Commun. Korean Math. Soc. **36** (2021), No. 4, pp. 771–782
https://doi.org/10.4134/CKMS.c200278
pISSN: 1225-1763 / eISSN: 2234-3024

ON GENERALIZED WEAKLY SEMI-CONFORMALLY SYMMETRIC MANIFOLDS

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Abstract. In this paper we introduce generalized weakly semi-conformally symmetric manifold, a generalization of weakly symmetric manifold. We study some basic properties and obtain the forms of the scalar curvature of such manifold. In the last section an example is given to ensure the existence of such manifold.

1. Introduction

The notion of weakly symmetric Riemannian manifold has been introduced by Támassy and Binh [20], defined as follows:

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a weakly symmetric manifold if the curvature tensor R of type (0, 4) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ & + H(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ & + E(V)R(Y,Z,U,X)\end{aligned}$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$, where A, B, H, D and E are 1-forms (not simultaneously zero) and ∇ is the operator of covariant differentiation with respect to the Riemannian metric g, $\chi(M^n)$ being the set of all smooth vector fields on M^n. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. The defining condition of a $(WS)_n$ can always be expressed in the form

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ & + B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ & + D(V)R(Y,Z,U,X),\end{aligned}$$

where A, B, D are non-zero 1-forms defined by $A(X) = g(X,\theta)$, $B(X) = g(X,\rho_1)$ and $D(X) = g(X,\phi_1)$ for all X and $R(Y,U,V,W) = g(R(Y,U)V,W)$.

Received August 3, 2020; Accepted January 20, 2021.
2010 *Mathematics Subject Classification.* Primary 53C15, 53C25.
Key words and phrases. Weakly symmetric manifold, generalized weakly symmetric manifold, semi-conformal curvature tensor, generalized weakly conformally symmetric manifold.

Later many authors works on $(\mathrm{WS})_n$ in different approach. For these, we may refer to the readers [3, 5, 8, 11–15, 18], etc.

The Kulkarni-Nomizu product ([16], [17]) $E \wedge F$ of two $(0,2)$ tensors E and F is defined by

$$(E \wedge F)(X_1, X_2, X_3, X_4) = E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3),$$

$X_i \in \chi(M)$, $i = 1, 2, 3, 4$. As a special subgroup of the conformal transformation group, Ishii [4] introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function. The conharmonic curvature tensor $\overline{C}$ of type (0,4) on a Riemannian manifold (M^n, g) of dimension $n \geq 4$ was defined as follows:

$$\overline{C} = R - \frac{1}{n-2} g \wedge S, \tag{1}$$

which remains invariant under conharmonic transformation where R and S are the Riemannian curvature and Ricci curvature tensor, respectively.

In [9], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetric and skew symmetric properties of the Riemannian curvature tensor as well as cyclic ones. The conharmonic curvature tensor has many applications in the theory of general relativity.

The conformal transformation on a Riemannian manifold is a transformation under which the angle between two curves remains invariant.

The Weyl conformal curvature tensor C of type $(0,4)$ of an n-dimensional Riemannian manifold M, $n > 3$, is defined by

$$C = R - \frac{1}{n-2}(g \wedge S) + \frac{r}{(n-1)(n-2)} G, \tag{2}$$

where $G = \frac{1}{2} g \wedge g$.

In [6], Kim introduced a type of curvature-like tensor called semi-conformal curvature tensor such that its $(1,3)$ components remain invariant under conharmonic transformation. More precisely, the semi-conformal curvature tensor $\widetilde{P}$ of type $(0,4)$ on a Riemannian manifold (M^n, g) is defined as follows:

$$\widetilde{P}(X, Y, Z, U) = -(n-2)bC(X, Y, Z, U) + [a + (n-2)b]\overline{C}(X, Y, Z, U) \tag{3}$$

for all X, Y, $Z \in \chi(M^n)$, where a, b are real constants not simultaneously zero. Using (1) and (2) in (3), $\widetilde{P}$ takes the form

$$\widetilde{P} = aR - \frac{a}{n-2}(g \wedge S) - \frac{rb}{(n-1)} G. \tag{4}$$

In particular if $a = 1$ and $b = -\frac{1}{n-2}$, then the semi-conformal curvature tensor reduces to conformal curvature tensor whereas for $a = 1$ and $b = 0$, such a tensor turns into conharmonic curvature tensor. The semi-conformal curvature tensor $\widetilde{P}$ of type $(0,4)$ possesses the several symmetric and skew symmetric properties as well as the cyclic ones. Recently Kim [7] studied pseudo semi-conformal symmetric manifolds.

