Commun. Korean Math. Soc. **36** (2021), No. 4, pp. 829–833
https://doi.org/10.4134/CKMS.c200380
pISSN: 1225-1763 / eISSN: 2234-3024

THE ACTION OF SPECIAL LINEAR GROUP ON THE SET OF MUTUALLY DISTINCT TRIPLE POINTS OF CIRCLE AND INVARIANT MEASURE

Sanghoon Kwon

Abstract. We investigate the Möbius transformation action of $PSL(2,\mathbb{R})$ on the set of mutually distinct ordered triple points of $\mathbb{R}\cup\{\infty\}$.

1. Introduction

The upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ has a large group of conformal automorphisms, consisting of Möbius transformations of the form

$$z \mapsto \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. These symmetries form the group

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}.$$

Under the $PSL(2,\mathbb{R})$-action, $\mathbb{H}$ has an invariant metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

called the Poincaré metric. Denoting by $z = x + yi$, we have a corresponding measure

$$dA(z) = \frac{dxdy}{y^2}$$

and the distance function

$$d(z,w) = \log \frac{|z-\overline{w}| + |z-w|}{|z-\overline{w}| - |z-w|}.$$

Let us denote by $\mathrm{T}^1\mathbb{H}$ the unit tangent bundle of $\mathbb{H}$, which is the bundle $\{(z,\mathbf{v}) : z \in \mathbb{H}, \mathbf{v} \in \mathrm{T}_z\mathbb{H} \text{ with } \|\mathbf{v}\| = 1\}$ of unit-length tangent vectors on the upper half-plane.

Received October 15, 2020; Accepted December 22, 2020.
2010 *Mathematics Subject Classification.* Primary 37E35; Secondary 28D05.
Key words and phrases. Upper-half plane, invariant measure, projective frame.

The action of the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$ on the unit tangent bundle $\mathrm{T}^1\mathbb{H}$ of the upper half plane $\mathbb{H}$ is given by (for example, see Chapter 9 of [2])

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] : (z, \mathbf{v}) \mapsto \left(\frac{az+b}{cz+d}, \frac{\mathbf{v}}{(cz+d)^2}\right) \text{ for } z \in \mathbb{H}, \mathbf{v} \in \mathrm{T}_z\mathbb{H}.$$

In particular, the induced action of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathrm{T}^1\mathbb{H}$ serves as a key example of ergodic theory on homogeneous spaces. (See also [1] for various concepts and applications for Riemann surfaces other than modular surface.)

The boundary at infinity $\partial_\infty\mathbb{H}$ may be identified with $\mathbb{R} \cup \{\infty\}$ and hence with S^1. Let us say that a mutually distinct ordered triple points $(w_1, w_2, w_3) \in (S^1)^3$ is *positively ordered* if one reaches w_2 before w_3 when starting counterclockwise from w_1.

We note that there is a bijection between the unit tangent bundle $\mathrm{T}^1\mathbb{H}$ of the upper half plane and the set of mutually distinct positively ordered triple points (w_1, w_2, w_3) of S^1 by the following manner. Take $w_1 = \ell(-\infty)$ and $w_3 = \ell(\infty)$ for the bi-infinite geodesic ℓ for which $\ell(0) = z$ and $\ell'(0) = \mathbf{v}$. Let $\mathbf{w} \in \mathrm{T}_z\mathbb{H}$ be the tangent vector obtained by rotating $\mathbf{v}$ by $\frac{\pi}{2}$ clockwise. Let $w_2 = \ell'(\infty)$ where ℓ' is the bi-infinite geodesic whose tangent vector at z is $\mathbf{w}$.

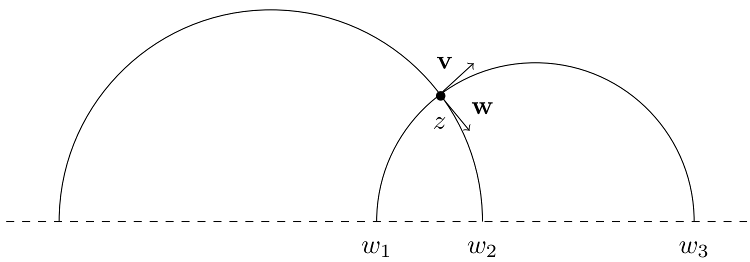

FIGURE 1. $(z, \mathbf{v}) \leftrightarrow (w_1, w_2, w_3)$

In this article, we investigate the invariant measure of the Möbius transformation action of the modular group $PSL(2,\mathbb{R})$ on the set of mutually distinct ordered triple points $(w_1, w_2, w_3) \in (S^1)^3$.

Let $(S^1)^3_{\mathrm{pos}}$ be the set of positively ordered triple points of S^1. Here and throughout, we identify S^1 with $\mathbb{R} \cup \{\infty\}$ and pull the real distance on $\mathbb{R} \cup \{\infty\}$ back to S^1.

Theorem 1.1. *The measure λ on $(S^1)^3_{pos}$ given by*

$$d\lambda = \frac{dw_1 dw_2 dw_3}{(w_1 - w_2)(w_2 - w_3)(w_3 - w_1)}$$

is the $SL(2,\mathbb{R})$-invariant measure.

