Commun. Korean Math. Soc. **36** (2021), No. 4, pp. 671–684
https://doi.org/10.4134/CKMS.c200168
pISSN: 1225-1763 / eISSN: 2234-3024

HYPERGEOMETRIC DISTRIBUTION SERIES AND ITS APPLICATION OF CERTAIN CLASS OF ANALYTIC FUNCTIONS BASED ON SPECIAL FUNCTIONS

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Abstract. The tenacity of the current paper is to find connections between various subclasses of analytic univalent functions by applying certain convolution operator involving generalized hypergeometric distribution series. To be more specific, we examine such connections with the classes of analytic univalent functions $k-\mathcal{UCV}^*(\beta)$, $k-\mathcal{S}_p^*(\beta)$, $\mathcal{R}(\beta)$, $\mathcal{R}^\tau(A,B)$, $k-\mathcal{PUCV}^*(\beta)$ and $k-\mathcal{PS}_p^*(\beta)$ in the open unit disc $\mathbb{U}$.

1. Introduction

Denote by $\mathcal{A}$ the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$. Also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ that are univalent in $\mathbb{U}$. A function f of $\mathcal{S}$ belongs to the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, $0 \leq \alpha < 1$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{U}. \tag{1.2}$$

A function f of $\mathcal{S}$ belongs to the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha, 0 \leq \alpha < 1$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{U}. \tag{1.3}$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were initially studied by Robertson [17]. We note that $f \in \mathcal{K}(\alpha) \Leftrightarrow f \in \mathcal{S}^*(\alpha)$. In 1975, Silverman [19] familiarized a subclass $\mathcal{T}$

Received May 16, 2020; Revised May 11, 2021; Accepted July 1, 2021.
2010 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Analytic, univalent functions, uniformly convex functions, uniformly starlike functions, generalized hypergeometric function.

of $\mathcal{S}$ consists of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \tag{1.4}$$

and gave a fruitful direction of research by investigating the necessary and sufficient condition for functions $f \in \mathcal{S}^*(\alpha)$ and $f \in \mathcal{K}(\alpha)$. Denote by $\mathcal{V}$ the subclass of $\mathcal{S} \subset \mathcal{A}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| \, z^n, \quad z \in \mathbb{U}. \tag{1.5}$$

Analogous to $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, for $1 < \beta \leq \frac{4}{3}$ Uralegaddi *et al.* [21] introduced a new class of functions with positive coefficients satisfying the analytic criteria

$$\mathcal{L}(\beta) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) < \beta \right\}$$

and

$$\mathcal{M}(\beta) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta \right\},$$

respectively and opened up a new and interesting direction of research in Geometric Function Theory. In fact, they considered the functions of the form (1.1), where the coefficients a_n are positive. Note that $\mathcal{U}(\beta) \equiv \mathcal{L}(\beta) \cap \mathcal{V}$ and $\mathcal{V}(\beta) \equiv \mathcal{M}(\beta) \cap \mathcal{V}$. Inspired by the work of Uralegaddi *et al.* [21], numerous researchers (see [3, 5–7, 12, 14, 22]) familiarized new subclasses of analytic functions with positive coefficients and observed certain characteristic properties. In 2010, Porwal and Dixit [12] introduced the classes $k-\mathcal{UCV}^*(\beta)$ and $k-\mathcal{S}_p^*(\beta)$ as

$$k - \mathcal{UCV}^*(\beta) = \left\{ f \in \mathcal{A} : \Re\left\{ 1 + \left(1 + ke^{i\varphi}\right) \frac{zf''(z)}{f'(z)} \right\} < \beta \right\}$$

and

$$k - \mathcal{S}_p^*(\beta) = \left\{ f \in \mathcal{A} : \Re\left\{ \left(1 + ke^{i\varphi}\right) \frac{zf'(z)}{f(z)} - ke^{i\varphi} \right\} < \beta \right\},$$

where $0 \leq k < \infty$, $1 < \beta \leq \frac{4+k}{3}$. Further, let $k-\mathcal{PUCV}^*(\beta) \equiv k-\mathcal{UCV}^*(\beta) \cap \mathcal{V}$, and $k - \mathcal{PS}_p^*(\beta) \equiv k - \mathcal{S}_p^*(\beta) \cap \mathcal{V}$. It is worthy to note that for $k = 0$, the classes $k - \mathcal{UCV}^*(\beta)$, $k - \mathcal{S}_p^*(\beta)$, $k - \mathcal{PUCV}^*(\beta)$ and $k - PS_p^*(\beta)$ are reduced to the classes $\mathcal{M}(\beta)$, $\mathcal{L}(\beta)$, $\mathcal{V}(\beta)$ and $\mathcal{U}(\beta)$, respectively. The classes $\mathcal{M}(\beta)$, $\mathcal{L}(\beta)$, $\mathcal{V}(\beta)$ and $\mathcal{U}(\beta)$ had been extensively studied by Uralegaddi *et al.* [21].

