Differential identities with automorphisms and antiautomorphisms. I*, J. Algebra **149** (1992), no. 2, 371–404. https://doi.org/10.1016/0021-8693(92)90023-F
[14] ______, *Differential identities with automorphisms and antiautomorphisms. II*, J. Algebra **160** (1993), no. 1, 130–171. https://doi.org/10.1006/jabr.1993.1181
[15] C.-L. Chuang, M.-C. Chou, and C.-K. Liu, *Skew derivations with annihilating Engel conditions*, Publ. Math. Debrecen **68** (2006), no. 1-2, 161–170.
[16] C.-L. Chuang and C.-K. Liu, *Extended Jacobson density theorem for rings with skew derivations*, Comm. Algebra **35** (2007), no. 4, 1391–1413. https://doi.org/10.1080/00927870601142207
[17] B. Davvaz and M. A. Raza, *A note on automorphisms of Lie ideals in prime rings*, Math. Slovaca **68** (2018), no. 5, 1223–1229. https://doi.org/10.1515/ms-2017-0179
[18] N. J. Divinsky, *On commuting automorphisms of rings*, Trans. Roy. Soc. Canada Sect. III **49** (1955), 19–22.
[19] T. S. Erickson, W. S. Martindale, 3rd, and J. M. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975), no. 1, 49–63. http://projecteuclid.org/euclid.pjm/1102868622
[20] M. Fošner, *A result concerning additive mappings in semiprime rings*, Math. Slovaca **65** (2015), no. 6, 1271–1276. https://doi.org/10.1515/ms-2015-0088
[21] M. Fošner, B. Marcen, and N. Rehman, *On skew-commuting mappings in semiprime rings*, Math. Slovaca **66** (2016), no. 4, 811–814. https://doi.org/10.1515/ms-2015-0183
[22] N. Jacobson, *Structure of rings*, American Mathematical Society Colloquium Publications, Vol. 37. Revised edition, American Mathematical Society, Providence, RI, 1964.
[23] ______, PI-*algebras*, Lecture Notes in Mathematics, Vol. 441, Springer-Verlag, Berlin, 1975.
[24] A. Kaya, *A theorem on semicentralizing derivations of prime rings*, Math. J. Okayama Univ. **27** (1985), 11–12.
[25] A. Kaya and C. Koç, *Semicentralizing automorphisms of prime rings*, Acta Math. Acad. Sci. Hungar. **38** (1981), no. 1-4, 53–55. https://doi.org/10.1007/BF01917518
[26] N. Krupnik, S. Roch, and B. Silbermann, *On C^*-algebras generated by idempotents*, J. Funct. Anal. **137** (1996), no. 2, 303–319. https://doi.org/10.1006/jfan.1996.0048
[27] C. Lanski and S. Montgomery, *Lie structure of prime rings of characteristic* 2, Pacific J. Math. **42** (1972), 117–136. http://projecteuclid.org/euclid.pjm/1102968014
[28] P. H. Lee and T. L. Wong, *Derivations cocentralizing Lie ideals*, Bull. Inst. Math. Acad. Sinica **23** (1995), no. 1, 1–5.
[29] P.-K. Liau and C.-K. Liu, *On automorphisms and commutativity in semiprime rings*, Canad. Math. Bull. **56** (2013), no. 3, 584–592. https://doi.org/10.4153/CMB-2011-185-5

[30] C.-K. Liu, *An Engel condition with automorphisms for left ideals*, J. Algebra Appl. **13** (2014), no. 2, 1350092, 14 pp. https://doi.org/10.1142/S0219498813500928
[31] V. Müller, *Nil, nilpotent and PI-algebras*, in Functional analysis and operator theory (Warsaw, 1992), 259–265, Banach Center Publ., 30, Polish Acad. Sci. Inst. Math., Warsaw, 1994.
[32] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Inc., Boston, MA, 1990.
[33] M. Nadeem, M. Aslam, and M. A. Javed, *On 2-skew commuting mappings of prime rings*, Gen. Math. Notes **31** (2015), 1–9.
[34] A. Najati and M. M. Saem, *Skew-commuting mappings on semiprime and prime rings*, Hacet. J. Math. Stat. **44** (2015), no. 4, 887–892.
[35] M. A. Raza and N. ur Rehman, *An identity on automorphisms of Lie ideals in prime rings*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **62** (2016), no. 1, 143–150. https://doi.org/10.1007/s11565-016-0240-4
[36] N. Rehman and M. Arif Raza, *On m-commuting mappings with skew derivations in prime rings*, St. Petersburg Math. J. **27** (2016), no. 4, 641–650; translated from Algebra i Analiz **27** (2015), no. 4, 74–86. https://doi.org/10.1090/spmj/1411
[37] J. Vukman, *Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **109** (1990), no. 1, 47–52. https://doi.org/10.2307/2048360
[38] Y. Wang, *Power-centralizing automorphisms of Lie ideals in prime rings*, Comm. Algebra **34** (2006), no. 2, 609–615. https://doi.org/10.1080/00927870500387812

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