In the sense of Dubey [2], recently Baishya [1] introduced a new type of space called generalized weakly symmetric manifold, denoted by $(GWS)_n$-space and defined as follows

$$\begin{aligned}(\nabla_X R)(Y,U,V,W) &= A(X)R(Y,U,V,W) + B(Y)R(X,U,V,W)\\ &\quad + B(U)R(Y,X,V,W) + D(V)R(Y,U,X,W)\\ &\quad + D(W)R(Y,U,V,X) + \alpha(X)G(Y,U,V,W)\\ &\quad + \beta(Y)G(X,U,V,W) + \beta(U)G(Y,X,V,W)\\ &\quad + \gamma(V)G(Y,U,X,W) + \gamma(W)G(Y,U,V,X),\end{aligned} \tag{5}$$

where A, B, D, α, β and γ are non-zero 1-forms defined as $A(X) = g(X,\theta_1)$, $B(X) = g(X,\phi_1)$ and $D(X) = g(X,\pi_1)$, $\alpha(X) = g(X,\theta_2)$, $\beta(X) = g(X,\phi_2)$ and $\gamma(X) = g(X,\pi_2)$. Generalizing the notion of weakly symmetric manifold by Támassy and Binh [20], in this paper we introduce the notion of generalized weakly conformally symmetric manifolds, generalized weakly conharmonically manifolds and generalized weakly semi-conformally symmetric manifolds. A Riemannian manifold (M^n, g), $n > 2$, is said to be generalized weakly semi-conformally symmetric manifold if its semi-conformal curvature tensor $\widetilde{P}$ satisfies the following relation:

$$\begin{aligned}(\nabla_X \widetilde{P})(Y,Z,U,V) &= A(X)\widetilde{P}(Y,Z,U,V) + B(Y)\widetilde{P}(X,Z,U,V)\\ &\quad + B(Z)\widetilde{P}(Y,X,U,V) + D(U)\widetilde{P}(Y,Z,X,V)\\ &\quad + D(V)\widetilde{P}(Y,Z,U,X) + \alpha(X)G(Y,Z,U,V)\\ &\quad + \beta(Y)G(X,Z,U,V) + \beta(Z)G(Y,X,U,V)\\ &\quad + \gamma(U)G(Y,Z,X,V) + \gamma(V)G(Y,Z,U,X),\end{aligned} \tag{6}$$

where A, B, D, α, β and γ are non-zero 1-forms defined as $A(X) = g(X,\overline{\theta_1})$, $B(X) = g(X,\overline{\phi_1})$ and $D(X) = g(X,\overline{\pi_1})$, $\alpha(X) = g(X,\overline{\theta_2})$, $\beta(X) = g(X,\overline{\phi_2})$ and $\gamma(X) = g(X,\overline{\pi_2})$. Such an n-dimensional manifold is denoted by $(GWSCS)_n$. If in (6), $\widetilde{P}$ is replaced by C and $\overline{C}$, then the manifold M, $n > 3$, is called a generalized weakly conformally symmetric and generalized weakly conharmonically symmetric manifold and are denoted by $(GWCS)_n$ and $(GWCHS)_n$, respectively.

The local expression of (6) is

$$\begin{aligned}\widetilde{P}_{mnpq,k} &= A_k\widetilde{P}_{mnpq} + B_m\widetilde{P}_{knpq} + B_n\widetilde{P}_{mkpq} + D_p\widetilde{P}_{mnkq} + D_q\widetilde{P}_{mnpk}\\ &\quad + \alpha_k G_{mnpq} + \beta_m G_{knpq} + \beta_n G_{mkpq} + \gamma_p G_{mnkq} + \gamma_q G_{mnpk},\end{aligned} \tag{7}$$

where A_i, B_i, D_i, α_i, β_i and γ_i are non-zero co-vectors. The beauty of such $(GWSCS)_n$-space is that for suitable choice of 1-forms and the real constants a and b we get different space such as

(i) weakly conformally symmetric manifolds [19] (for $a = 1$, $b = -\frac{1}{n-2}$ and $\alpha = \beta = \gamma = 0$),
(ii) weakly conharmonically symmetric manifolds [9], [10] (for $a = 1, b = 0$ and $\alpha = \beta = \gamma = 0$),
(iii) generalized weakly conformally symmetric manifolds (for $a = 1$, $b = -\frac{1}{n-2}$),
(iv) generalized weakly conharmonically symmetric manifolds (for $a = 1$ and $b = 0$).