For $\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$, let $\mathcal{R}^\tau(A, B)$ denote the subclass of $\mathcal{A}$ given by

$$\mathcal{R}^\tau(A, B) = \left\{ f \in \mathcal{A} : \left| \frac{f'(z) - 1}{(A - B)\tau - B(f'(z) - 1)} \right| < 1, z \in \mathbb{U} \right\},$$

and $\mathcal{R}(\beta)$ by

$$\mathcal{R}(\beta) = \{ f \in \mathcal{A} : \Re\{f'(z)\} < \beta, \, z \in \mathbb{U}, \, 1 < \beta \leq 2 \}.$$

The classes $\mathcal{R}^{\tau}(A,B)$ and $\mathcal{R}(\beta)$ were introduced and studied by Dixit and Pal [4] and Uralegaddi *et al.* [22], respectively.

For the complex numbers $a_1, a_2, \dots, a_p$ and $b_1, b_2, \dots, b_q$ with $b_j \neq 0, -1, -2, \dots, j = 1, 2, \dots, q$, the generalized hypergeometric functions ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$ is defined by

$$(1.6)\qquad {}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \quad z \in \mathbb{U},$$

where $p \leq q+1$ and $(a)_n$ is the Pochhammer symbol defined by

$$(1.7)\qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

The convergence condition of the series defined by (1.6) is given below:

(1) If $p < q+1$, then the series converges absolutely in the entire complex plane.
(2) If $p \leq q$, then the series converges absolutely for every finite z.
(3) If $p = q+1$, then the series converges absolutely when $|z| < 1$.
(4) If $p = q+1$ and $|z| = 1$, then the series converges when

$$\Re\left\{\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right\} > 0.$$

For a detailed study one may refer [16].

Now, we define for $a_i, b_j, m > 0$, where $i = 1, 2, \dots, p$, and $j = 1, 2, \dots, q$ such that the series

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; m) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{m^n}{n!} = {}_pF_q(a_1; b_1; m)$$

is convergent.

Throughout this article, we will frequently use the notation

$${}_pF_q(a_1; b_1; z) = {}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$$

and for $s \in \mathbb{N}$

$${}_pF_q(a_1+s; b_1+s; z) = {}_pF_q(a_1+s, a_2+s, \dots, a_p+s; b_1+s, b_2+s, \dots, b_q+s; z).$$

We note that

$$(1.8)\qquad {}_pF_q(a_1; b_1; m) - 1 - m\, \frac{a_1 \cdots a_p}{b_1 \cdots b_q} = \sum_{n=2}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{m^n}{n!},$$

$$(1.9)\qquad {}_pF_q(a_1; b_1; m) - 1 = \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!},$$

$$(1.10)\qquad m\, \frac{a_1 \cdots a_p}{b_1 \cdots b_q}\, {}_pF_q(a_1+1; b_1+1; m) = \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-2)!},$$

$$\begin{aligned} &m^2\, \frac{a_1(a_1+1)\cdots a_p(a_p+1)}{b_1(b_1+1)\cdots b_q(b_q+1)}\, {}_pF_q(a_1+2;b_1+2;m) \\ &= \sum_{n=3}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-3)!}. \end{aligned} \tag{1.11}$$

Recently, Themangani et al. [20] introduced the generalized hypergeometric distribution whose probability mass function is $\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{m^n}{n!}\frac{1}{{}_pF_q(a_1;b_1;m)}$, $n = 0,1,2,\ldots$.

By specializing the parameters in the generalized hypergeometric distribution it reduces to the following probability distribution:

(1) If $p = 2$, $q = 1$, then it reduces to the hypergeometric type probability distribution studied by Porwal and Gupta [15].
(2) If $p = q = 1$, then it reduces to the confluent hypergeometric distribution studied by Porwal [11].
(3) If $p = q = 1$ and $a_1 = b_1$, then it reduces to the well-known Poisson distribution.

Next, we introduce the generalized hypergeometric distribution series whose coefficients are probabilities of generalized hypergeometric distribution

$$ {}_p\Phi_q(a_1;b_1;m,z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{z^n}{{}_pF_q(a_1;b_1;m)}, \tag{1.12}$$

where $a_i, b_j > 0$, $i = 1,2,\ldots,p$, $j = 1,2,\ldots,q$.

