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A CAMERON–STORVICK THEOREM ON $C^2_{a,b}[0,T]$ WITH APPLICATIONS

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Abstract. The purpose of this paper is to establish a very general Cameron–Storvick theorem involving the generalized analytic Feynman integral of functionals on the product function space $C^2_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$ can be induced by the generalized Brownian motion process associated with continuous functions a and b. To do this we first introduce the class $\mathcal{F}^{a,b}_{A_1,A_2}$ of functionals on $C^2_{a,b}[0,T]$ which is a generalization of the Kallianpur and Bromley Fresnel class $\mathcal{F}_{A_1,A_2}$. We then proceed to establish a Cameron–Storvick theorem on the product function space $C^2_{a,b}[0,T]$. Finally we use our Cameron–Storvick theorem to obtain several meaningful results and examples.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m_w denote Wiener measure. Then, as is well-known, $(C_0[0,T],\mathcal{M},m_w)$ is a complete measure space.

In [1] Cameron established an integration by parts formula for the Wiener measure m_w. More precisely, in [1], Cameron introduced the first variation (a kind of Gâteaux derivative) of functionals on the classical Wiener space $C_0[0,T]$ and established a formula involving the Wiener integral of the first variation. This was the first infinite dimensional integration by parts formula. In [15] Donsker also established this formula using a different method, and applied it to study Fréchet–Volterra differential equations. In [22, 23] Kuo and Lee established an integration by parts formula for abstract Wiener space which they then used to evaluate various functional integrals.

In [2], Cameron and Storvick also established an integration by parts formula involving the analytic Feynman integral of functionals on $C_0[0,T]$. They also

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applied their celebrated parts formula to establish the existence of the analytic Feynman integral of unbounded functionals on $C_0[0,T]$. The integration by parts formula on $C_0[0,T]$ suggested in [2] was improved in [24] to study the parts formulas involving the analytic Feynman integral and the analytic Fourier–Feynman transform. Since then the parts formula for the analytic Feynman integral is called the Cameron–Storvick theorem by many mathematicians.

The Cameron–Storvick theorem and related topics were also developed for functionals on the very general function space $C_{a,b}[0,T]$ in [5,10]. The function space $C_{a,b}[0,T]$, induced by generalized Brownian motion process (GBMP), was introduced by J. Yeh [27,28].

A GBMP on a probability space (Ω,Σ,P) and a time interval $[0,T]$ is a Gaussian process $Y\equiv\{Y_t\}_{t\in[0,T]}$ such that $Y_0=c$ almost surely for some constant $c\in\mathbb{R}$, and for any set of time moments $0=t_0<t_1<\cdots<t_n\le T$ and any Borel set $B\subset\mathbb{R}^n$, the measure $P(I_{t_1,\dots,t_n,B})$ of the cylinder set $I_{t_1,\dots,t_n,B}$ of the form

$$I_{t_1,\dots,t_n,B}=\big\{\omega\in\Omega:(Y(t_1,\omega),\dots,Y(t_n,\omega))\in B\big\}$$

is equal to

$$\begin{aligned}&\bigg((2\pi)^n\prod_{j=1}^{n}\big(b(t_j)-b(t_{j-1})\big)\bigg)^{-1/2}\\ &\quad\times\int_B\exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}\frac{((\eta_j-a(t_j))-(\eta_{j-1}-a(t_{j-1})))^2}{b(t_j)-b(t_{j-1})}\bigg\}d\eta_1\cdots d\eta_n,\end{aligned}$$

where $\eta_0=c$, $a(t)$ is a continuous real-valued function on $[0,T]$, and $b(t)$ is a increasing continuous real-valued function on $[0,T]$. Thus, the GBMP Y is determined by the continuous functions $a(\cdot)$ and $b(\cdot)$. For more details, see [27, 28]. Note that when $c=0$, $a(t)\equiv0$ and $b(t)=t$ on $[0,T]$, the GBMP is the standard Brownian motion (Wiener process). In this paper we set $c=a(0)=b(0)=0$. Then the function space $C_{a,b}[0,T]$ induced by the GBMP Y determined by the $a(\cdot)$ and $b(\cdot)$ can be considered as the space of continuous sample paths of Y.

In this paper, we establish a general Cameron–Storvick theorem involving the generalized analytic Feynman integral of functionals on the product function space $C_{a,b}^2[0,T]$. We also present a meaningful example to which our Cameron–Storvick theorem can be applied. To do this, we introduce the class $\mathcal{F}_{A_1,A_2}^{a,b}$ of functionals on $C_{a,b}^2[0,T]$ which is a generalization of the Kallianpur and Bromley Fresnel class $\mathcal{F}_{A_1,A_2}$, see [21].

2. Preliminaries

In this section we first give a brief background of some ideas and results which are needed to establish our new results in Sections 3, 4, and 5 below.

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$ and $a'(t) \in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. The generalized Brownian motion process Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$.

We consider the function space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ induced by the generalized Brownian motion process Y, where $C_{a,b}[0,T]$ denotes the set of continuous sample paths of the generalized Brownian motion process Y and $\mathcal{W}(C_{a,b}[0,T])$ is the σ-field of all μ-Carathéodory measurable subsets of $C_{a,b}[0,T]$. For the precise procedure used to construct this function space, we refer to the references [3–5, 9, 10, 27, 28].

We note that the coordinate process defined by $e_t(x) = x(t)$ on $C_{a,b}[0,T] \times [0,T]$ is also the generalized Brownian motion process determined by $a(t)$ and $b(t)$, i.e., for each $t \in [0,T]$, $e_t(x) \sim N(a(t), b(t))$, and the process $\{e_t : 0 \le t \le T\}$ has nonstationary and independent increments.

