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RATIONALIZED EVALUATION SUBGROUPS OF THE COMPLEX HOPF FIBRATION

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Abstract. In this paper, we compute the rational evaluation subgroup of the Hopf fibration $S^{2n+1} \hookrightarrow \mathbb{C}P(n)$. We show that, for the Sullivan model $\phi : A \to B$, where A and B are the minimal Sullivan models of $\mathbb{C}P(n)$ and S^{2n+1} respectively, the evaluation subgroup $G_n(A, B; \phi)$ and the relative evaluation subgroup $G_n^{rel}(A, B; \phi)$ of ϕ are generated by single elements.

1. Introduction

Let X be a based CW-complex. An element $a \in \pi_n(X)$ is a Gottlieb element if there is a continuous map $H : X \times S^n \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} X \vee S^n & \xrightarrow{id_X \vee h} & X \vee X \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle \bigtriangledown} \\ X \times S^n & \xrightarrow{\quad H \quad} & X, \end{array}$$

where $h : S^n \to X$ is a representative of a and $\bigtriangledown$ is the folding map. Moreover, $G_n(X)$ is the set of all Gottlieb elements $a \in \pi_n(X)$ and is called the n-th Gottlieb group of X or the n-th evaluation subgroup of $\pi_n(X)$ [4]. Gottlieb groups play an important role in the study of problems in topology, fixed point theory and homotopy theory of fibrations.

Let $f : X \to Y$ be a based map of simply connected finite CW-complexes, in [5], the evaluation at the basepoint of X gives the *evaluation map* $\omega : \mathrm{Map}(X, Y; f) \to Y$, where $\mathrm{Map}(X, Y; f)$ is the component of f in the space of mappings from X to Y. The image of the homomorphism induced in homotopy groups

$$\omega_\sharp : \pi_* \mathrm{Map}(X, Y; f) \to \pi_*(Y)$$

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is called the n-th *evaluation subgroup of* p and it is denoted by $G_n(Y, X; p)$. Moreover, if $f = id_X$, the space $\mathrm{Map}(X, Y; f)$ is the monoid $\mathrm{aut}_1(X)$ of self-equivalences of X homotopic to the identity of X, then $ev : \mathrm{aut}_1(X) \to X$ is the evaluation map and the image of the induced homomorphism

$$ev_\sharp : \pi_*(\mathrm{aut}_1(X)) \to \pi_*(X)$$

is $G_n(X)$, i.e., the n-th Gottlieb group. Moreover, in [8], Woo and Lee studied the relative evaluation subgroups $G_n^{rel}(X, Y; p)$ and proved that they fit in a sequence

$$\cdots \to G_{n+1}^{rel}(X, Y; f) \to G_n(X) \to G_n(X, Y; f) \to \cdots$$

called the G-sequence of f. Further, in [5], Smith and Lupton identify the homomorphism induced on rational homotopy groups by the evaluation map $\omega : \mathrm{Map}(X, Y; f) \to Y$, in terms of a map of complexes of derivations constructed directly from the Sullivan minimal model of f. In this paper, we use a map of complexes of derivations of minimal Sullivan models of mapping spaces to compute rational relative Gottlieb groups of the Hopf fibration $S^{2n+1} \hookrightarrow \mathbb{C}P(n)$.

2. Preliminaries

Here we fix terminology and recall some standard facts on differential graded algebras. All vector spaces and algebras are taken over a field $\mathbb{Q}$ of rational numbers.

Definition 2.1. A commutative graded differential algebra (cdga) is a graded algebra (A, d) such that $xy = (-1)^{|x||y|}yx$ and $d(xy) = (dx)y + (-1)^{|pq|}x(dy)$ for all $x \in A^p, y \in A^q$. It is said to be connected if $H^0(A) \cong \mathbb{Q}$. If $V = \oplus_{i\geq 0} V^i$ with $V^{\mathrm{even}} := \oplus_{i\geq 0} V^{2i}$ and $V^{\mathrm{odd}} := \oplus_{i\geq 1} V^{2i-1}$, then $\wedge V$ denotes the free commutative graded algebra defined by the tensor product

$$\wedge V = S(V^{\mathrm{even}}) \otimes E(V^{\mathrm{odd}}),$$

where $S(V^{\mathrm{even}})$ is the symmetric algebra on V^{even} and $E(V^{\mathrm{odd}})$ is the exterior algebra on V^{odd}.

Definition 2.2. A Sullivan algebra is a commutative differential graded algebra $(\wedge V, d)$ where $V = \cup_{k\geq 0} V(k)$ and $V(0) \subset V(1) \cdots$ such that $dV(0) = 0$ and $dV(k) \subset \wedge V(k-1)$. It is called minimal if $dV \subset \wedge^{\geq 2} V$.