The convolution (or, Hadamard product) of two power series $f \in \mathcal{A}$ of the form (1.1) and $g \in \mathcal{A}$ is given by $\sum_{n=0}^{\infty} b_n z^n$, then

$$(f*g)(z) = f(z)*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \qquad z \in \mathbb{U}.$$

Now, we consider a linear operator $\Omega_{p,q}^m : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} \Omega_{p,q}^m f(z) &= {}_p\Phi_q(a_1;b_1;m,z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{a_n z^n}{{}_pF_q(a_1;b_1;m)}. \end{aligned} \tag{1.13}$$

The purpose of the present paper is to establish connections between distribution function and Geometric Function Theory based on special functions. Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions, generalized Bessel functions and distribution series (see [1,2,8,9,12,13,18]), we establish a number of connections between the classes $k-\mathcal{UCV}^*(\beta)$, $k-\mathcal{S}_p^*(\beta)$, $\mathcal{R}(\beta)$, $\mathcal{R}^\tau(A,B)$, $k-\mathcal{PUCV}^*(\beta)$ and $k-\mathcal{PS}_p^*(\beta)$ by applying the convolution operator $\Omega_{p,q}^m$.

2. Main results

In order to establish connection between various subclasses of $\mathcal{A}$ we need the following lemmas.

Lemma 2.1 ([4]). *If $f \in \mathcal{R}^{\tau}(A,B)$ is of the form* (1.1)*, then*

$$|a_n| \le (A-B)\frac{|\tau|}{n},\ n \ge 2.$$

Lemma 2.2 ([12]). *Let $f(z) \in \mathcal{A}$ be of the form* (1.1) *and*

$$\sum_{n=2}^{\infty} n\left(n - \frac{k+\beta}{1+k}\right)|a_n| \le \frac{k+\beta}{1+k} - 1.$$

Then $f \in k-\mathcal{UCV}^*(\beta) \equiv \mathcal{U}\left(\frac{k+\beta}{1+k}\right)$, where $0 \le k < \infty$, $1 < \beta \le \frac{4+k}{3}$.

Lemma 2.3 ([22]). *Let $f(z) \in \mathcal{A}$ be of the form* (1.5) *and $f \in \mathcal{R}(\beta)$. Then $|a_n| \le \frac{\beta-1}{n}$, where $1 < \beta \le 2$.*

Lemma 2.4 ([12]). *Let $f(z) \in \mathcal{A}$ be of the form* (1.1) *and*

$$\sum_{n=2}^{\infty} \left(n - \frac{k+\beta}{1+k}\right)|a_n| \le \frac{k+\beta}{1+k} - 1.$$

Then $f \in k-\mathcal{S}_p^*(\beta) \equiv \mathcal{V}\left(\frac{k+\beta}{1+k}\right)$, where $0 \le k < \infty$, $1 < \beta \le \frac{4+k}{3}$.

Lemma 2.5 ([12]). *Let $f(z) \in \mathcal{A}$ be of the form* (1.5) *and $f \in k-\mathcal{PS}_p^*(\beta)$. Then*

$$|a_n| \le \frac{\beta-1}{n+nk-k-\beta},$$

where $0 \le k < \infty$, $1 < \beta \le \frac{4+k}{3}$

Lemma 2.6 ([12]). *Let $f(z) \in \mathcal{A}$ be of the form* (1.5) *and $f \in k-\mathcal{PUCV}^*(\beta)$. Then*

$$|a_n| \le \frac{\beta-1}{n(n+nk-k-\beta)},$$

where $0 \le k < \infty$, $1 < \beta \le \frac{4+k}{3}$.

Theorem 2.7. *If $a_i,\ b_j > 0$ $(i = 1,2,\ldots,p; j = 1,2,\ldots,q)$ and the inequality*

$$(2.1)\quad (k+1)\frac{a_1\cdots a_p}{b_1\cdots b_q} m\, {}_pF_q(a_1+1;\ b_1+1;\ m) \le (\beta-1)[2\,{}_pF_q(a_1;b_1;m)-1]$$

holds with either one of the conditions

(1) *$p \le q$ and $m > 0$,*
(2) *$p = q+1$ and $m < 1$,*
(3) *$p = q+1$, $m = 1$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1$,*

then ${}_p\Phi_q(a_1;b_1;m,\ z)$ be defined by (1.12) *is in the class $k-\mathcal{S}_p^*(\beta)$.*