We organized this paper as follows: Section 2 is concerned with some basic geometric properties of $(GWSCS)_n$. In Section 3, it is shown that a $(GWSCS)_n$ is $(GWCS)_n$ as well as $(GWCHS)_n$ and conversely. In Section 4, we study projectively flat $(GWSCS)_n$. The last section is concerned with an example of $(GWSCS)_n$.

2. Some basic geometric properties of $(GWSCS)_n$

In this section, we consider a Riemannian manifold (M^n, g) $n > 2$, which is a generalized weakly semi-conformally symmetric manifold. Now, contracting (6) over U and Z we have

$$
\begin{aligned}
(8)\qquad & -\frac{[a+(n-2)b]}{n-2}g(Y,V)dr(X) \\
= & -\frac{[a+(n-2)b]}{n-2}A(X)rg(Y,V) \\
& -\frac{[a+(n-2)b]}{n-2}rg(X,V)B(Y) \\
& - aB(R(Y,X)V) - \frac{a}{n-2}[B(X)S(Y,V) + B(QX)g(Y,V) \\
& - B(Y)S(X,V) - g(X,V)B(QY)] \\
& - \frac{br}{n-1}[B(X)g(Y,V) - B(Y)g(X,V)] \\
& + aD(R(X,V)Y) - \frac{a}{n-2}[D(X)S(Y,V) + D(QX)g(Y,V) \\
& - g(Y,X)D(QV) - D(V)S(Y,X)] \\
& - \frac{br}{n-1}[D(X)g(Y,V) - D(V)g(Y,X)] \\
& - \frac{[a+(n-2)b]}{n-2}D(V)rg(Y,X) \\
& + (n-1)[\alpha(X)g(Y,V) + \beta(Y)g(X,V) + \gamma(V)g(Y,X)] \\
& + [\beta(X)g(Y,V) - \beta(Y)g(X,V)] + [\gamma(X)g(Y,V) - \gamma(V)g(Y,X)].
\end{aligned}
$$

Taking contraction of (8) over Y and V we have

$$
(9)\qquad \frac{[a+(n-2)b]}{n-2}ndr(X)
$$

$$= \frac{[a+(n-2)b]}{n-2} nrA(X) + \frac{2[a+(n-2)b]}{n-2} rB(X) + \frac{2[a+(n-2)b]}{n-2} rD(X) - (n-1)[n\alpha(X) + 2\beta(X) + 2\gamma(X)].$$

Again taking contraction of (8) over Y and X we get

$$-\frac{[a+(n-2)b]}{n-2} dr(V) = -\frac{[a+(n-2)b]}{n-2} rA(V) - \frac{[a+(n-2)b]}{n-2} rB(V) - \frac{[a+(n-2)b]}{n-2} r(n-1)D(V) + (n-1)[\alpha(V) + \beta(V) + (n-1)\gamma(V)]. \tag{10}$$

Contracting (8) over X and V we have

$$-\frac{[a+(n-2)b]}{n-2} dr(Y) = -\frac{[a+(n-2)b]}{n-2} rA(Y) - r\frac{[a+(n-2)b](n-1)}{n-2} B(Y) - \frac{[a+(n-2)b]}{n-2} rD(Y) + (n-1)[\alpha(Y) + (n-1)\beta(Y) + \gamma(Y)]. \tag{11}$$

From (9) and (10) we have

$$r = \frac{(n-1)(n-2)}{[a+(n-2)b]} \frac{[\beta(X) + (n+1)\gamma(X)]}{[B(X) + (n+1)D(X)]}. \tag{12}$$