Recall that the process $\{e_t : 0 \le t \le T\}$ on $C_{a,b}[0,T]$ is a continuous process. Thus the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$, considered in papers [1, 2, 18, 20, 24] if and only if $a(t) \equiv 0$ and $b(t) = t$ for all $t \in [0,T]$.

In [5, 8, 10], the generalized analytic Feynman integral and the generalized analytic Fourier–Feynman transform of functionals on $C_{a,b}[0,T]$ were investigated. The functionals considered in [5, 8, 10] are associated with the separable Hilbert space

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < +\infty \ \text{ and } \int_0^T v^2(s)d|a|(s) < +\infty \right\},$$

where $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$. The inner product on $L^2_{a,b}[0,T]$ is given by the formula

$$(u,v)_{a,b} = \int_0^T u(s)v(s)dm_{|a|,b}(s) \equiv \int_0^T u(s)v(s)d[b(s) + |a|(s)],$$

where $m_{|a|,b}$ is the Lebesgue–Stieltjes measure induced by the increasing function $|a|(\cdot) + b(\cdot)$ on $[0,T]$. We note that $\|u\|_{a,b} \equiv \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$.

The following linear subspace of $C_{a,b}[0,T]$ plays an important role throughout this paper.

Let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T Dw_1(s)Dw_2(s)db(s) = \int_0^T z_1(s)z_2(s)db(s)$$

is a separable Hilbert space.

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the linear operator given by the equation (2.1). The inverse operator of D is given by

$$(D^{-1}z)(t) = \int_0^t z(s)db(s), \quad t \in [0,T]. \tag{2.2}$$

For a more detailed study of the inverse operator D^{-1} of D, see [7].

Recall that above, as well as in papers [5,8,10], we require that $a : [0,T] \to \mathbb{R}$ is an absolutely continuous function with $a(0) = 0$ and with $\int_0^T |a'(t)|^2 dt < \infty$. Our conditions on $b : [0,T] \to \mathbb{R}$ imply that $0 < \delta < b'(t) < M$ for some positive real numbers δ and M, and all $t \in [0,T]$. In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty. \tag{2.3}$$

One can see that the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.3) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$. Under the condition (2.3), we also observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w, a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)db(t) = \int_0^T z(t)a'(t)dt = \int_0^T z(t)da(t).$$

For each $w \in C'_{a,b}[0,T]$, the Paley–Wiener–Zygmund (PWZ) stochastic integral $(w,x)^\sim$ is given by the formula

$$(w,x)^\sim = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (w, g_j)_{C'_{a,b}} Dg_j(t)dx(t) \tag{2.4}$$

for μ-a.e. $x \in C_{a,b}[0,T]$ where $\{g_j\}_{j=1}^\infty$ is a complete orthonormal set of functions in $C'_{a,b}[0,T]$ such that for each $j \in \mathbb{N}$, Dg_j is of bounded variation on $[0,T]$. For a more detailed study of the space $C'_{a,b}[0,T]$ and the PWZ stochastic integral given by (2.4), see [4,9].

In [25], Pierce and Skoug used the the inner product $(\cdot,\cdot)_{a,b}$ on $L^2_{a,b}[0,T]$ rather than the inner product $(\cdot,\cdot)_{C'_{a,b}}$ on $C'_{a,b}[0,T]$ to study the PWZ stochastic integral and the related integration formula on the function space $C_{a,b}[0,T]$.

The generalized analytic Feynman integral on the function space $C_{a,b}[0,T]$ was first defined and studied in [5, 10], and the study of this integral has continued in [3, 4, 8, 12]. We assume familiarity with [4, 5, 8, 10, 12] and adopt the concepts and the definitions of the generalized analytic Feynman integral on $C_{a,b}[0,T]$.

Based on the references [19–21], we present several concepts which involve the scale-invariant measurability to define a generalized analytic Feynman integral of functionals on the product space $C^2_{a,b}[0,T]$.

Let $(C^2_{a,b}[0,T], \mathcal{W}(C^2_{a,b}[0,T]), \mu^2)$ be the product function space, where

$$C^2_{a,b}[0,T] = C_{a,b}[0,T] \times C_{a,b}[0,T],$$

$\mathcal{W}(C^2_{a,b}[0,T]) \equiv \mathcal{W}(C_{a,b}[0,T]) \otimes \mathcal{W}(C_{a,b}[0,T])$ denotes the σ-field generated by measurable rectangles $A \times B$ with $A, B \in \mathcal{W}(C_{a,b}[0,T])$, and $\mu^2 \equiv \mu \times \mu$. A subset B of $C^2_{a,b}[0,T]$ is said to be scale-invariant measurable provided $\{(\rho_1 x_1, \rho_2 x_2) : (x_1, x_2) \in B\}$ is $\mathcal{W}(C^2_{a,b}[0,T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$, and a scale-invariant measurable subset N of $C^2_{a,b}[0,T]$ is said to be scale-invariant null provided $(\mu \times \mu)(\{(\rho_1 x_1, \rho_2 x_2) : (x_1, x_2) \in N\}) = 0$ for every $\rho_1 > 0$ and $\rho_2 > 0$. A property that holds except on a scale-invariant null set is said to hold s-a.e. on $C^2_{a,b}[0,T]$. A functional F on $C^2_{a,b}[0,T]$ is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho_1\,\cdot\,, \rho_2\,\cdot\,)$ is $\mathcal{W}(C^2_{a,b}[0,T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$. If two functionals F and G defined on $C^2_{a,b}[0,T]$ are equal s-a.e., then we write $F \approx G$.

