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CHORD AND AREA PROPERTIES OF STRICTLY CONVEX CURVES

Dong-Soo Kim and Incheon Kim

Abstract. Ellipses have a lot of interesting geometric properties. It is quite natural to ask whether such properties of ellipses and some related ones characterize ellipses. In this paper, we study some chord properties and area properties of ellipses. As a result, using the curvature and the support function of a strictly convex curve, we establish four characterization theorems of ellipses and hyperbolas centered at the origin.

1. Introduction

We study strictly convex plane curves. Recall that a regular plane curve $\mathcal{X} : I \to \mathbb{R}^2$ in the plane $\mathbb{R}^2$ defined on an open interval I, is called *convex* if, for all $s \in I$ the trace $\mathcal{X}(I)$ of $\mathcal{X}$ lies entirely on one side of the closed half-plane determined by the tangent line at s ([4]). We will say that a convex curve $\mathcal{X}$ in the plane $\mathbb{R}^2$ is *strictly convex* if the curve is smooth (that is, of class $C^{(2)}$) and is of positive curvature κ with respect to the unit normal N pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s))\rangle > 0$, where $\mathcal{X}(s)$ is an arc-length parametrization of $\mathcal{X}$.

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, we will also say that f is *strictly convex* if the graph of f has positive curvature κ with respect to the upward unit normal N. This condition is equivalent to the positivity of $f''(x)$ on I.

Suppose that $\mathcal{X}$ is a strictly convex curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. For a fixed point $P \in \mathcal{X}$ and a sufficiently small $t > 0$, we consider the line m passing through $P + tN(P)$ which is parallel to the tangent ℓ to $\mathcal{X}$ at P and the points A and B where the line m intersects the curve $\mathcal{X}$. We denote by $\mathcal{L}_P(t)$, $\mathcal{T}_P(t)$ and $\mathcal{A}_P(t)$ the length of the chord AB, the area of the triangle ABP and the area of the region bounded by the curve $\mathcal{X}$ and the chord AB, respectively. Then, we have the following ([10]):

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Lemma 1.1. *Suppose that $\mathcal{X}$ is a smooth strictly convex curve in the plane $\mathbb{R}^2$. Then for a point $P \in \mathcal{X}$ we have*

$$\mathcal{A}'_P(t) = \mathcal{L}_P(t), \tag{1.1}$$

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} \mathcal{L}_P(t) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}} \tag{1.2}$$

and

$$\lim_{t\to 0} \frac{1}{t\sqrt{t}} \mathcal{A}_P(t) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}, \tag{1.3}$$

where $\kappa(P)$ denotes the curvature of $\mathcal{X}$ at P with respect to the unit normal vector N pointing to the convex side.

Proof. It follows from [10] that (1.1) and (1.2) hold. Hence L'hospital's rule gives a proof of (1.3). □

For a higher dimensional analogue of Lemma 1.1, see [5].

First of all, for the area $\mathcal{A}_P(t)$ of parabolic sections we have the following ([10]).

Proposition 1.2. *Suppose that $\mathcal{X}$ is a smooth strictly convex curve in the plane $\mathbb{R}^2$. Then the following are equivalent.*

1) *For arbitrary point $P \in \mathcal{X}$ and sufficiently small $t > 0$, $\mathcal{X}$ satisfies*

$$\mathcal{A}_P(t) = \frac{4}{3}\mathcal{T}_P(t). \tag{1.4}$$

2) *$\mathcal{X}$ is a parametrization of an open arc of a parabola.*

Actually, Archimedes showed that parabolas satisfy (1.4) ([17]). In [10], it was shown that (1.4) is a characteristic property of parabolas and some other characterizations of parabolas were established, which are the converses of well-known properties of parabolas originally due to Archimedes ([17]). For some properties and characterizations of parabolas with respect to the area of triangles associated with a curve, see [3, 12, 14, 15]. For the higher dimensional analogues of some results in [10], see [8] and [9].

Now, we consider an ellipse $\mathcal{E} : x^2/a^2 + y^2/b^2 = 1$ centered at the origin $O \in \mathbb{R}^2$. For a fixed point $P \in \mathcal{E}$ and a sufficiently small $h > 0$, we consider the point $Q = Q(h) \in \overrightarrow{OP}$ satisfying $OP : OQ = 1 : 1 - h$. The line m passing through Q which is parallel to the tangent ℓ to $\mathcal{E}$ at P intersects the ellipse $\mathcal{E}$ at two points A and B. We denote by $\mathcal{T}^r_P(h)$ and $\mathcal{A}^r_P(h)$ the area of the triangle ABP and the area of the region bounded by the ellipse $\mathcal{E}$ and the chord AB, respectively. Then, we have the following:

Proposition 1.3. *The ellipse $\mathcal{E} : x^2/a^2 + y^2/b^2 = 1$ centered at the origin $O \in \mathbb{R}^2$ satisfies the following.*