Again from (9) and (11) we get

$$r = \frac{(n-1)(n-2)}{[a+(n-2)b]} \frac{[(n+1)\beta(X) + \gamma(X)]}{[(n+1)B(X) + D(X)]}. \tag{13}$$

From (10) and (11) we get

$$r = \frac{(n-1)(n-2)}{[a+(n-2)b]} \frac{[\beta(X) - \gamma(X)]}{[B(X) - D(X)]}. \tag{14}$$

Also from (12) and (14) we get

$$\gamma(X)B(X) = \beta(X)D(X). \tag{15}$$

This leads the following:

Theorem 2.1. *In a $(GWSCS)_n$, the 1-forms B, D, β and γ are related by the expression* (15)*.*

Using (15) in (14) we have

$$rD(X) = \frac{(n-1)(n-2)\gamma(X)}{[a+(n-2)b]} \quad \forall X. \tag{16}$$

Again using (15) in (13) we get

$$rB(X) = \frac{(n-1)(n-2)\beta(X)}{[a+(n-2)b]} \quad \forall X. \tag{17}$$

From (16)

$$rg(X, \overline{\pi_1}) = \frac{(n-1)(n-2)g(X, \overline{\pi_2})}{[a+(n-2)b]} \quad \forall X. \tag{18}$$

Putting $X = \overline{\pi_2}$ in above we get

$$rg(\overline{\pi_2}, \overline{\pi_1}) = \frac{(n-1)(n-2)\|\overline{\pi_2}\|^2}{[a+(n-2)b]}. \tag{19}$$

Similarly putting $X = \overline{\pi_1}$ in (18) we get

$$r\|\overline{\pi_1}\|^2 = \frac{(n-1)(n-2)}{[a+(n-2)b]} g(\overline{\pi_1}, \overline{\pi_2}). \tag{20}$$

From (19) and (20) we easily obtain

$$r = \pm \frac{(n-1)(n-2)\|\overline{\pi_2}\|}{[a+(n-2)b]\|\overline{\pi_1}\|}. \tag{21}$$

Similarly from (17) we obtain

$$r = \pm \frac{(n-1)(n-2)\|\overline{\phi_2}\|}{[a+(n-2)b]\|\overline{\phi_1}\|}. \tag{22}$$

From (21) and (22) we have

$$\frac{\|\overline{\pi_2}\|}{\|\overline{\pi_1}\|} = \frac{\|\overline{\phi_2}\|}{\|\overline{\phi_1}\|} = \pm \frac{[a+(n-2)b]}{(n-1)(n-2)} r. \tag{23}$$

Thus we can state the following:

Theorem 2.2. *The scalar curvature of a $(GWSCS)_n$ is given by* (21) *and* (22)*, provided $a + (n-2)b \neq 0$ and the associated vector fields $\overline{\pi_1}$, $\overline{\pi_2}$, $\overline{\phi_1}$ and $\overline{\phi_2}$ are related by* (23)*.*

Corollary 2.3. *If the associated vector fields $\overline{\pi_1}$, $\overline{\pi_2}$, $\overline{\phi_1}$ and $\overline{\phi_2}$ are of unit vector fields, then the scalar curvature of the $(GWSCS)_n$ is constant and is equal to $\pm\frac{(n-1)(n-2)}{a+(n-1)b}$.*

Now using (21) in (20) we get

$$g(\overline{\pi_1}, \overline{\pi_2}) = \pm\|\overline{\pi_1}\|\|\overline{\pi_2}\|,$$

or, $\cos\theta = \pm 1$ where θ is the angle between the associated vector fields $\overline{\pi_1}$ and $\overline{\pi_2}$ which implies that $\theta = 0$ or $\theta = \pi$, i.e., $\overline{\pi_1}$ and $\overline{\pi_2}$ are parallel vector fields. Similar result holds for $\overline{\phi_1}$ and $\overline{\phi_1}$. Thus we can conclude that:

Theorem 2.4. *In a $(GWSCS)_n$, vector fields $\overline{\pi_1}$ is parallel to $\overline{\pi_2}$ and $\overline{\phi_1}$ is parallel to $\overline{\phi_2}$.*

Again using (16) and (17) in (9) we get

$$\frac{[a+(n-2)b]}{n-2}ndr(X) = \frac{[a+(n-2)b]}{n-2}nrA(X) - (n-1)n\alpha(X). \tag{24}$$