We denote the product function space integral of a $\mathcal{W}(C^2_{a,b}[0,T])$-measurable functional F by

$$E[F] \equiv E_{\vec{x}}[F(x_1, x_2)] = \int_{C^2_{a,b}[0,T]} F(x_1, x_2)\, d(\mu \times \mu)(x_1, x_2)$$

whenever the integral exists.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the complex numbers, the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part, respectively. Furthermore, for each $\lambda \in \widetilde{\mathbb{C}}_+$, $\lambda^{1/2}$ denotes the principal square root of λ; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$. We also assume that every functional F on $C^2_{a,b}[0,T]$ we consider is s-a.e. defined and is scale-invariant measurable.

The following definition is due to Choi, Skoug and Chang [9, 13].

Definition 2.1. Let $\mathbb{C}^2_+ = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \mathrm{Re}(\lambda_j) > 0 \text{ for } j = 1, 2\}$ and let $\widetilde{\mathbb{C}}^2_+ = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_j \neq 0 \text{ and } \mathrm{Re}(\lambda_j) \geq 0 \text{ for } j = 1, 2\}$. Let $F : C^2_{a,b}[0,T] \to \mathbb{C}$ be a scale-invariant measurable functional such that the function space integral

$$J(\lambda_1, \lambda_2) = \int_{C^2_{a,b}[0,T]} F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2)\, d(\mu \times \mu)(x_1, x_2)$$

exists and is finite for each $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists a function $J^*(\lambda_1, \lambda_2)$ analytic on $\mathbb{C}_+^2$ such that $J^*(\lambda_1, \lambda_2) = J(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J^*(\lambda_1, \lambda_2)$ is defined to be the analytic function space integral of F over $C_{a,b}^2[0, T]$ with parameter $\vec{\lambda} = (\lambda_1, \lambda_2)$, and for $\vec{\lambda} \in \mathbb{C}_+^2$ we write

$$\begin{aligned} E^{\mathrm{an}_{\vec{\lambda}}}[F] &\equiv E_{\vec{x}}^{\mathrm{an}_{\vec{\lambda}}}[F(x_1, x_2)] \\ &\equiv E_{x_1,x_2}^{\mathrm{an}_{(\lambda_1,\lambda_2)}}[F(x_1, x_2)] = J^*(\lambda_1, \lambda_2). \end{aligned}$$

Let q_1 and q_2 be nonzero real numbers. Let F be a functional such that $E^{\mathrm{an}_{\vec{\lambda}}}[F]$ exists for all $\vec{\lambda} \in \mathbb{C}_+^2$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter $\vec{q} = (q_1, q_2)$ and we write

$$\begin{aligned} E^{\mathrm{anf}_{\vec{q}}}[F] &\equiv E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}[F(x_1, x_2)] \\ &\equiv E_{x_1,x_2}^{\mathrm{anf}_{(q_1,q_2)}}[F(x_1, x_2)] = \lim_{\substack{\vec{\lambda} \to -i\vec{q} \\ \vec{\lambda} \in \mathbb{C}_+^2}} E^{\mathrm{an}_{\vec{\lambda}}}[F]. \end{aligned}$$

3. A Cameron–Storvick theorem on $C_{a,b}^2[0, T]$

In [1], Cameron (also see [2, Theorem A]) expressed the Wiener integral of the first variation of a functional F on the Wiener space $C_0[0, T]$ in terms of the Wiener integral of the product of F by a linear functional, and in [2, Theorem 1], Cameron and Storvick obtained a similar result for the analytic Feynman integral on $C_0[0, T]$. In [11, Theorem 2.4], Chang, Song and Yoo also obtained a Cameron–Storvick theorem on abstract Wiener spaces. In [10], Chang and Skoug developed these results for functionals on the function space $C_{a,b}[0, T]$.

In this section, we establish a Cameron–Storvick theorem for the generalized analytic Feynman integral of functionals on the product function space $C_{a,b}^2[0, T]$. To do this we first give the definition of the first variation of a functional F on $C_{a,b}^2[0, T]$.

Definition 3.1. Let F be a functional on $C_{a,b}^2[0, T]$ and let g_1 and g_2 be functions in $C_{a,b}[0, T]$. Then

$$\begin{aligned} \delta F(x_1, x_2|g_1, g_2) &= \frac{\partial}{\partial h}\Big(F(x_1 + hg_1, x_2) + F(x_1, x_2 + hg_2)\Big)\Big|_{h=0} \\ &= \frac{\partial}{\partial h} F(x_1 + hg_1, x_2)\Big|_{h=0} + \frac{\partial}{\partial h} F(x_1, x_2 + hg_2)\Big|_{h=0} \end{aligned} \tag{3.1}$$

(if it exists) is called the first variation of F in the direction of (g_1, g_2).

Throughout this section, when working with $\delta F(x_1, x_2|g_1, g_2)$, we will always require g_1 and g_2 to be functions in $C'_{a,b}[0, T]$.

We first quote the translation theorem [10] for the function space integral using our notations.