Proof. Let ${}_p\Phi_q(a_1;b_1;m,\ z)$ defined by (1.12). By virtue of Lemma 2.4, it is sufficient to prove that

$$\begin{aligned}L_1(k,\beta,m) &= \sum_{n=2}^{\infty}[n(1+k)-(k+\beta)]\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(a_1;b_1;m)}\\ &\leq \beta-1.\end{aligned}$$

Now,

$$\begin{aligned}&L_1(k,\beta,m)\\ &= \sum_{n=2}^{\infty}(n+nk-k-\beta)\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(a_1;b_1;m)}\\ &= \sum_{n=2}^{\infty}\Big[\ (k+1)(n-1)-(\beta-1)\ \Big]\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(a_1;b_1;m)}\\ &= \frac{1}{{}_pF_q(a_1;b_1;m)}\Bigg[(k+1)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-2)!}\\ &\qquad -(\beta-1)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\Bigg].\end{aligned}$$

Using (1.9)-(1.10), we get

$$\begin{aligned}L_1(k,\beta,m) = \frac{1}{{}_pF_q(a_1;b_1;m)}\Bigg[&m(k+1)\frac{a_1\cdots a_p}{b_1\cdots b_q}\,{}_pF_q(a_1+1;\ b_1+1;\ m)\\ &-(\beta-1)\left({}_pF_q(a_1;b_1;m)-1\right)\Bigg].\end{aligned}$$

But this expression is bounded above by $\beta-1$ if and only if (2.1) holds, and the proof is complete. □

Theorem 2.8. *If $a_i,\ b_j>0$ $(i=1,2,\ldots,p; j=1,2,\ldots,q)$ and the inequality*

$$\begin{aligned}&(1+k)\frac{a_1(a_1+1)\cdots a_p(a_p+1)}{b_1(b_1+1)\cdots b_q(b_q+1)}m^2{}_pF_q(a_1+2;\ b_1+2;\ m)\\ &+(3+2k-\beta)\frac{a_1\cdots a_p}{b_1\cdots b_q}m{}_pF_q(a_1+1;\ b_1+1;\ m)\\ &\leq(\beta-1)[2{}_pF_q(a_1;b_1;m)-1] \qquad (2.2)\end{aligned}$$

holds with either one of the conditions

1. *$p\leq q$ and $m>0$,*
2. *$p=q+1$ and $m<1$,*
3. *$p=q+1$, $m=1$ and $\sum_{j=1}^{q}b_j>\sum_{i=1}^{p}a_i+2$,*

then ${}_p\Phi_q(a_1;b_1;m,\ z)$ defined by (1.12) is in the class $k-\mathcal{UCV}^*(\beta)$.

Proof. By virtue of Lemma 2.2, it is sufficient to prove that

$$L_2(k,\beta,m) = \sum_{n=2}^{\infty} n[n(1+k)-(k+\beta)] \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{1}{{}_pF_q(a_1;b_1;m)} \leq \beta - 1.$$

Now, by writing $n = (n-1)+1$, and $n^2 = (n-1)(n-2)+3(n-1)+1$, we can rewrite the above term as

$$\begin{aligned} &L_2(k,\beta,m) \\ &= \sum_{n=2}^{\infty} n[n(1+k)-(k+\beta)] \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{1}{{}_pF_q(a_1;b_1;m)} \\ &= \frac{1}{{}_pF_q(a_1;b_1;m)} \Bigg[\sum_{n=2}^{\infty} [(n-1)(n-2)(1+k)+(3+2k-\beta)(n-1)-(\beta-1)] \\ &\qquad \times \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \Bigg] \\ &= \frac{1}{{}_pF_q(a_1;b_1;m)} \Bigg[(1+k) \sum_{n=3}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-3)!} \\ &\qquad + (3+2k-\beta) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-2)!} \\ &\qquad - (\beta-1) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \Bigg]. \end{aligned}$$

Using (1.9)-(1.11), we get

$$\begin{aligned} &L_2(k,\beta,m) \\ &= \frac{1}{{}_pF_q(a_1;b_1;m)} \Bigg[(1+k) \frac{a_1(a_1+1)\cdots a_p(a_p+1)}{b_1(b_1+1)\cdots b_q(b_q+1)} m^2 \; {}_pF_q(a_1+2;\; b_1+2;\; m) \\ &\qquad + (3+2k-\beta) \frac{a_1\cdots a_p}{b_1\cdots b_q} m \; {}_pF_q(a_1+1;\; b_1+1;\; m) \\ &\qquad - (\beta-1)\left({}_pF_q(a_1;b_1;m)-1\right) \Bigg] \end{aligned}$$

which is bounded above by $\beta - 1$ if and only if (2.2) holds.This completes the proof of Theorem 2.8. □