If (A, d) is a cdga of which the cohomology is connected and finite dimensional in each degree, then there always exists a quasi-isomorphism from a Sullivan algebra $(\wedge V, d)$ to (A, d) [2]. To each simply connected space, Sullivan associates a cdga $A_{PL}(X)$ of rational polynomial differential forms on X that uniquely determines the rational homotopy type of X [7]. A minimal Sullivan model of X is a minimal Sullivan model of $A_{PL}(X)$. More precisely, $H^*(\wedge V, d) \cong H^*(X; \mathbb{Q})$ as graded algebras and $V \cong \pi_*(X) \otimes \mathbb{Q}$ as graded vector spaces.

3. Evaluation subgroups of a map

Consider the Hopf fibration $f : S^{2n+1} \hookrightarrow \mathbb{C}P(n)$. The Hopf fibration plays an important role in the study of Sasakian manifolds. More precisely, the most basic example of a simply connected compact regular Sasakian manifold is the odd dimensional sphere S^{2n+1} considered as the total space of the Hopf fibration $S^{2n+1} \hookrightarrow \mathbb{C}P(n)$ (see [1]). In [3], the minimal model of $\mathbb{C}P(n)$ is given by $(\wedge(x_2, y_{2n+1}), d)$, where $dx_2 = 0$, $dy_{2n+1} = x_2^{n+1}$ and the minimal model of S^{2n+1} is given by $(\wedge x_{2n+1}, 0)$. Moreover, the minimal Sullivan model of f is given by

$$\phi : (\wedge(x_2, y_{2n+1}), d) \to (\wedge x_{2n+1}, 0),$$

where $\phi(x_2) = 0$ and $\phi(y_{2n+1}) = x_{2n+1}$.

We study the evaluation subgroups of ϕ. Let (A, d) be a commutative differential graded algebra. A derivation θ of degree k is a linear mapping $\theta : A^n \to A^{n-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{k|a|}a\theta(b)$.

Denote by $\mathrm{Der}_k A$ the vector space of all derivation of degree k and $\mathrm{Der}\, A = \oplus_k \mathrm{Der}_k A$. The commutator bracket induces a graded Lie algebra structure on $\mathrm{Der}\, A$. Moreover, $(\mathrm{Der}\, A, \delta)$ is a differential graded Lie algebra [7], with the differential δ defined in the usual way by

$$\delta\theta = d \circ \theta + (-1)^{k+1}\theta \circ d.$$

Let $(\wedge V, d)$ be a Sullivan algebra where V is spanned by $\{v_1, \dots, v_k\}$. Then, $\mathrm{Der} \wedge V$ is spanned by $\theta_1, \dots, \theta_k$, where θ_i is the unique derivation of $\wedge V$ defined by $\theta_i(v_j) = \delta_{ij}$. The derivation θ_i will be denoted by $(v_i, 1)$. Moreover, an element $v \in V \cong \pi_*(X) \otimes \mathbb{Q}$ is a Gottlieb element of $\pi_*(X) \otimes \mathbb{Q}$ if and only if there is a derivation θ of $\wedge V$ satisfying $\theta(v) = 1$ and such that $\delta\theta = 0$ [2].

Let $\phi : (A, d) \to (B, d)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping $\theta : A^n \to B^{n-k}$ for which $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$.

We consider only derivations of positive degree. Denote by $\mathrm{Der}_n(A, B; \phi)$ the vector space of ϕ-derivations of degree n for $n > 0$ and by $\mathrm{Der}(A, B; \phi) = \oplus_n \mathrm{Der}_n(A, B; \phi)$ the $\mathbb{Z}$-graded vector space of all ϕ-derivations. The differential graded vector space of ϕ-derivations is denoted by $(\mathrm{Der}(A, B; \phi), \partial)$, where the differential ∂ is defined by $\partial\theta = d_B \circ \theta + (-1)^{k+1}\theta \circ d_A$. In case $A = B$ and $\phi = 1_B$, then $(\mathrm{Der}(B, B; 1), \partial)$ is just the usual differential graded Lie algebra of derivations on the cdga B [5]. We note that, there is an isomorphism of graded vector spaces

$$\mathrm{Der}(A, B; \phi) \cong \mathrm{Hom}(V, B).$$

If $\{v_i\}$ is a basis of V, then the vector space $\mathrm{Der}(A, B; \phi)$, is spanned by the unique ϕ-derivation θ denoted by (v_i, b_i) such that $\theta_i(v_i) = b_i$, where $b_i \in B$ and $\theta_i(v_j) = 0$ for $i \neq j$. Moreover, in [5], pre-composition with ϕ gives a chain complex map $\phi^* : \mathrm{Der}(B, B; 1) \to \mathrm{Der}(A, B; \phi)$ and post-composition with the augmentation $\varepsilon : B \to \mathbb{Q}$ gives a chain complex map $\varepsilon_* : \mathrm{Der}(A, B; \phi) \to \mathrm{Der}(A, \mathbb{Q}; \varepsilon)$. The evaluation subgroup of ϕ is defined as follows:

$$G_n(A, B; \phi) = \mathrm{Im}\{H(\varepsilon_*) : H_n(\mathrm{Der}(A, B; \phi)) \to H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