Lemma 3.2 (Translation Theorem)**.** *Let* $F \in L^1(C_{a,b}[0,T])$ *and let* $w_0 \in C'_{a,b}[0,T]$*. Then*

$$\begin{aligned}(3.2)\qquad &\int_{C_{a,b}[0,T]} F(x+w_0)d\mu(x)\\ &= \exp\bigg\{-\frac{1}{2}\|w_0\|^2_{C'_{a,b}} - (w_0,a)_{C'_{a,b}}\bigg\}\int_{C_{a,b}[0,T]} F(x)\exp\{(w_0,x)^\sim\}d\mu(x).\end{aligned}$$

Theorem 3.3 (An integration by parts formula)**.** *Let* g_1 *and* g_2 *be nonzero functions in* $C'_{a,b}[0,T]$*. Let* $F(x_1,x_2)$ *be* $\mu\times\mu$*-integrable over* $C^2_{a,b}[0,T]$*. Assume that* F *has a first variation* $\delta F(x_1,x_2|g_1,g_2)$ *for all* $(x_1,x_2)\in C^2_{a,b}[0,T]$ *such that for some* $\gamma>0$*,*

$$\sup_{|h|\le\gamma}\big|\delta F(x_1+hg_1,x_2+hg_2|g_1,g_2)\big|$$

is $\mu\times\mu$*-integrable over* $C^2_{a,b}[0,T]$ *as a functional of* $(x_1,x_2)\in C^2_{a,b}[0,T]$*. Then*

$$\begin{aligned}(3.3)\qquad E_{\vec{x}}[\delta F(x_1,x_2|g_1,g_2)] &= E_{\vec{x}}\big[F(x_1,x_2)\big\{(g_1,x_1)^\sim+(g_2,x_2)^\sim\big\}\big]\\ &\quad -\big\{(g_1,a)_{C'_{a,b}}+(g_2,a)_{C'_{a,b}}\big\}E_{\vec{x}}[F(x_1,x_2)].\end{aligned}$$

Proof. First note that

$$\begin{aligned}&\delta F(x_1+hg_1,x_2+hg_2|g_1,g_2)\\ &= \frac{\partial}{\partial\lambda}F(x_1+hg_1+\lambda g_1,x_2+hg_2)\bigg|_{\lambda=0} + \frac{\partial}{\partial\lambda}F(x_1+hg_1,x_2+hg_2+\lambda g_2)\bigg|_{\lambda=0}\\ &= \frac{\partial}{\partial\lambda}F(x_1+(h+\lambda)g_1,x_2+hg_2)\bigg|_{\lambda=0} + \frac{\partial}{\partial\lambda}F(x_1+hg_1,x_2+(h+\lambda)g_2)\bigg|_{\lambda=0}\\ &= \frac{\partial}{\partial\mu}F(x_1+\mu g_1,x_2+hg_2)\bigg|_{\mu=h} + \frac{\partial}{\partial\mu}F(x_1+hg_1,x_2+\mu g_2)\bigg|_{\mu=h}\\ &= 2\frac{\partial}{\partial h}F(x_1+hg_1,x_2+hg_2).\end{aligned}$$

But since

$$\sup_{|h|\le\gamma}\bigg|\frac{\partial}{\partial h}F(x_1+hg_1,x_2+hg_2)\bigg|$$

is $\mu\times\mu$-integrable,

$$\frac{\partial}{\partial h}F(x_1+hg_1,x_2+hg_2)$$

is $\mu\times\mu$-integrable for sufficiently small values of h. Hence by the Fubini theorem and the equation (3.2), it follows that

$$\begin{aligned}&E_{\vec{x}}[\delta F(x_1,x_2|g_1,g_2)]\\ &= E_{\vec{x}}\bigg[\frac{\partial}{\partial h}\Big(F(x_1+hg_1,x_2)+F(x_1,x_2+hg_2)\Big)\bigg|_{h=0}\bigg]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial h} E_{\vec{x}}\big[F(x_1+hg_1,x_2)+F(x_1,x_2+hg_2)\big]\Big|_{h=0}\\
&= \frac{\partial}{\partial h} E_{\vec{x}}\big[F(x_1+hg_1,x_2)\big]\Big|_{h=0} + \frac{\partial}{\partial h} E_{\vec{x}}\big[F(x_1,x_2+hg_2)\big]\Big|_{h=0}\\
&= \frac{\partial}{\partial h} E_{x_2}\big[E_{x_1}\big[F(x_1+hg_1,x_2)\big]\big]\Big|_{h=0} + \frac{\partial}{\partial h} E_{x_1}\big[E_{x_2}\big[F(x_1,x_2+hg_2)\big]\big]\Big|_{h=0}\\
&= \frac{\partial}{\partial h}\bigg(\exp\bigg\{-\frac{h^2}{2}\|g_1\|^2_{C'_{a,b}} - h(g_1,a)_{C'_{a,b}}\bigg\}\\
&\qquad\qquad \times E_{x_2}\big[E_{x_1}\big[F(x_1,x_2)\exp\{h(g_1,x_1)^\sim\}\big]\big]\bigg)\bigg|_{h=0}\\
&\quad + \frac{\partial}{\partial h}\bigg(\exp\bigg\{-\frac{h^2}{2}\|g_2\|^2_{C'_{a,b}} - h(g_2,a)_{C'_{a,b}}\bigg\}\\
&\qquad\qquad \times E_{x_1}\big[E_{x_2}\big[F(x_1,x_2)\exp\{h(g_2,x_2)^\sim\}\big]\big]\bigg)\bigg|_{h=0}\\
&= E_{\vec{x}}\big[F(x_1,x_2)\big\{(g_1,x_1)^\sim + (g_2,x_2)^\sim\big\}\big]\\
&\quad - \big\{(g_1,a)_{C'_{a,b}} + (g_2,a)_{C'_{a,b}}\big\}E_{\vec{x}}\big[F(x_1,x_2)\big]
\end{aligned}
$$