If r is constant, then from (24) we have

$$rA(X) = \frac{(n-1)(n-2)}{[a+(n-2)b]}\alpha(X). \tag{25}$$

This gives the following:

Theorem 2.5. *If the scalar curvature of a $(GWSCS)_n$ is constant, then its forms is given by* (25)*.*

3. Some spaces which are $(GWSCS)_n$ under certain condition

We consider a Riemannian manifold (M^n, g) $n > 3$ which is weakly conformally symmetric [19]. Then we have

$$\begin{aligned}(\nabla_X C)(Y,Z,U,V) = {} & A(X)C(Y,Z,U,V) + B(Y)C(X,Z,U,V) \\ & + B(Z)C(Y,X,U,V) + D(U)C(Y,Z,X,V) \\ & + D(V)C(Y,Z,U,X),\end{aligned} \tag{26}$$

where A, B, D are non-zero 1-forms defined by $A(X) = g(X,\theta)$, $B(X) = g(X,\rho_1)$ and $D(X) = g(X,\phi_1)$ for all X and $R(Y,U,V,W) = g(R(Y,U)V,W)$. Using (1) and (2) in (26) we get

$$\begin{aligned}(\nabla_X \widetilde{P})(Y,Z,U,V) = {} & A_1(X)\widetilde{P}(Y,Z,U,V) + B_1(Y)\widetilde{P}(X,Z,U,V) \\ & + B_1(Z)\widetilde{P}(Y,X,U,V) + D_1(U)\widetilde{P}(Y,Z,X,V) \\ & + D_1(V)\widetilde{P}(Y,Z,U,X) + \alpha_1(X)G(Y,Z,U,V) \\ & + \beta_1(Y)G(X,Z,U,V) + \beta_1(Z)G(Y,X,U,V) \\ & + \gamma_1(U)G(Y,Z,X,V) + \gamma_1(V)G(Y,Z,U,X),\end{aligned} \tag{27}$$

where $A_1 = A$, $B_1 = B$, $D_1 = D$, $\alpha_1 = \frac{[a+(n-2)b]}{(n-1)(n-2)}[rA - dr]$, $\beta_1 = \frac{[a+(n-2)b]rB}{(n-1)(n-2)}$ and $\gamma_1 = \frac{[a+(n-2)b]rD}{(n-1)(n-2)}$ are non-zero 1-forms. This leads to the following:

Theorem 3.1. *A weakly conformally symmetric space is necessarily a $(GWSCS)_n$ space, provided $a \neq 0$.*

However, the converse of the above theorem may not be true.

Again if we consider the space is weakly conharmonically symmetric [9], then in the similar way we can show that the space is a $(GWSCS)_n$. In this case the 1-forms are $A_1 = A$, $B_1 = B$, $D_1 = D$, $\alpha_1 = \frac{b[rA-dr]}{(n-1)}$, $\beta_1 = \frac{rbB}{(n-1)}$ and $\gamma_1 = \frac{rbD}{(n-1)}$. Thus we can state the following:

Theorem 3.2. *A weakly conharmonically symmetric space is necessarily a $(GWSCS)_n$ space provided $a \neq 0$.*

However, the converse of the above theorem may not be true.

From (2) and (4) we have

$$aC = \widetilde{P} + a\frac{[a+(n-2)b]rG}{(n-1)(n-2)}, \tag{28}$$

which gives

$$a(\nabla_X C) = (\nabla_X \widetilde{P}) + a\frac{[a+(n-2)b]dr(X)G}{(n-1)(n-2)}. \tag{29}$$

If the space is $(GWSCS)_n$, then using (28) and (29) in (6) we can easily deduce that the space is $(GWCS)_n$ and conversely if the space is $(GWCS)_n$, then it is also $(GWSCS)_n$. Thus we can state the following:

Theorem 3.3. *A $(GWSCS)_n$ is a $(GWCS)_n$ and vice-versa, provided $a \neq 0$.*

Again from (1) and (3) we get

$$\widetilde{P} = a\overline{C} - \frac{rbG}{(n-1)}, \tag{30}$$

which gives

$$(\nabla_X \widetilde{P}) = a(\nabla_X \overline{C}) - \frac{dr(X)bG}{(n-1)}. \tag{31}$$