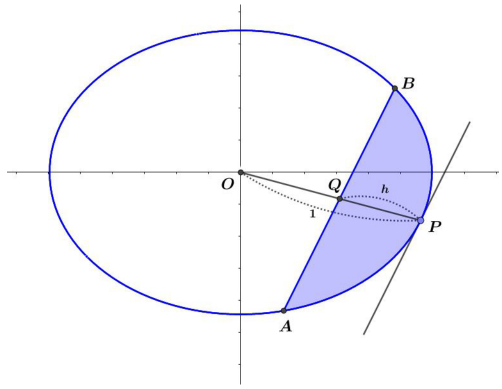

FIGURE 1. $\mathcal{A}_P^r(h)$ of an ellipse at a point P.

(1) *For any chord AB of $\mathcal{E}$, let us denote by P the point where tangent ℓ to $\mathcal{E}$ is parallel to the chord AB, then the ray $\overrightarrow{OP}$ bisects the chord AB.*
(2) *For two tangents to $\mathcal{E}$ at A and B in $\mathcal{X}$ which intersect at a point Q, the ray $\overrightarrow{OQ}$ bisects the chord AB.*
(3) *$\mathcal{A}_P^r(h)$ is a function of h only, which is independent of the point P.*
(4) *$\mathcal{T}_P^r(h)$ is a function of h only, which is independent of the point P.*

Proof. We consider the transformation T of the plane $\mathbb{R}^2$ defined by

$$T = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}. \tag{1.5}$$

Then the ellipse $\mathcal{E}$ is transformed to the unit circle of radius 1, the tangent at P to the tangent at the corresponding point P', etc.. Hence, well-known properties of unit circle complete the proof. For the proof of (3) and (4), see also Appendix. □

Conversely, it is reasonable to ask the following question:

Question 1.4. Are there any other curves in $\mathbb{R}^2$ satisfying the above mentioned properties?

In this article, we study whether the above properties of ellipses and some related ones characterize ellipses.

First of all, in Section 2 we prove the following:

Theorem A. *Suppose that $\mathcal{X} : I \to \mathbb{R}^2$ denotes a strictly convex $C^{(2)}$ curve defined on an open interval I. Then the following are equivalent.*

(1) *For any chord AB of $\mathcal{X}$, if we denote by P the point where tangent ℓ to $\mathcal{X}$ is parallel to the chord AB, then the ray $\overrightarrow{OP}$ bisects the chord AB.*
(2) *$\mathcal{X}$ parametrizes an open arc of either an ellipse or a hyperbola centered at the origin.*

Theorem B. *Suppose that $\mathcal{X} : I \to \mathbb{R}^2$ denotes a strictly convex $C^{(2)}$ curve defined on an open interval I. Then the following are equivalent.*

(1) *For two tangents to the curve $\mathcal{X}$ at A and B in $\mathcal{X}$ which intersect at a point Q, the ray $\overrightarrow{OQ}$ with O the origin bisects the chord AB.*
(2) *$\mathcal{X}$ is an open arc of either an ellipse or a hyperbola centered at the origin.*

Next, in Section 3 we use Theorem A in order to give an elementary proof (1st proof) of the following characterization theorem.

Theorem C. *Suppose that $\mathcal{X} : I \to \mathbb{R}^2$ denotes a strictly convex $C^{(2)}$ curve defined on an open interval I with the origin $O \in \mathbb{R}^2$. Then the following are equivalent.*

(1) *$\mathcal{A}_P^r(h)$ is a function of h only, which is independent of the point P.*
(2) *$\mathcal{X}$ is an open arc of either an ellipse or a hyperbola centered at the origin.*

In Section 4, with the help of Lemma 1.1 and a characterization theorem ([7]) we give a second proof of Theorem C.

Finally, in Section 5 we investigate some more properties of area of elliptic sections. As a result, we prove the following (For details, see Section 5):

Theorem D. *Suppose that $\mathcal{X} : I \to \mathbb{R}^2$ denotes a strictly convex $C^{(2)}$ curve defined on an open interval I with the origin O. Then the following are equivalent.*

(1) *$\mathcal{T}_P^r(h)$ is a function of h only, which is independent of the point P.*
(2) *$\mathcal{U}_P^r(h)$ is a function of h only, which is independent of the point P.*
(3) *$\mathcal{I}_P^r(h)$ is a function of h only, which is independent of the point P.*
(4) *$\kappa = c\mathfrak{h}^3$ for a constant c, where κ and $\mathfrak{h}$ are the curvature and support function of the curve $\mathcal{X}$, respectively.*
(5) *$\mathcal{X}$ is an open arc of an ellipse or a hyperbola centered at the origin O.*

In Appendix, we state some area formulae which will be used in the proof of Theorems C and D.