In case $A = B$ and $\phi = 1_B$, we get the Gottlieb group of (B, d), defined as follows

$$G_n(B) = \mathrm{Im}\{H(\varepsilon_*) : H_n(\mathrm{Der}(B, B; 1)) \to H_n(\mathrm{Der}(B, \mathbb{Q}; \varepsilon))\}.$$

In particular, $G_n(B) \cong G_n(X_{\mathbb{Q}})$, if B is the minimal Sullivan model of a simply connected space X [2, Proposition 29.8].

Proposition 3.1. *Let $B = (\wedge x_{2n+1}, 0)$. Then $G_n(B) = \langle [x^*_{2n+1}] \rangle$.*

Proof. We have that $\mathrm{Der}(B, B; 1) = (\mathbb{Q}\alpha_{2n+1}, 0)$ where α_{2n+1} is the derivation taking x_{2n+1} to one. Thus, $[\alpha_{2n+1}]$ is the non zero homology class in $H_*(\mathrm{Der}(B, B; 1))$. Moreover, $\varepsilon_*(\alpha_{2n+1}) = x^*_{2n+1}$. As S^{2n+1} is a finite CW-complex then $G_{even}(B) = 0$ [2, Page 379]. Hence $G_n(B) = \langle [x^*_{2n+1}] \rangle$. □

Proposition 3.2. *Consider the Hopf fibration $S^{2n+1} \hookrightarrow \mathbb{C}P(n)$ and $\phi : A \to B$ its Sullivan model. Then $G_n(A, B; \phi) = \langle [y^*_{2n+1}] \rangle$.*

Proof. Define $\theta_{2n+1} = (y_{2n+1}, 1)$ in $\mathrm{Der}(A, B; \phi)$. Then $\partial\theta_{2n+1} = 0$. Moreover, $[\theta_{2n+1}]$ is the non zero homology class in $H_*(\mathrm{Der}(A, B; \phi))$. A simple calculation shows that $\theta_2 = (x_2, 1)$ is not a cycle in $\mathrm{Der}(A, B; \phi)$. Moreover, $H(\varepsilon_*)([\theta_{2n+1}]) = [y^*_{2n+1}] \in G_{2n+1}(A, B; \phi)$. It then follows that $G_n(A, B; \phi) = \langle [y^*_{2n+1}] \rangle$. □

Definition 3.3 ([5, 6])**.** Let $\phi : A \to B$ be a map of differential graded vector spaces. A differential graded vector space, $Rel_*(\phi)$, called the mapping cone of ϕ is defined as follows. $Rel_n(\phi) = A_{n-1} \oplus B_n$ with the differential $\delta(a, b) = (-d_A(a), \phi(a) + d_B(b))$. There are inclusion and projection chain maps $J : B_n \to Rel_n(\phi)$ and $P : Rel_n(\phi) \to A_{n-1}$ defined by $J(w) = (0, w)$ and $P(a, b) = a$. These yields a short exact sequence of chain complexes

$$0 \to B_* \xrightarrow{J} Rel_*(\phi) \xrightarrow{P} A_{*-1} \to 0$$

and a long exact homology sequence of ϕ

$$\cdots \to H_{n+1}(Rel(\phi)) \xrightarrow{H(P)} H_n(A) \xrightarrow{H(\phi)} H_n(B) \xrightarrow{H(J)} H_n(Rel(\phi)) \to \cdots$$

whose connecting homomorphism is $H(\phi)$.

In following [5], we consider a commutative diagram of differential graded vector spaces;

$$\begin{array}{ccc} \mathrm{Der}(B, B; 1) & \xrightarrow{\phi^*} & \mathrm{Der}(A, B; \phi) \\ {\scriptstyle \varepsilon_*}\downarrow & & \downarrow{\scriptstyle \varepsilon_*} \\ \mathrm{Der}(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & \mathrm{Der}(A, \mathbb{Q}; \varepsilon), \end{array}$$

where ε is the augmentation of either A or B. On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder for $n \geq 2$,

$$\begin{array}{ccccc} \cdots \to H_{n+1}(Rel(\phi^*)) & \xrightarrow{H(P)} & H_n(\mathrm{Der}(B,B;1)) & \xrightarrow{H(\phi^*)} & H_n(\mathrm{Der}(A,B;\phi)) \to \cdots \\ \downarrow{\scriptstyle H(\varepsilon_*,\varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} \\ \cdots \to H_{n+1}(Rel(\widehat{\phi^*})) & \xrightarrow{H(\widehat{P})} & H_n(\mathrm{Der}(B,\mathbb{Q};\varepsilon)) & \xrightarrow{H(\widehat{\phi^*})} & H_n(\mathrm{Der}(A,\mathbb{Q};\varepsilon)) \to \cdots \end{array}$$