Theorem 2.9. *Let $m > 0$, if for some $k\,(0 \leq k < \infty)$ and $\beta\left(1 < \beta \leq \frac{3+k}{2}\right)$, $a_i, b_j > 0$ $(i = 1,2,\ldots,p; j = 1,2,\ldots,q)$, and $f \in \mathcal{R}^\tau(A,B)$ the inequality*

$$\frac{(A-B)|\tau|}{{}_pF_q(a_1;b_1;m)} \Bigg[(1+k) \frac{a_1\cdots a_p}{b_1\cdots b_q} m \; {}_pF_q(a_1+1;\; b_1+1;\; m) - (\beta-1)\left({}_pF_q(a_1;b_1;m)-1\right) \Bigg] \leq \beta - 1, \tag{2.3}$$

holds with either one of the conditions

(1) $p \le q$ *and* $m > 0$,
(2) $p = q + 1$ *and* $m < 1$,
(3) $p = q + 1$, $m = 1$ *and* $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1$,

then $\Omega_{p,q}^{m} f \in k - \mathcal{UCV}^*(\beta)$.

Proof. Let $f \in \mathcal{R}^{\tau}(A, B)$, where f is of the form (1.1). In view of Lemma 2.2, it is enough to prove $P_1 \le \beta - 1$ where

(2.4)
$$P_1 = \sum_{n=2}^{\infty} n\,(n + nk - k - \beta) \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{1}{{}_pF_q(a_1; b_1; m)} |a_n| \,.$$

In view of Lemma 2.1, substituting for $|a_n| \le (A - B)\,\frac{|\tau|}{n}$, $n \ge 2$ we have

$$\begin{aligned} P_1 &\le (A-B)|\tau| \left[\sum_{n=2}^{\infty} (n + nk - k - \beta) \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{1}{{}_pF_q(a_1; b_1; m)} \right] \\ &= (A-B)|\tau| \left[\sum_{n=2}^{\infty} \Big[(k+1)(n-1) - (\beta - 1) \Big] \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{1}{{}_pF_q(a_1; b_1; m)} \right] \\ &= \frac{(A-B)|\tau|}{{}_pF_q(a_1; b_1; m)} \left[(k+1) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-2)!} \right. \\ &\qquad \left. -(\beta - 1) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \right] \\ &= \frac{(A-B)|\tau|}{{}_pF_q(a_1; b_1; m)} \left[(k+1) \frac{a_1 \cdots a_p}{b_1 \cdots b_q} m \; {}_pF_q(a_1 + 1;\; b_1 + 1;\; m) \right. \\ &\qquad \left. - (\beta - 1) \left({}_pF_q(a_1; b_1; m) - 1 \right) \right] \end{aligned}$$

and this last expression is bounded above by $\beta - 1$ if and only if (2.3) holds. □

Theorem 2.10. *Let* $m > 0$ *and* $f \in \mathcal{R}(\beta_1)$. *If for some* $k\,(0 \le k < \infty)$, $\beta_1\,(1 < \beta_1 \le 2)$ *and* $a_i, b_j > 0$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$, $\beta \left(1 < \beta \le \frac{3+k}{2}\right)$, *the inequality*

$$\begin{aligned} \frac{\beta_1 - 1}{{}_pF_q(a_1; b_1; m)} &\left[(k+1) \frac{a_1 \cdots a_p}{b_1 \cdots b_q} m \; {}_pF_q(a_1 + 1;\; b_1 + 1;\; m) \right. \\ &\qquad \left. - (\beta - 1) \left({}_pF_q(a_1; b_1; m) - 1 \right) \right] \le \beta - 1, \end{aligned} \tag{2.5}$$

holds with either one of the conditions

(1) $p \le q$ *and* $m > 0$,
(2) $p = q + 1$ *and* $m < 1$,
(3) $p = q + 1$, $m = 1$ *and* $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1$,

is satisfied, then $\Omega_{p,q}^{m} f \in k - \mathcal{UCV}^*(\beta)$.