Among the graphs of functions, Á. Bényi et al. proved some characterizations of parabolas ([1,2]). In [16], B. Richmond and T. Richmond established a dozen necessary and sufficient conditions for the graph of a function to be a parabola by using elementary techniques. For some characterizations of parabolas or conic sections by properties of tangent lines, see [6] and [13]. In [7], it was proved that (4) and (5) in Theorem D are equivalent. See also [11] for some characterizations of quadrics.

Throughout this article, all curves are of class $C^{(2)}$ and connected, unless otherwise mentioned.

2. Theorems A and B

In this section, we prove Theorems A and B stated in Section 1.

It is well-known that for every chord AB of either an ellipse centered at the origin or a hyperbola centered at the origin, the ray from the origin through the midpoint of the chord AB meets the quadric $\mathcal{X}$ at the point where tangent ℓ to $\mathcal{X}$ is parallel to the chord AB. This shows that (2) implies (1).

Conversely, suppose that $\mathcal{X} : I \to \mathbb{R}^2$ denotes a strictly convex $C^{(2)}$ curve defined on an open interval I which satisfies (1). Around an arbitrary point C of $\mathcal{X}$, by a suitable rotation around the origin if necessary it is the graph of a $C^{(2)}$ function $f : J \to \mathbb{R}$ defined on an open interval J containing zero with $C = (0, f(0))$ and $f''(x) > 0$.

For distinct points $s, t \in J$, we put $A = (s, f(s))$ and $B = (t, f(t))$. If we denote by $P = (x, f(x))$ with $x = x(s,t)$ the point where the tangent line to the curve is parallel to the chord AB, then we have

$$(s-t)f'(x(s,t)) = f(s) - f(t). \tag{2.1}$$

By the assumption, the midpoint of the chord AB lies on the straight line through P and the origin. Hence we obtain

$$x(s,t)(f(s)+f(t)) = (s+t)f(x(s,t)). \tag{2.2}$$

Let us differentiate (2.1) with respect to s and t, respectively. Then, we get

$$x_s(s,t) = \frac{f'(s) - f'(x(s,t))}{(s-t)f''(x(s,t))} \tag{2.3}$$

and

$$x_t(s,t) = \frac{f'(x(s,t)) - f'(t)}{(s-t)f''(x(s,t))}. \tag{2.4}$$

Differentiating (2.2) with respect to s and t respectively also yields

$$x_s(s,t) = \frac{x(s,t)f'(s) - f(x(s,t))}{(s+t)f'(x(s,t)) - (f(s)+f(t))} \tag{2.5}$$

and

$$x_t(s,t) = \frac{x(s,t)f'(t) - f(x(s,t))}{(s+t)f'(x(s,t)) - (f(s)+f(t))}. \tag{2.6}$$

It follows from (2.3) and (2.5) that

$$\begin{aligned} &\{(s+t)f'(x(s,t)) - (f(s)+f(t))\}\{f'(s) - f'(x(s,t))\} \\ &= (s-t)f''(x(s,t))\{x(s,t)f'(s) - f(x(s,t))\}. \end{aligned} \tag{2.7}$$

From (2.4) and (2.6) we also get

$$\begin{aligned} &\{(s+t)f'(x(s,t)) - (f(s)+f(t))\}\{f'(x(s,t)) - f'(t)\} \\ &= (s-t)f''(x(s,t))\{x(s,t)f'(t) - f(x(s,t))\}. \end{aligned} \tag{2.8}$$

By eliminating $f''(x(s,t))$ from (2.7) and (2.8), we obtain

$$\begin{aligned} &A(s,t)\{f'(s) - f'(x(s,t))\}\{x(s,t)f'(t) - f(x(s,t))\} \\ &= A(s,t)\{f'(x(s,t)) - f'(t)\}\{x(s,t)f'(s) - f(x(s,t))\}, \end{aligned} \tag{2.9}$$

where we put

$$A(s,t) = (s+t)f'(x(s,t)) - (f(s)+f(t)). \tag{2.10}$$

Now, we substitute $f'(x(s,t))$ and $f(x(s,t))/x(s,t)$ given by (2.1) and (2.2) into (2.9) and (2.10), respectively. Then we obtain

$$\begin{aligned} &A(s,t)\{(s-t)f'(s) - (f(s)-f(t))\}\{(s+t)f'(t) - (f(s)+f(t))\} \\ &= A(s,t)\{(f(s)-f(t)) - (s-t)f'(t)\}\{(s+t)f'(s) - (f(s)+f(t))\} \end{aligned} \tag{2.11}$$

and

$$A(s,t) = (s+t)\frac{f(s)-f(t)}{s-t} - (f(s)+f(t)). \tag{2.12}$$

Suppose that $A(s,t)$ vanishes on a subinterval $J_1 \subset J$. Then, by letting t tend to $s \in J_1$ we get

$$sf'(s) - f(s) = 0, \tag{2.13}$$

which implies $f''(s) = 0$ on the subinterval J_1. Together with (2.11), this contradiction shows that

$$\begin{aligned} &\{(s-t)f'(s) - (f(s)-f(t))\}\{(s+t)f'(t) - (f(s)+f(t))\} \\ &= \{(f(s)-f(t)) - (s-t)f'(t)\}\{(s+t)f'(s) - (f(s)+f(t))\}. \end{aligned} \tag{2.14}$$