Again if we consider the space is a $(GWSCS)_n$, then using (30) and (31) in (6) we can easily show that the space is $(GWCHS)_n$ and conversely. Thus we can state the following:

Theorem 3.4. *A $(GWSCS)_n$ is a $(GWCHS)_n$ and vice-versa, provided $a \neq 0$.*

4. Projectively flat $(GWSCS)_n$

Let us consider a projectively flat $(GWSCS)_n$. We know for a projectively flat manifold the Riemannian curvature tensor R is given by

$$R(X,Y,Z,U) = \frac{1}{n-1}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)]. \tag{32}$$

After suitable contraction (32) reduces to

$$S(X,Y) = \frac{r}{n}g(X,Y), \tag{33}$$

i.e., the space is Einstein. So r is constant. By virtue of (33), (4) yields

$$\widetilde{P} = aR - \left[\frac{2a}{n(n-2)} + \frac{b}{n-1}\right] rG. \tag{34}$$

Using (34) in (6) we have

$$\begin{aligned} a(\nabla_X R)(Y,U,V,W) = {} & A_2(X)R(Y,U,V,W) + B_2(Y)R(X,U,V,W) \\ & + B_2(U)R(Y,X,V,W) + D_2(V)R(Y,U,X,W) \end{aligned} \tag{35}$$

$$+ D_2(W)R(Y,U,V,X) + \alpha_2(X)G(Y,U,V,W) + \beta_2(Y)G(X,U,V,W) + \beta_2(U)G(Y,X,V,W) + \gamma(V)_2G(Y,U,X,W) + \gamma_2(W)G(Y,U,V,X),$$

where $A_2 = A$, $B_2 = B$, $D_2 = D$, $\alpha_2 = \frac{1}{a}[\alpha + \{\frac{2a}{n(n-2)} + \frac{b}{n-1}\}\{dr - rA\}]$, $\beta_2 = \frac{1}{a}[\beta - \{\frac{2a}{n(n-2)} + \frac{b}{n-1}\}rB]$ and $\gamma_2 = \frac{1}{a}[\gamma - \{\frac{2a}{n(n-2)} + \frac{b}{n-1}\}rD]$ are non-zero 1-forms and hence the manifold under consideration is a $(GWS)_n$ [1].

This leads to the following:

Theorem 4.1. *A projectively flat $(GWSCS)_n$ is a $(GWS)_n$, provided $a \neq 0$.*

Corollary 4.2. *An Einstein $(GWSCS)_n$ is a $(GWS)_n$, provided $a \neq 0$.*

5. Example of a generalized weakly semi-conformally symmetric space

Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian manifold endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = -(dx^1)^2 + (x^1)^{\frac{4}{3}}(dx^2)^2 + (x^1)^{\frac{4}{3}}(dx^3)^2 + (x^1)^{\frac{4}{3}}(dx^4)^2, \tag{36}$$

with $x^1 > 0$, $(i,j = 1,2,3,4)$.

$$\begin{cases} \widetilde{P}_{1212} = \frac{2(a+2b)}{9(x^1)^{\frac{2}{3}}} = \widetilde{P}_{1313} = \widetilde{P}_{1414}, \\ \widetilde{P}_{2323} = -\frac{2(a+2b)(x^1)^{\frac{2}{3}}}{9} = \widetilde{P}_{2424} = \widetilde{P}_{3434}, \\ r = -\frac{4}{3(x^1)^2}. \end{cases} \tag{37}$$

With the help of (37), we can find out

$$\begin{cases} G_{1212} = (x^1)^{\frac{4}{3}} = G_{1313} = G_{1414}, \\ G_{2323} = -(x^1)^{\frac{8}{3}} = G_{2424} = G_{3434}. \end{cases} \tag{38}$$

The non-vanishing component of covariant derivatives of semi-conformal curvature tensors are

$$\begin{cases} \widetilde{P}_{1212,1} = -\frac{4(a+2b)}{9(x^1)^{\frac{5}{3}}} = \widetilde{P}_{1313,1} = \widetilde{P}_{1414,1}, \\ \widetilde{P}_{2323,1} = \frac{4(a+2b)}{9(x^1)^{\frac{1}{3}}} = \widetilde{P}_{2424,1} = \widetilde{P}_{3434,1}. \end{cases} \tag{39}$$