Proof. Let $f \in \mathcal{R}(\beta_1)$, where f is of the form (1.1). In view of Lemma 2.2, it suffices to prove $P_1 \le \beta - 1$, where P_1 is given by (2.4). In view of Lemma 2.3, we have

$$\begin{aligned}
P_1 &\le (\beta_1 - 1)\left[\sum_{n=2}^{\infty}(n + nk - k - \beta)\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\frac{1}{{}_pF_q(a_1;b_1;m)}\right]\\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)}\left[\sum_{n=2}^{\infty}((k+1)(n-1) - (\beta - 1))\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right]\\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)}\left[(k+1)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-2)!}\right.\\
&\qquad\left. -(\beta - 1)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\right]\\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)}\left[(k+1)\frac{a_1\cdots a_p}{b_1\cdots b_q}m\;{}_pF_q(a_1+1;\; b_1+1;\; m)\right.\\
&\qquad\left. -(\beta - 1)\left({}_pF_q(a_1;b_1;m) - 1\right)\right]\\
&\le \beta - 1,
\end{aligned}$$

by the given hypothesis.

This completes the proof of Theorem 2.10. □

Theorem 2.11. *Let $m > 0$ and $f \in k - \mathcal{PUCV}^*(\beta_1)$, $(1 < \beta_1 \le 2)$. If for some $k\,(0 \le k < \infty)$, and $a_i, b_j > 0$ $(i = 1, 2, \dots, p; j = 1, 2, \dots, q)$, $\beta\left(1 < \beta \le \frac{3+k}{2}\right)$, the inequality*

$$\frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)}\left({}_pF_q(a_1;b_1;m) - 1\right) \le \beta - 1$$

holds with either one of the conditions

(1) $p \le q$ *and* $m > 0$,
(2) $p = q + 1$ *and* $m < 1$,
(3) $p = q + 1$, $m = 1$ *and* $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i$,

then $\Omega^m_{p,q} f \in k - \mathcal{UCV}^*(\beta)$.

Proof. Let $f \in k - \mathcal{PUCV}^*(\beta_1)$, where f is of the form (1.5). In view of Lemma 2.2, it suffices to prove that $P_1 \le \beta - 1$, where P_1 is given by (2.4). In view of Lemma 2.6, we have

$$\begin{aligned}
P_1 &\le \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)}\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{m^{n-1}}{(n-1)!}\\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)}\left({}_pF_q(a_1;b_1;m) - 1\right)\\
&\le \beta - 1,
\end{aligned}$$

which is true for all $m > 0$. This completes the proof of Theorem 2.11. □

Theorem 2.12. *Let $m > 0$ and $f \in k - \mathcal{PS}_p^*(\beta_1), (1 < \beta_1 \leq 2)$. If for some $k\,(0 \leq k < \infty)$ and $\beta\left(1 < \beta \leq \frac{4+k}{3}\right)$, $a_i, b_j > 0$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$, the inequality*

$$\frac{\beta_1 - 1}{{}_pF_q(a_1; b_1; m)}\left[\frac{a_1 \cdots a_p}{b_1 \cdots b_q} m\, {}_pF_q(a_1+1; b_1+1; m) + ({}_pF_q(a_1; b_1; m) - 1)\right] \leq \beta - 1, \tag{2.6}$$

holds with either one of the conditions

1. $p \leq q$ *and* $m > 0$,
2. $p = q + 1$ *and* $m < 1$,
3. $p = q + 1$, $m = 1$ *and* $\sum_{j=1}^q b_j > \sum_{i=1}^p a_i + 1$,

is satisfied, then $\Omega_{p,q}^m f \in k - \mathcal{UCV}^*(\beta)$.

Proof. Let $f \in k - \mathcal{PS}_p^*(\beta_1)$, where f is of the form (1.5). In view of Lemma 2.2, it is enough to show that $P_1 \leq \beta - 1$, where P_1 is given by (2.4). In view of Lemma 2.5, we have

$$\begin{aligned} P_1 &\leq \frac{\beta_1 - 1}{{}_pF_q(a_1; b_1; m)} \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \\ &= \frac{\beta_1 - 1}{{}_pF_q(a_1; b_1; m)} \left[\sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-2)!} \right. \\ &\qquad \left. + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!}\right] \\ &= \frac{\beta_1 - 1}{{}_pF_q(a_1; b_1; m)} \left[\frac{a_1 \cdots a_p}{b_1 \cdots b_q} m\, {}_pF_q(a_1 + 1; b_1 + 1; m) + ({}_pF_q(a_1; b_1; m) - 1)\right] \end{aligned}$$

which is bounded above by $\beta - 1$, if and only if (2.6) holds. This completes the proof of Theorem 2.12. □