By differentiating (2.14) with respect to s, we have

$$f''(s)\{sf(s) - tf(t) - (s^2-t^2)f'(t)\} + \{f'(s) - f'(t)\}\{sf'(s) - f(s)\} = 0. \tag{2.15}$$

We put $s = 0$ in (2.15). Then we get

$$(at^2 + c)f'(t) = atf(t) + bc, \tag{2.16}$$

where we use

$$a = f''(0) > 0,\ b = f'(0),\ c = f(0) \neq 0. \tag{2.17}$$

Let us multiply both sides of the linear differential equation (2.16) by the integration factor $\mu = |at^2+c|^{-3/2}$. Then, after replacing t with x, for $y = f(x)$ we get

$$\frac{y}{\sqrt{|ax^2+c|}} = |c| \int \frac{bdx}{|ax^2+c|^{3/2}}. \tag{2.18}$$

We proceed as follows.

Case 1. Suppose that $b = 0$. Then, from (2.18) with (2.17) we obtain

$$y = \epsilon\sqrt{|acx^2 + c^2|}, \tag{2.19}$$

where $\epsilon = 1$ (if $c > 0$) or $\epsilon = -1$ (if $c < 0$). It follows from (2.19) that the curve $\mathcal{X}$ is locally an open arc of

$$acx^2 - y^2 + c^2 = 0, \tag{2.20}$$

which is an ellipse (if $c < 0$) centered at the origin or a hyperbola (if $c > 0$) centered at the origin.

Case 2. Suppose that $b \neq 0$ and $c > 0$. We put $\alpha^2 = c/a$ with $\alpha > 0$, then we have from (2.18)

$$\frac{y}{\sqrt{x^2+\alpha^2}} = \frac{bc}{a}\int \frac{dx}{(x^2+\alpha^2)^{3/2}}. \tag{2.21}$$

Integrating the right hand side of (2.21), from (2.17) we get

$$y = bx + \sqrt{acx^2 + c^2}. \tag{2.22}$$

This shows that the curve $\mathcal{X}$ is locally an open arc of

$$(b^2 - ac)x^2 - 2bxy + y^2 = c^2, \tag{2.23}$$

which is a hyperbola (because $c > 0$) centered at the origin.

Case 3. Suppose that $b \neq 0$ and $c < 0$. We put $\alpha^2 = -c/a$ with $\alpha > 0$, then we have from (2.18)

$$\frac{y}{\sqrt{\alpha^2 - x^2}} = \frac{-bc}{a}\int \frac{dx}{(\alpha^2 - x^2)^{3/2}}. \tag{2.24}$$

Integrating the right hand side of (2.24), from (2.17) we get

$$y = bx - \sqrt{acx^2 + c^2}. \tag{2.25}$$

This shows that the curve $\mathcal{X}$ is locally an open arc of

$$(b^2 - ac)x^2 - 2bxy + y^2 = c^2, \tag{2.26}$$

which is an ellipse (because $c < 0$) centered at the origin.

Combining Cases 1-3, we see that around an arbitrary point C of $\mathcal{X}$, $\mathcal{X}$ is locally an open arc of either an ellipse centered at the origin (if $c < 0$) or a hyperbola centered at the origin (if $c > 0$). This completes the proof of Theorem A.

Next, in the similar manner as in the proof of Theorem A, we prove Theorem B stated in Section 1 as follows.

It is well-known that either an ellipse centered at the origin or a hyperbola centered at the origin satisfies (1), respectively. This shows that (2) implies (1).

Conversely, suppose that $\mathcal{X} : I \to \mathbb{R}^2$ denotes a strictly convex $C^{(2)}$ curve defined on an open interval I which satisfies (1). Around an arbitrary point C of $\mathcal{X}$, just as in the proof of Theorem A, it is the graph of a $C^{(2)}$ function $f : J \to \mathbb{R}$ defined on an open interval J containing zero with $C = (0, f(0))$ and $f''(x) > 0$ on J.