as desired. □

Lemma 3.4. *Let g_1, g_2, and F be as in Theorem 3.3. For each $\rho_1 > 0$ and $\rho_2 > 0$, assume that $F(\rho_1 x_1, \rho_2 x_2)$ is $\mu\times\mu$-integrable. Furthermore assume that $F(\rho_1 x_1, \rho_2 x_2)$ has a first variation $\delta F(\rho_1 x_1, \rho_2 x_2|\rho_1 g_1, \rho_2 g_2)$ for all $(x_1, x_2) \in C^2_{a,b}[0,T]$ such that for some positive function $\gamma(\rho_1,\rho_2)$,*

$$\sup_{|h|\le\gamma(\rho_1,\rho_2)} \big|\delta F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2 + \rho_2 h g_2|\rho_1 g_1, \rho_2 g_2)\big|$$

is $\mu\times\mu$-integrable over $C^2_{a,b}[0,T]$ as a functional of $(x_1,x_2) \in C^2_{a,b}[0,T]$. Then

$$
\begin{aligned}
&E_{\vec{x}}\big[\delta F(\rho_1 x_1, \rho_2 x_2|\rho_1 g_1, \rho_2 g_2)\big]\\
&= E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big\{(g_1,x_1)^\sim + (g_2,x_2)^\sim\big\}\big]\\
&\quad - \big\{(g_1,a)_{C'_{a,b}} + (g_2,a)_{C'_{a,b}}\big\}E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big].
\end{aligned}
\tag{3.4}
$$

Proof. Given a pair (ρ_1,ρ_2) with $\rho_1 > 0$ and $\rho_2 > 0$, let $R_{(\rho_1,\rho_2)}(x_1,x_2) = F(\rho_1 x_1, \rho_2 x_2)$. Then we have that

$$R_{(\rho_1,\rho_2)}(x_1+hg_1,x_2) = F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2)$$

and

$$R_{(\rho_1,\rho_2)}(x_1,x_2+hg_2) = F(\rho_1 x_1, \rho_2 x_2 + \rho_2 h g_2)$$

and that

$$\frac{\partial}{\partial h} R_{(\rho_1,\rho_2)}(x_1+hg_1,x_2)\Big|_{h=0} = \frac{\partial}{\partial h} F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2)\Big|_{h=0}$$

and

$$\frac{\partial}{\partial h} R_{(\rho_1,\rho_2)}(x_1, x_2 + hg_2)\bigg|_{h=0} = \frac{\partial}{\partial h} F(\rho_1 x_1, \rho_2 x_2 + \rho_2 h g_2)\bigg|_{h=0}.$$

Thus we have

$$\begin{aligned}
&\delta F(\rho_1 x_1, \rho_2 x_2 | \rho_1 g_1, \rho_2 g_2) \\
&= \frac{\partial}{\partial h} F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2)\bigg|_{h=0} + \frac{\partial}{\partial h} F(\rho_1 x_1, \rho_2 x_2 + \rho_2 h g_2)\bigg|_{h=0} \\
&= \frac{\partial}{\partial h} R_{(\rho_1,\rho_2)}(x_1 + h w_1, x_2)\bigg|_{h=0} + \frac{\partial}{\partial h} R_{(\rho_1,\rho_2)}(x_1, x_2 + h g_2)\bigg|_{h=0} \\
&= \delta R_{(\rho_1,\rho_2)}(x_1, x_2 | g_1, g_2).
\end{aligned}$$

Hence by the equation (3.3) with F replaced with $R_{(\rho_1,\rho_2)}$, we have

$$\begin{aligned}
&E_{\vec{x}}\big[\delta F(\rho_1 x_1, \rho_2 x_2 | \rho_1 g_1, \rho_2 g_2)\big] \\
&= E_{\vec{x}}\big[\delta R(x_1, x_2 | g_1, g_2)\big] \\
&= E_{\vec{x}}\big[R(x_1, x_2)\big\{(g_1, x_1)^{\sim} + (g_2, x_2)^{\sim}\big\}\big] \\
&\quad - \big\{(g_1, a)_{C'_{a,b}} + (g_2, a)_{C'_{a,b}}\big\} E_{\vec{x}}\big[R(x_1, x_2)\big] \\
&= E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big\{(g_1, x_1)^{\sim} + (g_2, x_2)^{\sim}\big\}\big] \\
&\quad - \big\{(g_1, a)_{C'_{a,b}} + (g_2, a)_{C'_{a,b}}\big\} E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big]
\end{aligned}$$

which establishes the equation (3.4). □

Theorem 3.5. *Let g_1, g_2, and F be as in Lemma 3.4. Then if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds*:

$$\begin{aligned}
(3.5)\quad & E^{\mathrm{anf}_{(q_1,q_2)}}_{\vec{x}}\big[\delta F(x_1, x_2 | g_1, g_2)\big] \\
&= -i E^{\mathrm{anf}_{(q_1,q_2)}}_{\vec{x}}\big[F(x_1, x_2)\big\{q_1 (g_1, x_1)^{\sim} + q_2 (g_2, x_2)^{\sim}\big\}\big] \\
&\quad - \Big\{(-iq_1)^{1/2}(g_1, a)_{C'_{a,b}} + (-iq_2)^{1/2}(g_2, a)_{C'_{a,b}}\Big\} E^{\mathrm{anf}_{(q_1,q_2)}}_{\vec{x}}\big[F(x_1, x_2)\big].
\end{aligned}$$