We consider the 1-forms as follows:

$$\begin{cases} A(\partial_i) = A_i = \begin{cases} \frac{1}{x^1} & \text{for } i = 1, \\ 0 & \text{otherwise,} \end{cases} \\ B(\partial_i) = B_i = \begin{cases} 1 & \text{for } i = 2, \\ 0 & \text{otherwise,} \end{cases} \\ D(\partial_i) = D_i = \begin{cases} 3x^1 & \text{for } i = 3, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{40}$$

$$(41)\qquad \begin{cases} \alpha(\partial_i) = \alpha_i = \begin{cases} -\frac{6(a+2b)}{9(x^1)^3} & \text{for } i=1, \\ 0 & \text{otherwise}, \end{cases} \\ \beta(\partial_i) = \beta_i = \begin{cases} -\frac{2(a+2b)}{9(x^1)^2} & \text{for } i=2, \\ 0 & \text{otherwise}, \end{cases} \\ \gamma(\partial_i) = \gamma_i = \begin{cases} -\frac{2(a+2b)}{3(x^1)} & \text{for } i=3, \\ 0 & \text{otherwise}, \end{cases} \end{cases}$$

where $\partial_i = \frac{\partial}{\partial u^i}$, u^i being the local coordinates of $\mathbb{R}^4$.

In our $\mathbb{R}^4$, (36) reduces with these 1-forms to the following equations:

$$\begin{aligned} \widetilde{P}_{2323,k} = {} & A_k\widetilde{P}_{2323} + B_2\widetilde{P}_{k323} + B_2\widetilde{P}_{2k23} + D_1\widetilde{P}_{23k3} + D_2\widetilde{P}_{232k} \\ & + \alpha_k G_{2323} + \beta_2 G_{k323} + \beta_2 G_{2k23} + \gamma_3 G_{23k3} + \gamma_3 G_{232k}, \end{aligned}$$

$$\begin{aligned} \widetilde{P}_{2424,k} = {} & A_k\widetilde{P}_{2424} + B_2\widetilde{P}_{k424} + B_4\widetilde{P}_{2k24} + D_2\widetilde{P}_{24k4} + D_4\widetilde{P}_{242k} \\ & + \alpha_k G_{2424} + \beta_2 G_{k424} + \beta_4 G_{2k24} + \gamma_2 G_{24k4} + \gamma_4 G_{242k}, \end{aligned}$$

$$\begin{aligned} \widetilde{P}_{1414,k} = {} & A_k\widetilde{P}_{1414} + B_1\widetilde{P}_{k414} + B_4\widetilde{P}_{1k14} + D_1\widetilde{P}_{14k4} + D_4\widetilde{P}_{141k} \\ & + \alpha_k G_{1414} + \beta_1 G_{k414} + \beta_4 G_{1k14} + \gamma_1 G_{14k4} + \gamma_4 G_{141k}, \end{aligned}$$

$$\begin{aligned} \widetilde{P}_{1212,k} = {} & A_k\widetilde{P}_{1212} + B_1\widetilde{P}_{k212} + B_2\widetilde{P}_{1k12} + D_1\widetilde{P}_{12k2} + D_2\widetilde{P}_{121k} \\ & + \alpha_k G_{1212} + \beta_1 G_{k212} + \beta_2 G_{1k12} + \gamma_1 G_{12k2} + \gamma_2 G_{121k}, \end{aligned}$$

$$\begin{aligned} \widetilde{P}_{1313,k} = {} & A_k\widetilde{P}_{1313} + B_1\widetilde{P}_{k313} + B_3\widetilde{P}_{1k13} + D_1\widetilde{P}_{13k3} + D_3\widetilde{P}_{131k} \\ & + \alpha_k G_{1313} + \beta_1 G_{k313} + \beta_3 G_{1k13} + \gamma_1 G_{13k3} + \gamma_3 G_{131k}, \end{aligned}$$

$$\begin{aligned} \widetilde{P}_{3434,k} = {} & A_k\widetilde{P}_{3434} + B_3\widetilde{P}_{k434} + B_4\widetilde{P}_{3k34} + D_3\widetilde{P}_{34k4} + D_4\widetilde{P}_{343k} \\ & + \alpha_k G_{3434} + \beta_3 G_{k434} + \beta_4 G_{3k34} + \gamma_3 G_{34k4} + \gamma_4 G_{343k}, \end{aligned}$$

where $k = 1, 2, 3, 4$. By virtue of (37)–(41) it can be easily check that the above relations hold and hence we can state the following:

Theorem 5.1. *Let $M = (\mathbb{R}^4, g)$ be a Riemannian manifold equipped with the metric given by* (36). *Then M is a $(GWSCS)_4$ with non-vanishing and non-constant scalar curvature.*

References

[1] K. K. Baishya, *On generalized weakly symmetric manifolds*, Bull. Transilv. Univ. Braşov Ser. III **10(59)** (2017), no. 1, 31–38.

[2] R. S. D. Dubey, *Generalized recurrent spaces*, Indian J. Pure Appl. Math. **10** (1979), no. 12, 1508–1513.

[3] S. K. Hui, *On weak concircular symmetries of trans-Sasakian manifolds*, Cubo **13** (2011), no. 3, 141–152. `https://doi.org/10.4067/s0719-06462011000300008`

[4] Y. Ishii, *On conharmonic transformations*, Tensor (N.S.) **7** (1957), 73–80.

[5] S. K. Jana and A. A. Shaikh, *On quasi-conformally flat weakly Ricci symmetric manifolds*, Acta Math. Hung. **115** (2007), no. 3,197–214; *Erratum to*: *On quasi-conformal flat weakly Ricci symmetric manifolds*, Acta Math. Hungarica, **122** (2009), no. 1-2, 201–202. https://doi.org/10.1007/s10474-008-8001-1

[6] J. Kim, *A type of conformal curvature tensor*, Far East J. Math. Sci. **99** (2016), no. 1, 61–74.

[7] ———, *On pseudo semiconformally symmetric manifolds*, Bull. Korean Math. Soc. **54** (2017), no. 1, 177–186. https://doi.org/10.4134/BKMS.b151007

[8] A. A. Shaikh and K. K. Baishya, *On weakly quasi-conformally symmetric manifolds*, Soochow J. Math. **31** (2005), no. 4, 581–595.

[9] A. A. Shaikh and S. K. Hui, *On weakly conharmonically symmetric manifolds*, Tensor (N.S.) **70** (2008), no. 2, 119–134.

[10] ———, *On decomposable weakly conharmonically symmetric manifolds*, Lobachevskii J. Math. **29** (2008), no. 4, 206–215. https://doi.org/10.1134/S1995080208040021

[11] ———, *On weakly concircular symmetric manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **55** (2009), no. 1, 167–186.

[12] ———, *On weakly projective symmetric manifolds*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **25** (2009), no. 2, 247–269.

[13] A. A. Shaikh and S. K. Jana, *On weakly cyclic Ricci symmetric manifolds*, Ann. Polon. Math. **89** (2006), no. 3, 273–288. https://doi.org/10.4064/ap89-3-4

[14] ———, *On weakly quasi-conformally symmetric manifolds*, SUT J. Math. **43** (2007), no. 1, 61–83.

[15] ———, *On weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen **71** (2007), no. 1-2, 27–41.

[16] A. A. Shaikh and H. Kundu, *On weakly symmetric and weakly Ricci symmetric warped product manifolds*, Publ. Math. Debrecen **81** (2012), no. 3-4, 487–505. https://doi.org/10.5486/PMD.2012.5361

[17] ———, *On equivalency of various geometric structures*, J. Geom. **105** (2014), no. 1, 139–165. https://doi.org/10.1007/s00022-013-0200-4

[18] A. A. Shaikh, I. Roy, and S. K. Hui, *On totally umbilical hypersurfaces of weakly conharmonically symmetric spaces*, Global J. Science Frontier Research **10** (2010), no. 4, 28–31.

[19] A. A. Shaikh, M. H. Shahid, and S. K. Hui, *On weakly conformally symmetric manifolds*, Mat. Vesnik **60** (2008), no. 4, 269–284.

[20] L. Tamássy and T. Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, in Differential geometry and its applications (Eger, 1989), 663–670, Colloq. Math. Soc. János Bolyai, 56, North-Holland, Amsterdam, 1992.

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