For sufficiently close two values $s, t \in J$, we put $A = (s, f(s))$ and $B = (t, f(t))$. If we denote by $Q = (x(s,t), y(s,t))$ the point where the two tangents to $\mathcal{X}$ at A and B respectively meets, then we have

$$\begin{aligned} x(s,t) &= \frac{sf'(s) - tf'(t) - f(s) + f(t)}{f'(s) - f'(t)}, \\ y(s,t) &= \frac{(s-t)f'(s)f'(t) + f'(s)f(t) - f(s)f'(t)}{f'(s) - f'(t)}. \end{aligned} \tag{2.27}$$

By the assumption, we obtain

$$x(s,t)(f(s) + f(t)) = (s+t)y(s,t). \tag{2.28}$$

It follows from (2.27) and (2.28) that

$$\begin{aligned} &sf(s)f'(s) - tf(t)f'(t) - f(s)^2 + f(t)^2 \\ &= (s^2 - t^2)f'(s)f'(t) + tf'(s)f(t) - sf(s)f'(t). \end{aligned} \tag{2.29}$$

Differentiating (2.29) with respect to s gives

$$\begin{aligned} &s\{f'(s)^2 + f(s)f''(s)\} - f(s)f'(s) \\ &= sf'(s)f'(t) + (s^2 - t^2)f''(s)f'(t) + tf''(s)f(t) - f(s)f'(t). \end{aligned} \tag{2.30}$$

Let us put $s = 0$ in (2.30). Then we obtain

$$(at^2 + c)f'(t) = atf(t) + bc, \tag{2.31}$$

where we use

$$a = f''(0) > 0, b = f'(0), c = f(0). \tag{2.32}$$

Hence the proof of Theorem A shows that (1) $\Rightarrow$ (2). This completes the proof of Theorem B.

3. 1st proof of Theorem C

In this section, we consider a strictly convex $C^{(2)}$ curve in $\mathbb{R}^2$ (not necessarily closed) with the origin $O \in \mathbb{R}^2$ which is not on $\mathcal{X}$. Suppose that for every point $P \in \mathcal{X}$, the ray $\overrightarrow{OP}$ starting from O passes the curve $\mathcal{X}$ transversally at P. Then, for a point $P \in \mathcal{X}$, the tangent ℓ to $\mathcal{X}$ at P divide the plane $\mathbb{R}^2$. In case the origin O and $\mathcal{X}$ lie in the same closed half plane, we put $\epsilon = 1$. Otherwise, we put $\epsilon = -1$. For an ellipse with center O, we have $\epsilon = 1$ and for a hyperbola with center O we have $\epsilon = -1$.

For a point $P \in \mathcal{X}$ and a sufficiently small positive constant h, we define a point $Q = Q(h)$ on the ray $\overrightarrow{OP}$ as follows:

(1) When $\epsilon = 1$, $Q = Q(h)$ satisfies $OP : OQ = 1 : 1 - h$.
(2) When $\epsilon = -1$, $Q = Q(h)$ satisfies $OP : OQ = 1 : 1 + h$.

In any case, we see that the point $Q = Q(h)(\in \overrightarrow{OP})$ satisfies $OP : OQ = 1 : 1 - \epsilon h$. The line m passing through Q and parallel to the tangent ℓ to $\mathcal{X}$ at P meet the curve $\mathcal{X}$ at two points (say, A and B). Then we denote by $\mathcal{A}_P^r(h)$ the area of the region bounded by the chord AB and the arc of $\mathcal{X}$ containing P.

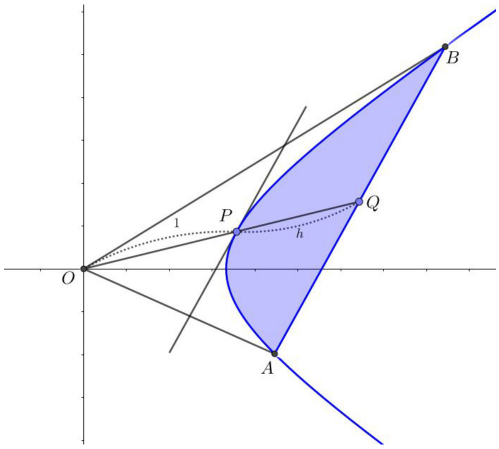

FIGURE 2. $\mathcal{A}_P^r(h)$ of a hyperbola at a point P.

Now, using Theorem A we give an elementary proof (1st proof) of Theorem C stated in Section 1.

It is well-known (or see Appendix) that either an ellipse or a hyperbola centered at the origin satisfies (1), respectively. This shows that (2) implies (1).

Conversely, suppose that $\mathcal{X} : I \to \mathbb{R}^2$ denotes a strictly convex $C^{(2)}$ curve defined on an open interval I which satisfies (1). Around an arbitrary point C of $\mathcal{X}$, after a suitable rotation around the origin $O \in \mathbb{R}^2$ if necessary, we may assume that the curve $\mathcal{X}$ is the graph of a $C^{(2)}$ function $f : J \to \mathbb{R}$ defined on an open interval J containing zero with $C = (0, f(0))$ and $f''(x) > 0$ on J.