We note that $(w_1-w_2)(w_2-w_3)(w_3-w_1)$ is always positive since (w_1, w_2, w_3) is positively oriented.

2. Proof of Main Theorem

Given $(z, \mathbf{v})$ in $\mathrm{T}^1\mathbb{H}$, let $z = x + yi$ $(y > 0)$ and θ be the angle between $\mathbf{v}$ and the x-axis. We may identify the elements $(z, \mathbf{v})$ of $\mathrm{T}^1\mathbb{H}$ with $(x + yi, \theta)$. The Liouville measure

$$dm = \frac{dxdyd\theta}{y^2}$$

is the unique (up to scalar) $PSL(2, \mathbb{R})$-invariant measure on $\mathrm{T}^1\mathbb{H}$ (see Chapter 9 of [2]).

Lemma 2.1. *The one-to-one correspondence* $(x, y, \theta) \leftrightarrow (w_1, w_2, w_3)$ *between* $T^1\mathbb{H}$ *and* $(\mathbb{R} \cup \{\infty\})^3_{pos}$ *is explicitly given by*

$$\begin{aligned} w_1 &= x + y\tan\theta - y|\sec\theta|, \\ w_2 &= x - y\cot\theta\ + y|\csc\theta|, \\ w_3 &= x + y\tan\theta + y|\sec\theta|, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} x &= \frac{w_1^2w_3 + w_1w_2^2 - 4w_1w_2w_3 + w_1w_3^2 + w_2^2w_3}{(w_1 - w_2)^2 + (w_2 - w_3)^2}, \\ y &= \frac{(w_1 - w_2)(w_3 - w_2)^2}{(w_1 - w_2)^2 + (w_2 - w_3)^2}, \\ \theta &= \arctan\left(\frac{(w_1 - w_3)^2(w_1 - 2w_2 + w_3)}{2(w_1 - w_2)(w_2 - w_3)^2}\right). \end{aligned} \tag{2.2}$$

Proof. First, we note that Equation (2.1) follows directly from Figure 2.

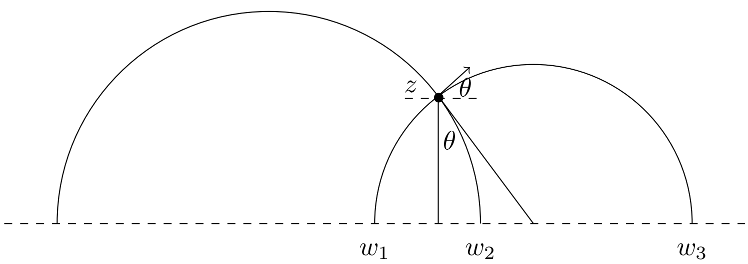

FIGURE 2. $(x + iy, \theta) \leftrightarrow (w_1, w_2, w_3)$

Now let us recall that two Euclidean circles with Cartesian equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{and} \quad x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

are orthogonal if and only if $2gg' + 2ff' = c + c'$. For the unique geodesic ℓ with $\ell(0) = z$ and $\ell(\infty) = w_2$, let us denote by $\ell(-\infty) = w_4$. Let O_1 be the circle centered at $\frac{w_2+w_4}{2}$ with radius $\frac{|w_2-w_4|}{2}$ and let O_2 be the circle centered at $\frac{w_1+w_3}{2}$ with radius $\frac{|w_1-w_3|}{2}$. Since they cut one another at right angles on $z = x + yi$, it follows that

$$\frac{(x^2+y^2-w_2^2)(w_1+w_3)}{x-w_2} = 2w_1w_3 + \frac{2w_2(x^2+y^2-w_2^2)}{x-w_2} - 2w_2^2$$

and

$$y = \sqrt{-(x-w_1)(x-w_3)}.$$

Solving these equations yields the formula of x and y in Equation (2.2). From the relation

$$\tan\theta = \frac{w_1+w_3-2x}{2y}$$

we obtain the formula of θ in Equation (2.2). □

Now we will give the proof of Theorem 1.1. Since

$$dm = \frac{dxdyd\theta}{y^2}$$

is the $PSL(2,\mathbb{R})$-invariant measure on $\mathrm{T}^1\mathbb{H}$ and the one-to-one correspondence $(x,y,\theta) \leftrightarrow (w_1,w_2,w_3)$ is a diffeomorphism almost everywhere, the measure on $(\mathbb{R}\cup\{\infty\})^3_{\mathrm{pos}}$ locally given by

$$\frac{1}{y^2}\left|\frac{\partial(x,y,\theta)}{\partial(w_1,w_2,w_3)}\right| dw_1dw_2dw_3$$

is invariant under $PSL(2,\mathbb{R})$.