Theorem 2.13. *Let $m > 0$ and $f \in \mathcal{R}^\tau(A, B)$. If for some $k\,(0 \leq k < \infty)$ and $\beta\left(1 < \beta \leq \frac{4+k}{3}\right)$, and $a_i, b_j > 1$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$, the inequality*

$$\begin{aligned} &\frac{(A - B)|\tau|}{{}_pF_q(a_1; b_1; m)} \left[(k + 1)\left({}_pF_q(a_1; b_1; m) - 1\right) - \frac{(b_1 - 1) \cdots (b_q - 1)(k + \beta)}{(a_1 - 1) \cdots (a_p - 1)m} \right. \\ &\left. \left({}_pF_q(a_1 - 1; b_1 - 1; m) - \frac{(a_1 - 1) \cdots (a_p - 1)m}{(b_1 - 1) \cdots (b_q - 1)} - 1\right)\right] \leq \beta - 1, \end{aligned} \tag{2.7}$$

holds with either one of the conditions

1. $p \leq q$ *and* $m > 0$,
2. $p = q + 1$ *and* $m < 1$,
3. $p = q + 1$, $m = 1$ *and* $\sum_{j=1}^q b_j > \sum_{i=1}^p a_i$,

then $\Omega^m_{p,q} f \in k - \mathcal{S}^*_p(\beta)$.

Proof. Let $f \in \mathcal{R}^\tau(A,B)$, where f is of the form (1.1). In view of Lemma 2.4, it is enough to show that $P_2 \le \beta - 1$, where

(2.8)
$$P_2 = \sum_{n=2}^{\infty} (n + nk - k - \beta) \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \frac{1}{{}_pF_q(a_1;b_1;m)} |a_n| .$$

In view of Lemma 2.1, fixing $|a_n| \le (A-B)\,\frac{|\tau|}{n}$, $n \ge 2$, we get

$$\begin{aligned}
P_2 &\le \frac{(A-B)|\tau|}{{}_pF_q(a_1;b_1;m)} \sum_{n=2}^{\infty} \left[(k+1) - \frac{k+\beta}{n}\right] \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \\
&= \frac{(A-B)|\tau|}{{}_pF_q(a_1;b_1;m)} \left[(k+1) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \right. \\
&\qquad \left. - (k+\beta) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n-1}}{(n)!} \right] \\
&= \frac{(A-B)|\tau|}{{}_pF_q(a_1;b_1;m)} \Big[(k+1) \left({}_pF_q(a_1;b_1;m) - 1\right) \\
&\qquad - \frac{k+\beta}{m} \Big(\sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_p)_{n-1}}{(b_1)_{n-1} \cdots (b_q)_{n-1}} \frac{m^{n}}{(n)!} \Big) \Big] \\
&= \frac{(A-B)|\tau|}{{}_pF_q(a_1;b_1;m)} \left[(k+1) \left({}_pF_q(a_1;b_1;m) - 1\right) - \frac{(b_1-1)\ldots(b_q-1)(k+\beta)}{(a_1-1)\cdots(a_p-1)m} \right. \\
&\qquad \left. \left({}_pF_q(a_1-1;b_1-1;m) - \frac{(a_1-1)\cdots(a_p-1)m}{(b_1-1)\cdots(b_q-1)} - 1\right)\right]
\end{aligned}$$

is bounded above by $\beta - 1$, if and only if (2.7) holds, which completes the proof of Theorem 2.13. □

Theorem 2.14. *Let* $m > 0$ *and* $f \in \mathcal{R}(\beta_1), (1 < \beta_1 \le 2)$. *If for some* k $(0 \le k < \infty)$, $a_i, b_j > 1$ $(i = 1,2,\ldots,p; j = 1,2,\ldots,q)$, *and* $\beta \left(1 < \beta \le \frac{4+k}{3}\right)$, *then the inequality*

(2.9)
$$\begin{aligned}
&\left[(k+1) \left({}_pF_q(a_1;b_1;m) - 1\right) - \frac{(b_1-1)\cdots(b_q-1)(k+\beta)}{(a_1-1)\cdots(a_p-1)m} \right. \\
&\quad \left. \times \left({}_pF_q(a_1-1;b_1-1;m) - \frac{(a_1-1)\cdots(a_p-1)m}{(b_1-1)\cdots(b_q-1)} - 1\right)\right] \\
&\le \frac{(\beta-1)}{\beta_1 - 1} {}_pF_q(a_1;b_1;m),
\end{aligned}$$

holds with either one of the conditions

(1) $p \le q$ *and* $m > 0$,
(2) $p = q+1$ *and* $m < 1$,

(3) $p = q+1$, $m = 1$ *and* $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i$,

is satisfied, then $\Omega_{p,q}^{m} f \in k - \mathcal{S}_p^*(\beta)$.