For any distinct s, t in J with $s < t$, we consider $A(s, f(s)), B = (t, f(t)) \in \mathcal{X}$. We denote by $P(u, f(u)), u = u(s,t) \in (s,t)$ the point where the tangent ℓ to $\mathcal{X}$ is parallel to the chord AB. By definition, for a sufficiently small $h > 0$ the point $Q = Q(h)$ is given by

$$Q = (ku, kf(u)), \quad k = 1 - \epsilon h. \tag{3.1}$$

Since the tangent ℓ to $\mathcal{X}$ at P is parallel to AB, we get

$$(s - t)f'(u(s,t)) = f(s) - f(t) \tag{3.2}$$

The line m through A and B is given by

$$m : y = f'(u)(x - ku) + kf(u). \tag{3.3}$$

Hence, we also obtain

$$\begin{aligned} f(s) &= f'(u)(s - ku) + kf(u), \\ f(t) &= f'(u)(t - ku) + kf(u). \end{aligned} \tag{3.4}$$

Now, we have the area function $\mathcal{A}_P^r(h)$ as follows.

$$\mathcal{A}_P^r(h) = \frac{1}{2}(f(s)+f(t))(t-s) - \int_s^t f(x)dx. \tag{3.5}$$

Since $\mathcal{A}_P^r(h)$ is a function of only h, which is independent of the point P, differentiating (3.5) with respect to u, we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial u}\mathcal{A}_P^r(h) \\ &= \frac{\partial}{\partial s}\mathcal{A}_P^r(h) \times \frac{\partial s}{\partial u} + \frac{\partial}{\partial t}\mathcal{A}_P^r(h) \times \frac{\partial t}{\partial u}. \end{aligned} \tag{3.6}$$

Let us differentiate (3.5) with respect to s and t, respectively. Then we have

$$\begin{aligned} \frac{\partial}{\partial s}\mathcal{A}_P^r(h) &= \frac{1}{2}(t-s)(f'(s)-f'(u)), \\ \frac{\partial}{\partial t}\mathcal{A}_P^r(h) &= \frac{1}{2}(t-s)(f'(t)-f'(u)). \end{aligned} \tag{3.7}$$

On the other hand, differentiating two equations in (3.4) respectively, with respect to u gives

$$\frac{\partial s}{\partial u} = \frac{f''(u)(s-ku)}{f'(s)-f'(u)}, \quad \frac{\partial t}{\partial u} = \frac{f''(u)(t-ku)}{f'(t)-f'(u)}. \tag{3.8}$$

Together with (3.7) and (3.8), (3.6) implies

$$\begin{aligned} 0 &= \frac{1}{2}(t-s)(f'(s)-f'(u)) \times \frac{f''(u)(s-ku)}{f'(s)-f'(u)} \\ &\quad + \frac{1}{2}(t-s)(f'(t)-f'(u)) \times \frac{f''(u)(t-ku)}{f'(t)-f'(u)} \\ &= \frac{1}{2}(t-s)f''(u)(s+t-2ku). \end{aligned} \tag{3.9}$$

Since $s<t$ and $f''(x)>0$ on J, it follows from (3.9) that

$$s+t = 2ku. \tag{3.10}$$

This shows that the point $Q = Q(h)$ is the midpoint of the chord AB. Since the chord AB is an arbitrary chord of $\mathcal{X}$, Theorem A completes the proof of (1) $\Rightarrow$ (2). This completes the proof of Theorem C.

4. 2nd proof of Theorem C

In this section, using Lemma 1.1 and the main theorem of [7] we give a second proof of Theorem C as follows.

We may assume that the origin $O \in \mathbb{R}^2$ is not on the curve $\mathcal{X}$ with unit normal N pointing toward the convex side of $\mathcal{X}$. Suppose that for every point $P \in \mathcal{X}$, the ray $\overrightarrow{OP}$ starting from O passes through the curve $\mathcal{X}$ transversally at the point P (See Section 3). For a point $P \in \mathcal{X}$ and a sufficiently small positive constant h, we define a point $Q = Q(h)$ on the ray $\overrightarrow{OP}$ satisfying OP :

$OQ = 1 : 1 - \epsilon h$, $\epsilon = \pm 1$. Hence the point $P + tN(P)$ with $t = h|\langle P, N(P)\rangle|$ lies on the line m passing through Q and parallel to the tangent ℓ to $\mathcal{X}$ at P. Thus we obtain

$$\mathcal{A}_P^r(h) = \mathcal{A}_P(t), \tag{4.1}$$

where t is given by

$$t = h|\langle P, N(P)\rangle|. \tag{4.2}$$

Note that the support function $\mathfrak{h}(P)$ of $\mathcal{X}$ is defined by

$$\mathfrak{h}(P) = \langle P, N(P)\rangle, \tag{4.3}$$

from which we get

$$t = h|\mathfrak{h}(P)|. \tag{4.4}$$

It follows from Lemma 1.1 that

$$\lim_{h\to 0} \frac{1}{h\sqrt{h}} \mathcal{A}_P^r(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}} |\mathfrak{h}(P)|^{3/2}. \tag{4.5}$$