Proof. Let $\rho_1 > 0$ and $\rho_2 > 0$ be given. Let $y_1 = \rho_1^{-1} g_1$ and $y_2 = \rho_2^{-1} g_2$. By the equation (3.4),

$$\begin{aligned}
&E_{\vec{x}}\big[\delta F(\rho_1 x_1, \rho_2 x_2 | g_1, g_2)\big] \\
&= E_{\vec{x}}\big[\delta F(\rho_1 x_1, \rho_2 x_2 | \rho_1 y_1, \rho_2 y_2)\big] \\
&= E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big\{(y_1, x_1)^{\sim} + (y_2, x_2)^{\sim}\big\}\big] \\
&\quad - \big\{(y_1, a)_{C'_{a,b}} + (y_2, a)_{C'_{a,b}}\big\} E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big] \\
&= E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big\{\rho_1^{-2}(g_1, \rho_1 x_1)^{\sim} + \rho_2^{-2}(g_2, \rho_2 x_2)^{\sim}\big\}\big] \\
&\quad - \big\{\rho_1^{-1}(g_1, a)_{C'_{a,b}} + \rho_2^{-1}(g_2, a)_{C'_{a,b}}\big\} E_{\vec{x}}\big[F(\rho_1 x_1, \rho_2 x_2)\big].
\end{aligned} \tag{3.6}$$

Now let $\rho_1 = \lambda_1^{-1/2}$ and $\rho_2 = \lambda_2^{-1/2}$. Then the equation (3.6) becomes

$$\begin{aligned} &E_{\vec{x}}\big[\delta F(\lambda_1^{-1/2}x_1, \lambda_2^{-1/2}x_2|g_1,g_2)\big] \\ (3.7) \quad &= E_{\vec{x}}\big[F(\lambda_1^{-1/2}x_1, \lambda_2^{-1/2}x_2)\ \big\{\lambda_1(g_1, \lambda_1^{-1/2}x_1)^{\sim} + \lambda_2(g_2, \lambda_2^{-1/2}x_2)^{\sim}\big\}\big] \\ &\quad - \big\{\lambda_1^{1/2}(g_1,a)_{C'_{a,b}} + \lambda_2^{1/2}(g_2,a)_{C'_{a,b}}\big\}E_{\vec{x}}\big[F(\lambda_1^{-1/2}x_1, \lambda_1^{-1/2}x_2)\big]. \end{aligned}$$

Since $\rho_1 > 0$ and $\rho_2 > 0$ were arbitrary, we have that the equation (3.7) holds for all $\lambda_1 > 0$ and $\lambda_2 > 0$. We now use Definition 2.1 to obtain our desired conclusion. □

Corollary 3.6. *Let H be a μ-integrable functional on $C_{a,b}[0,T]$. Let g be a function in $C'_{a,b}[0,T]$. Then if any two of the three generalized analytic Feynman integrals on $C_{a,b}[0,T]$ in the following equation exist, then the third one also exists, and equality holds*:

$$E_x^{\mathrm{anf}_q}\big[\delta H(x|g)\big] = -iqE_x^{\mathrm{anf}_q}\big[H(x)(g,x)^{\sim}\big] - (-iq)^{1/2}(g,a)_{C'_{a,b}}E_x^{\mathrm{anf}_q}\big[H(x)\big],$$

where $E_x^{\mathrm{anf}_q}[H(x)] = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} H(x)d\mu(x)$ means the generalized analytic Feynman integral of functionals H on $C_{a,b}[0,T]$, see [3–5, 8, 10, 12].

Proof. Simply choose $F(x_1,x_2) = H(x_1)$. □

4. Functionals in the generalized Fresnel type class $\mathcal{F}_{A_1,A_2}^{a,b}$

In this section we introduce the generalized Fresnel type class $\mathcal{F}_{A_1,A_2}^{a,b}$ to which we apply our Cameron–Storvick theorem.

Let $\mathcal{M}(C'_{a,b}[0,T])$ be the space of complex-valued, countably additive (and hence finite) Borel measures on $C'_{a,b}[0,T]$. The space $\mathcal{M}(C'_{a,b}[0,T])$ is a Banach algebra under the total variation norm and with convolution as multiplication, see [14, 26].

Definition 4.1. Let A_1 and A_2 be bounded, nonnegative self-adjoint operators on $C'_{a,b}[0,T]$. The generalized Fresnel type class $\mathcal{F}_{A_1,A_2}^{a,b}$ of functionals on $C^2_{a,b}[0,T]$ is defined as the space of all functionals F on $C^2_{a,b}[0,T]$ of the form

$$F(x_1,x_2) = \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}w, x_j)^{\sim}\Big\}df(w) \tag{4.1}$$

for s-a.e. $(x_1,x_2) \in C^2_{a,b}[0,T]$, where f is in $\mathcal{M}(C'_{a,b}[0,T])$. More precisely, since we identify functionals which coincide s-a.e. on $C^2_{a,b}[0,T]$, $\mathcal{F}_{A_1,A_2}^{a,b}$ can be regarded as the space of all s-equivalence classes of functionals of the form (4.1). For more details, see [9, 13].

Remark 4.2. (1) Note that in case $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. In this case, the generalized Fresnel type class $\mathcal{F}^{\,a,b}_{A_1,A_2}$ reduces to the Kallianpur and Bromley Fresnel class $\mathcal{F}_{A_1,A_2}$, see [21].

(2) In addition, if we choose $A_1 = I$ (identity operator) and $A_2 = 0$ (zero operator), then the class $\mathcal{F}^{\,a,b}_{A_1,A_2}$ reduces to the Fresnel class $\mathcal{F}(C_0[0,T])$. It is known, see [18], that $\mathcal{F}(C_0[0,T])$ forms a Banach algebra over the complex field.