Proof. Let $f \in \mathcal{R}(\beta_1)$, where f is of the form (1.5). In view of Lemma 2.4, it is enough to show that $P_2 \leq \beta - 1$, where P_2 is defined by (2.8).

In view of Lemma 2.3, we have

$$\begin{aligned}
P_2 &\leq \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)} \sum_{n=2}^{\infty} \left[(k+1) - \frac{k+\beta}{n}\right] \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)} \left[(k+1) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \right. \\
&\qquad \left. -(k+\beta) \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n)!} \right] \\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)} \Big[(k+1)\left({}_pF_q(a_1;b_1;m) - 1\right) \\
&\qquad - \frac{(b_1-1)\cdots(b_q-1)(k+\beta)}{(a_1-1)\cdots(a_p-1)m} \sum_{n=2}^{\infty} \frac{(a_1-1)_n\cdots(a_p-1)_n}{(b_1-1)_n\cdots(b_q-1)_n} \frac{m^n}{(n)!} \Big] \\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)} \left[(k+1)\left({}_pF_q(a_1;b_1;m) - 1\right) - \frac{(b_1-1)\cdots(b_q-1)(k+\beta)}{(a_1-1)\cdots(a_p-1)m} \right. \\
&\qquad \left. \left({}_pF_q(a_1-1;b_1-1;m) - \frac{(a_1-1)\cdots(a_p-1)m}{(b_1-1)\cdots(b_q-1)} - 1\right)\right].
\end{aligned}$$

The above P_2 is confined above by $\beta - 1$, if and only if (2.9)holds. This completes the proof of Theorem 2.14. □

Theorem 2.15. *For all* $m > 0$, *and* $f \in k-\mathcal{PS}_p^*(\beta_1)$, $(1 < \beta_1 \leq 2)$. *If for some* $a_i, b_j > 1$ $(i = 1,2,\ldots,p; j = 1,2,\ldots,q)$, $\beta\left(1 < \beta \leq \frac{4+k}{3}\right)$, *and the inequality*

$$\frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)} \left({}_pF_q(a_1;b_1;m) - 1\right) \leq \beta - 1 \tag{2.10}$$

holds, then $\Omega_{p,q}^{m}(f) \in k - \mathcal{S}_p^*(\beta)$.

Proof. Let $f \in k - \mathcal{PS}_p^*(\beta_1)$, where f is of the form (1.5). In view of Lemma 2.4, it is enough to show that $P_2 \leq \beta - 1$, where P_2 is defined by (2.8). In view of Lemma 2.5, we have

$$\begin{aligned}
P_2 &\leq \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)} \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}} \frac{m^{n-1}}{(n-1)!} \\
&= \frac{\beta_1 - 1}{{}_pF_q(a_1;b_1;m)} \left({}_pF_q(a_1;b_1;m) - 1\right).
\end{aligned}$$

P_2 is bounded above by $\beta - 1$, if and only if (2.10) holds, which completes the proof of Theorem 2.15. □

We state the following result without proof.

Theorem 2.16. *Let* $m > 0$*, and* $f \in k - \mathcal{PUCV}^*(\beta)$*. If* $a_i, b_j > 1$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$*,* k $(0 \le k < \infty)$ *and* $\beta \left(1 < \beta \le \frac{4+k}{3}\right)$*, and the inequality*

$$\frac{(b_1-1)\cdots(b_q-1)(k+\beta)}{(a_1-1)\cdots(a_p-1)m} \times \left({}_pF_q(a_1-1;b_1-1;m) - \frac{(a_1-1)\cdots(a_p-1)m}{(b_1-1)\cdots(b_q-1)} - 1\right) \le {}_pF_q(a_1;b_1;m)$$

holds with either one of the conditions

(1) $p \le q$ *and* $m > 0$,
(2) $p = q + 1$ *and* $m < 1$,
(3) $p = q + 1$, $m = 1$ *and* $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i - 1$,

is satisfied, then $\Omega_{p,q}^m(f) \in k - \mathcal{S}_p^*(\beta)$.

Remark 2.17. If we put $p = q = 1$ and $a_1 = b_1$ in Theorems 2.7-2.16, then we obtain the corresponding results of Porwal [10].

Remark 2.18. If we put $p = 2, q = 1$ in Theorems 2.7-2.16, then we obtain the results related to hypergeometric type distribution series.

Remark 2.19. If we put $p = q = 1$ in Theorems 2.7-2.16, then we obtain the results related to confluent hypergeometric distribution series.

Acknowledgement. The authors are thankful to the referee for his/her valuable comments and observations which helped in improving the paper.

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