By hypothesis, the left side of (4.5) is a constant α which is independent of the point P. Together with (4.5), this shows that

$$\kappa(P) = \pm \frac{32}{9\alpha^2} \mathfrak{h}(P)^3. \tag{4.6}$$

Finally, we use the following ([7]):

Proposition 4.1. *Let $\mathcal{X} : I \to \mathbb{E}^2$ be a unit-speed curve of class $C^{(2)}$ in $\mathbb{E}^2$ whose curvature function κ does not vanish identically. Then $\mathcal{X}$ satisfies condition $\kappa = c\mathfrak{h}^3$ for a constant c if and only if $\mathcal{X}$ parametrizes a connected open subset of either an ellipse centered at the origin or a hyperbola centered at the origin.*

With the help of the above Proposition, (4.6) completes the proof of Theorem C.

5. Proof of Theorem D

In this section, we give a proof of Theorem D stated in Section 1 as follows.

We may assume that the origin $O \in \mathbb{R}^2$ is not on the curve $\mathcal{X}$ with unit normal N pointing toward the convex side of $\mathcal{X}$. Suppose that for every point $P \in \mathcal{X}$, the ray $\overrightarrow{OP}$ starting from O passes through the curve $\mathcal{X}$ transversally at the point P (See Section 3). For a point $P \in \mathcal{X}$ and a sufficiently small positive constant h, we define a point $Q = Q(h)$ on the ray $\overrightarrow{OP}$ which satisfies $OP : OQ = 1 : 1 - \epsilon h$, $\epsilon = \pm 1$. Hence the point $P + tN(P)$ with $t = h|\langle P, N(P)\rangle|$ lies on the line m passing through Q and parallel to the tangent ℓ to $\mathcal{X}$ at P. We denote by A and B the two points where the line m meets the curve $\mathcal{X}$.

We also denote by $\mathcal{A}_P^r(h)$, $\mathcal{T}_P^r(h)$, $\mathcal{U}_P^r(h)$ and $\mathcal{I}_P^r(h)$ the area of the region surrounded by the chord AB and the arc of $\mathcal{X}$ containing P, the area of the

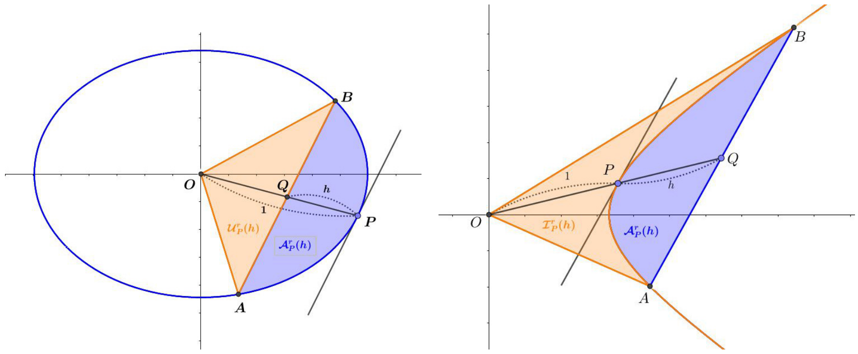

FIGURE 3. $\mathcal{A}_P^r(h), \mathcal{U}_P^r(h)$ and $\mathcal{T}_P^r(h)$ at a point $P \in \mathcal{X}$ with $\epsilon = \pm 1$.

triangle ABP, the area of the triangle ABO, the area of the ice cream cone-shaped region (if $\epsilon = 1$) bounded by OA, OB and the arc of $\mathcal{X}$, respectively. Then we obtain

$$\mathcal{A}_P^r(h) = \mathcal{A}_P(t), \tag{5.1}$$

where t is given by

$$t = h|\mathfrak{h}(P)|. \tag{5.2}$$

(1) $\Rightarrow$ (4). It follows from (5.2) that

$$2\mathcal{T}_P^r(h) = \mathcal{L}_P(t)t = h|\mathfrak{h}(P)|\mathcal{L}_P(t). \tag{5.3}$$

Hence we obtain

$$\frac{\mathcal{T}_P^r(h)}{h\sqrt{h}} = \frac{\mathcal{L}_P(t)}{\sqrt{t}} \frac{|\mathfrak{h}(P)|^{3/2}}{2}. \tag{5.4}$$

Thus it follows from Lemma 1.1 that

$$\beta = \lim_{h\to 0} \frac{1}{h\sqrt{h}} \mathcal{T}_P^r(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}} |\mathfrak{h}(P)|^{3/2}, \tag{5.5}$$

which implies

$$\kappa(P) = \pm \frac{2}{\beta^2} \mathfrak{h}(P)^3. \tag{5.6}$$

This completes the proof of (1) $\Rightarrow$ (4).