(3) The map $f \mapsto F$ defined by (4.1) sets up an algebra isomorphism between $\mathcal{M}(C'_{a,b}[0,T])$ and $\mathcal{F}^{\,a,b}_{A_1,A_2}$ if $\mathrm{Ran}(A_1 + A_2)$ is dense in $C'_{a,b}[0,T]$ where Ran indicates the range of an operator. In this case, $\mathcal{F}^{\,a,b}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\| = \|f\|$. For more details, see [21].

As discussed in [13, Remark 7], for a functional F in $\mathcal{F}^{\,a,b}_{A_1,A_2}$ and a vector $\vec{q} = (q_1, q_2)$ with $q_1 \neq 0$ and $q_2 \neq 0$, the generalized analytic Feynman integral $E^{\mathrm{anf}_{\vec{q}}}[F]$ might not exist. By a simple modification of the example illustrated in [3], we can construct an example for the functional whose generalized analytic Feynman integral $E^{\mathrm{anf}_{\vec{q}}}[F]$ does not exist. Thus we need to impose additional restrictions on the functionals F in $\mathcal{F}^{\,a,b}_{A_1,A_2}$.

Given a positive real number $q_0 > 0$, and bounded, nonnegative self-adjoint operators A_1 and A_2 on $C'_{a,b}[0,T]$, let

$$
\begin{aligned}
k(q_0;\vec{A};w) &\equiv k(q_0;A_1,A_2;w) \\
&= \exp\left\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}\|_o\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\right\},
\end{aligned}
\tag{4.2}
$$

where $\|A_j^{1/2}\|_o$ means the operator norm of $A_j^{1/2}$ for $j \in \{1,2\}$. For the existence of the generalized analytic Feynman integral of F, we define a subclass $\mathcal{F}^{\,q_0}_{A_1,A_2}$ of $\mathcal{F}^{\,a,b}_{A_1,A_2}$ by $F \in \mathcal{F}^{\,q_0}_{A_1,A_2}$ if and only if

$$
\int_{C'_{a,b}[0,T]} k(q_0;\vec{A};w)\,d|f|(w) < +\infty,
$$

where f and F are related by the equation (4.1) and k is given by the equation (4.2).

The following theorem is due to Choi, Skoug and Chang [13].

Theorem 4.3. *Let q_0 be a positive real number and let F be an element of $\mathcal{F}^{\,q_0}_{A_1,A_2}$. Then for all real numbers q_1 and q_2 with $|q_j| > q_0$, $j \in \{1,2\}$, the generalized analytic Feynman integral $E^{\mathrm{anf}_{\vec{q}}}[F]$ of F exists and is given by the formula*

$$
E^{\mathrm{anf}_{\vec{q}}}[F] = \int_{C'_{a,b}[0,T]} \psi(-i\vec{q};\vec{A};w)\,df(w),
\tag{4.3}
$$

where $\psi(-i\vec{q};\vec{A};w)$ *is given by*

$$\psi(-i\vec{q};\vec{A};w) = \exp\bigg\{\sum_{j=1}^{2}\Big[-\frac{i(A_jw,w)_{C'_{a,b}}}{2q_j} + i(-iq_j)^{-1/2}(A_j^{1/2}w,a)_{C'_{a,b}}\Big]\bigg\}. \tag{4.4}$$

For $j\in\{1,2\}$, let $g_j\in C'_{a,b}[0,T]$ and let F be an element of $\mathcal{F}^{a,b}_{A_1,A_2}$ whose associated measure f, see the equation (4.1), satisfies the inequality

$$\int_{C'_{a,b}[0,T]}\|w\|_{C'_{a,b}}d|f|(w) < +\infty. \tag{4.5}$$

Then using the equation (3.1), we obtain that

$$\begin{aligned} &\delta F(x_1,x_2|g_1,g_2)\\ &= \sum_{k=1}^{2}\bigg[\frac{\partial}{\partial h}\bigg(\int_{C'_{a,b}[0,T]}\exp\bigg\{\sum_{j=1}^{2}i(A_j^{1/2}w,x_j)^{\sim}\\ &\qquad\qquad + ih(A_k^{1/2}w,g_k)^{\sim}\bigg\}df(w)\bigg)\bigg|_{h=0}\bigg]\\ &= \int_{C'_{a,b}[0,T]}\Big[\sum_{k=1}^{2}i(A_k^{1/2}w,g_k)_{C'_{a,b}}\Big]\exp\bigg\{\sum_{j=1}^{2}i(A_j^{1/2}w,x_j)^{\sim}\bigg\}df(w)\\ &= \int_{C'_{a,b}[0,T]}\exp\bigg\{\sum_{j=1}^{2}i(A_j^{1/2}w,x_j)^{\sim}\bigg\}d\sigma^{\vec{A},\vec{g}}(w), \end{aligned} \tag{4.6}$$

where the complex measure $\sigma^{\vec{A},\vec{g}}$ is defined by

$$\sigma^{\vec{A},\vec{g}}(B) = \int_B\Big[\sum_{k=1}^{2}i(A_k^{1/2}w,g_k)_{C'_{a,b}}\Big]df(w), \quad B\in\mathcal{B}(C'_{a,b}[0,T]).$$