By Lemma 2.1, the Jacobian matrix $\frac{\partial(x,y,\theta)}{\partial(w_1,w_2,w_3)}$ is given by

$$\begin{bmatrix} 1 & \tan\theta + |\sec\theta| & y\sec^2\theta + y|\sec\theta|\tan\theta \\ 1 & -\cot\theta + |\csc\theta| & y\csc^2\theta - y|\csc\theta|\cot\theta \\ 1 & \tan\theta - |\sec\theta| & y\csc^2\theta - y|\csc\theta|\cot\theta \end{bmatrix}$$

of which the determinant is $y\sec^2\left(\frac{\theta}{2}\right)\sec^2\theta$. Applying the formula of y and θ of Equation (2.2), we get

$$\frac{1}{y^2}\left|\frac{\partial(x,y,\theta)}{\partial(w_1,w_2,w_3)}\right| dw_1dw_2dw_3 = \frac{dw_1dw_2dw_3}{(w_1-w_2)(w_2-w_3)(w_3-w_1)}.$$

This completes the proof of Theorem 1.1.

3. More on parametrization and fundamental domain

Hopf parametrization

There is another parametrization of $(z,\mathbf{v}) \in \mathrm{T}^1\mathbb{H}$, called *Hopf parametrization*, by two distinct boundary points ξ and η at infinity together with a real number $t \in \mathbb{R}$. Given $(x+yi,\theta) \in \mathrm{T}^1\mathbb{H}$, there is a unique bi-infinite parametrized geodesic α of $\mathbb{H}$ such that $\alpha(0) = x+yi$ and $\alpha'(0) = \mathbf{v}$.

Let $\xi = \alpha(\infty)$ and $\eta = \alpha(-\infty)$. Let o be the orthogonal projection point of $i \in \mathbb{H}$ onto the geodesic α and t be the real number for which $\alpha(t) = o$.

Under this parametrization $(x+yi,\theta) \leftrightarrow (\xi,\eta,t)$, the similar argument gives

$$\frac{dxdyd\theta}{y^2} = \frac{2d\xi d\eta dt}{(\eta-\xi)^2}.$$

Fundamental domain

For each $(z,\mathbf{v})$ in $\mathrm{T}^1\mathbb{H}$, we can find a neighborhood of $(z,\mathbf{v})$ which does not contain any other element of the $\mathrm{PSL}(2,\mathbb{Z})$-orbit of $(z,\mathbf{v})$. This enables us to construct fundamental domains, which contain exactly one representative for the $\mathrm{PSL}(2,\mathbb{Z})$-orbit of every $(z,\mathbf{v})$ in $\mathrm{T}^1\mathbb{H}$.

There are various ways of constructing a strong fundamental domain, but a common choice is the union

$$\{(z,\mathbf{v})\colon z\in R, \mathbf{v}\in \mathrm{T}^1_z\mathbb{H}\} \cup \left\{(w_1,\mathbf{v})\in \mathrm{T}^1\mathbb{H}\colon 0\leqslant \arg(\mathbf{v}) < \frac{2\pi}{3}\right\}$$
$$\cup \left\{(w_2,\mathbf{v})\in \mathrm{T}^1\mathbb{H}\colon 0\leqslant \arg(\mathbf{v}) < \pi\right\}$$

for two branched points $w_1 = \frac{1}{2}+\frac{\sqrt{3}i}{2}$ and $w_2 = i$ and the region

$$R = \left\{z\in\mathbb{H}\colon |z|>1, -\frac{1}{2}\leqslant \mathrm{Re}(z) < \frac{1}{2}\right\} \cup \left\{z\in\mathbb{H}\colon |z|=1, -\frac{1}{2} < \mathrm{Re}(z) < 0\right\}$$

bounded by the vertical lines $\mathrm{Re}(z) = -\frac{1}{2}$ and $\mathrm{Re}(z)=\frac{1}{2}$ and the circle $|z|=1$. It would be interesting to construct explicitly a fundamental domain for the action of $PSL(2,\mathbb{R})$ on $(S^1)^3_{\mathrm{pos}}$.

We remark that the positive characteristic analogue case is attained in [3]. Namely, the author considers $PGL(2,\mathbb{F}_q[t])$-action on the $(q+1)$-regular tree $\mathcal{T}$ and its fundamental domain as a subset of $\partial_\infty\mathcal{T}^3_{\mathrm{dist}}$, the set of mutually distinct ordered triple points on $\partial_\infty\mathcal{T}$.

References

[1] D. Borthwick, *Spectral theory of infinite-area hyperbolic surfaces*, second edition, Progress in Mathematics, 318, Birkhäuser/Springer, 2016. https://doi.org/10.1007/978-3-319-33877-4

[2] M. Einsiedler and T. Ward, *Ergodic theory with a view towards number theory*, Graduate Texts in Mathematics, 259, Springer-Verlag London, Ltd., London, 2011. https://doi.org/10.1007/978-0-85729-021-2

[3] S. Kwon, *A fundamental domain for* $PGL(2,\mathbb{F}_q[t])\backslash PGL(2,\mathbb{F}_q(\!(t^{-1})\!))$, Bull. Korean Math. Soc. **57** (2020), no. 6, 1491–1499. https://doi.org/10.4134/BKMS.b200021

Sanghoon Kwon
Department of Mathematics Education
Catholic Kwandong University
Gangneung-si 25601, Korea
Email address: skwon@cku.ac.kr