(2) $\Rightarrow$ (4). It follows from (5.2) that

$$2\mathcal{U}_P^r(h) = (1 - \epsilon h)|\mathfrak{h}(P)|\mathcal{L}_P(t). \tag{5.7}$$

Hence we obtain

$$\frac{\mathcal{U}_P^r(h)}{\sqrt{h}} = \frac{\mathcal{L}_P(t)}{\sqrt{t}} \frac{1 - \epsilon h}{2} |\mathfrak{h}(P)|^{3/2}. \tag{5.8}$$

It follows from (5.8) and Lemma 1.1 that

$$\gamma = \lim_{h\to 0} \frac{1}{\sqrt{h}} \mathcal{U}_P^r(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}} |\mathfrak{h}(P)|^{3/2}, \tag{5.9}$$

which implies

$$\kappa(P) = \pm \frac{2}{\gamma^2} \mathfrak{h}(P)^3. \tag{5.10}$$

This completes the proof of (2) $\Rightarrow$ (4).

(3) $\Rightarrow$ (4). First note that

$$\mathcal{I}_P^r(h) = \mathcal{U}_P^r(h) + \epsilon \mathcal{A}_P^r(h). \tag{5.11}$$

Hence, from (4.5), (5.9) and (5.11) we have

$$\delta = \lim_{h\to 0} \frac{1}{\sqrt{h}} \mathcal{I}_P^r(h) = \lim_{h\to 0} \frac{1}{\sqrt{h}} \mathcal{U}_P^r(h) \quad = \frac{\sqrt{2}}{\sqrt{\kappa(P)}} |\mathfrak{h}(P)|^{3/2}, \tag{5.12}$$

which implies

$$\kappa(P) = \pm \frac{2}{\delta^2} \mathfrak{h}(P)^3. \tag{5.13}$$

This completes the proof of (3) $\Rightarrow$ (4).

Note that Proposition 4.1 shows that (4) $\Leftrightarrow$ (5). Thus, Appendix completes the proof of Theorem D.

6. Appendix

In this section, we state some area formulae which were used in the proof of Theorems C and D.

Proposition A. *For the ellipse* ($\epsilon = 1$) *given by*

$$\frac{x^2}{a^2} + \epsilon \frac{y^2}{b^2} = 1, \quad a, b > 0, \tag{5.1}$$

we have the following.

(1)

$$\begin{aligned} \mathcal{A}_P^r(h) &= 2ab \int_{1-h}^{1} \sqrt{1-x^2}\,dx \\ &= ab\{\frac{\pi}{2} - \arcsin(1-h) - (1-h)\sqrt{2h-h^2}\}. \end{aligned} \tag{5.2}$$

(2)

$$\mathcal{T}_P^r(h) = abh\sqrt{2h-h^2}. \tag{5.3}$$

(3)

$$\mathcal{U}_P^r(h) = ab(1-h)\sqrt{2h-h^2}. \tag{5.4}$$

(4)

$$\mathcal{I}_P^r(h) = \mathcal{U}_P^r(h) + \epsilon \mathcal{A}_P^r(h) = ab\{\frac{\pi}{2} - \arcsin(1-h)\}. \tag{5.5}$$

Proposition B. *For the hyperbola ($\epsilon = -1$) given by*

$$\frac{x^2}{a^2} + \epsilon \frac{y^2}{b^2} = 1, \quad a, b > 0, \tag{5.6}$$

if we put

$$h_1 = 1 + h - \sqrt{2h + h^2}, \; h_2 = 1 + h + \sqrt{2h + h^2}, \tag{5.7}$$

we have the following.

(1)

$$\begin{aligned} \mathcal{A}_P^r(h) &= \frac{ab}{2} \int_{h_1 s}^{h_2 s} \{\frac{-1}{s^2} x + 2\frac{1+h}{s} - \frac{1}{x}\} dx \\ &= \frac{ab}{2}\{2(1+h)\sqrt{2h+h^2} - \ln\left(\frac{h_2}{h_1}\right)\}. \end{aligned} \tag{5.8}$$

(2)

$$\mathcal{T}_P^r(h) = abh\sqrt{2h + h^2}. \tag{5.9}$$

(3)

$$\mathcal{U}_P^r(h) = ab(1+h)\sqrt{2h + h^2}. \tag{5.10}$$

(4)

$$\mathcal{I}_P^r(h) = \mathcal{U}_P^r(h) + \epsilon \mathcal{A}_P^r(h) = \frac{ab}{2} \ln\left(\frac{h_2}{h_1}\right). \tag{5.11}$$

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Dong-Soo Kim
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
Email address: dosokim@chonnam.ac.kr

Incheon Kim
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
Email address: goatham@jnu.ac.kr