The n-th relative evaluation subgroup of ϕ is defined as follows:

$$G_n^{rel} = \mathrm{Im}\{H(\varepsilon_*,\varepsilon_*) : H_n(Rel(\phi^*)) \to H_n(Rel(\widehat{\phi^*}))\}.$$

The G-sequence of the map $\phi : A \to B$ is given by

$$\cdots \overset{H(\widehat{J})}{\to} G_{n+1}^{rel}(A,B;\phi) \overset{H(\widehat{P})}{\to} G_n(B) \overset{H(\widehat{\phi^*})}{\to} G_n(A,B;\phi) \overset{H(\widehat{J})}{\to} \cdots$$

which ends in $G_2(A,B;\phi)$. Moreover, in [5, Theorem 3.5], this can be applied to the Sullivan model $\phi : A \to B$ of the map $f : X \to Y$.

Theorem 3.4. *Consider the Hopf fibration $S^{2n+1} \hookrightarrow \mathbb{C}P(n)$ and $\phi : A \to B$ its Sullivan model. Then $G_n^{rel}(A,B;\phi) = \langle[(0, y_{2n+1}^*)]\rangle$.*

Proof. Consider the following diagram [5].

$$\begin{array}{ccccc} \mathrm{Der}(B,B;1) & \xrightarrow{\phi^*} & \mathrm{Der}(A,B;\phi) & \xrightarrow{J} & Rel(\phi^*) \\ \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle (\varepsilon_*,\varepsilon_*)} \\ \mathrm{Der}(B,\mathbb{Q};\varepsilon) & \xrightarrow{\widehat{\phi^*}} & \mathrm{Der}(A,\mathbb{Q};\varepsilon) & \xrightarrow{\widehat{J}} & Rel(\widehat{\phi^*}). \end{array}$$

Let $\alpha_{2n+1} = (x_{2n+1}, 1)$ in $\mathrm{Der}(B,B;1)$ and $\theta_{2n+1} = (y_{2n+1}, 1)$ in $\mathrm{Der}(A,B;\phi)$. Then $\phi^*(\alpha_{2n+1}) = \theta_{2n+1}$. Moreover, $D(\alpha_{2n+1}, 0) = (0, \theta_{2n+1})$ and $D(0, \theta_{2n+1}) = (0,0)$. Therefore, $[(0, \theta_{2n+1})]$ is the non zero homology class in $H_*(Rel(\phi^*))$. Further,

$$H(\varepsilon_*,\varepsilon_*)([(0,\theta_{2n+1})]) = [(0, y_{2n+1}^*)].$$

It is easily checked that $[(0, y_{2n+1}^*)]$ spans $H(\varepsilon_*,\varepsilon_*)$. □

The G-sequence reduces to

$$0 \to G_{2n+1}(A,B;\phi) \underset{\simeq}{\overset{H(J)}{\to}} G_{2n+1}^{rel}(A,B;\phi) \to 0,$$

$$0 \to G_{2n+1}(B) \underset{\simeq}{\overset{H(\phi^*)}{\to}} G_{2n+1}(A,B;\phi) \to 0,$$

$$0 \to G_{2n+1}^{rel}(A,B;\phi) \underset{\simeq}{\overset{H(P)}{\to}} G_{2n+1}(B) \to 0$$

and is exact.

Example 1. Consider the Hopf fibration $f : S^3 \hookrightarrow \mathbb{C}P(1)$. A Sullivan model of f is given by

$$\phi : A = (\wedge(x_2, y_3), d) \to (\wedge x_3, 0) = B,$$

where $dx_2 = 0$, $dy_3 = x_2^2$. We determine $G_n^{rel}(A, B; \phi)$ as follows. Consider $\alpha_3 = (x_3, 1) \in \mathrm{Der}(B, B; 1)$ and $\theta_3 = (y_3, 1) \in \mathrm{Der}(A, B; \phi)$. Then $\phi^*(\alpha_3) = \theta_3$. Moreover, $D(\alpha_3, 0) = (0, \theta_3)$ and $D(0, \theta_3) = (0, 0)$. Thus, $[(0, \theta_3]$ is the non zero homology class. Moreover, $(\varepsilon_*, \varepsilon_*)(0, \theta_3) = (0, y_3^*)$. Therefore,

$$G_n^{rel}(A, B; \phi) = \langle [(0, y_3^*)] \rangle.$$

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