The second equality of (4.6) follows from (4.5) and Theorem 2.27 in [17]. Also, $\delta F(x_1,x_2|g_1,g_2)$ is an element of $\mathcal{F}^{a,b}_{A_1,A_2}$ as a functional of (x_1,x_2), since by the Cauchy–Schwartz inequality and (4.5),

$$\begin{aligned} \|\sigma^{\vec{A},\vec{g}}\| &\le \int_{C'_{a,b}[0,T]}\sum_{j=1}^{2}|i(A_j^{1/2}w,g_j)_{C'_{a,b}}|d|f|(w)\\ &\le \int_{C'_{a,b}[0,T]}\sum_{j=1}^{2}\|A_j^{1/2}\|_o\|w\|_{C'_{a,b}}\|g_j\|_{C'_{a,b}}d|f|(w)\\ &\le \bigg(\sum_{j=1}^{2}\|A_j^{1/2}\|_o\|g_j\|_{C'_{a,b}}\bigg)\int_{C'_{a,b}[0,T]}\|w\|_{C'_{a,b}}d|f|(w) < +\infty, \end{aligned}$$

where $\|A_j^{1/2}\|_o$ is the operator norm of $A_j^{1/2}$.

For the existence of the generalized analytic Feynman integral of the first variation δF of a functional F in $\mathcal{F}^{\,a,b}_{A_1,A_2}$, we also define a subclass of $\mathcal{F}^{\,a,b}_{A_1,A_2}$ as follows: given a positive real number q_0, we define a subclass $\mathcal{G}^{\,q_0}_{A_1,A_2}$ of $\mathcal{F}^{\,a,b}_{A_1,A_2}$ by $F\in\mathcal{G}^{\,q_0}_{A_1,A_2}$ if and only if

$$\int_{C'_{a,b}[0,T]}\|w\|_{C'_{a,b}}k(q_0;\vec{A};w)d|f|(w)<+\infty,$$

where f is the associated measure of F by the equation (4.1) and $k(q_0;\vec{A};w)$ is given by the equation (4.2).

Our next theorem follows quite readily from the techniques developed in the proof of Theorem 9 of [13].

Theorem 4.4. *Let q_0 be a positive real number and let g_1 and g_2 be functions in $C'_{a,b}[0,T]$. Let F be an element of $\mathcal{G}^{\,q_0}_{A_1,A_2}$. Then for all real numbers q_1 and q_2 with $|q_j|>q_0$, $j\in\{1,2\}$, the generalized analytic Feynman integral of $\delta F(\cdot,\cdot|g_1,g_2)$ exists and is given by the formula*

$$E^{\mathrm{anf}_{\vec{q}}}_{\vec{x}}[\delta F(x_1,x_2|g_1,g_2)]=\int_{C'_{a,b}[0,T]}\Big[\sum_{j=1}^{2}i(A_j^{1/2}w,g_j)_{C'_{a,b}}\Big]df^{\vec{A}}_{\vec{q}}(w), \tag{4.7}$$

where $f^{\vec{A}}_{\vec{q}}$ is a complex measure on $\mathcal{B}(C'_{a,b}[0,T])$, the Borel σ-algebra of $C'_{a,b}[0,T]$, given by

$$f^{\vec{A}}_{\vec{q}}(B)=\int_B\psi(-i\vec{q};\vec{A};w)df(w),\qquad B\in\mathcal{B}(C'_{a,b}[0,T]) \tag{4.8}$$

and where $\psi(-i\vec{q};\vec{A};w)$ is given by (4.4).

5. Applications of the Cameron–Storvick theorem

Let A be a bounded self-adjoint operator on $C'_{a,b}[0,T]$. Then we can write

$$A=A_+-A_-, \tag{5.1}$$

where A_+ and A_- are each bounded, nonnegative and self-adjoint. Take $A_1=A_+$ and $A_2=A_-$ in the definition of $\mathcal{F}^{\,a,b}_{A_1,A_2}$ above. In this section we consider functionals in the generalized Fresnel type class $\mathcal{F}^{\,a,b}_{A}\equiv\mathcal{F}^{\,a,b}_{A_+,A_-}$ where A, A_+ and A_- are related by the equation (5.1) above.

Let $C^*_{a,b}[0,T]$ be the set of functions k in $C'_{a,b}[0,T]$ such that Dk is continuous except for a finite number of finite jump discontinuities and is of bounded variation on $[0,T]$. For any $w\in C'_{a,b}[0,T]$ and $k\in C^*_{a,b}[0,T]$, let the operation $\odot$ between $C'_{a,b}[0,T]$ and $C^*_{a,b}[0,T]$ be defined by

$$w\odot k=D^{-1}(DwDk) \tag{5.2}$$

so that $D(w\odot k)=DwDk$, where $DwDk$ denotes the pointwise multiplication of the functions Dw and Dk. In the equation (5.2), the operator D^{-1} is given by (2.2) above. Then $(C^*_{a,b}[0,T],\odot)$ is a commutative algebra with the identity b. Also we can observe that for any $w,w_1,w_2\in C'_{a,b}[0,T]$ and $k\in C^*_{a,b}[0,T]$,

$$w\odot k=k\odot w \tag{5.3}$$

and

$$(w_1,w_2\odot k)_{C'_{a,b}}=(w_1\odot k,w_2)_{C'_{a,b}}. \tag{5.4}$$

For a more detailed study of the class $C^*_{a,b}[0,T]$, see [4].

Next let ϑ be a function in $C^*_{a,b}[0,T]$ with $D\vartheta=\theta$. Define an operator $A:C'_{a,b}[0,T]\to C'_{a,b}[0,T]$ by

$$\begin{aligned}Aw(t)=(\vartheta\odot w)(t)&=\int_0^t D\vartheta(s)Dw(s)db(s)\\&=\int_0^t\theta(s)\frac{w'(s)}{b'(s)}db(s)=\int_0^t\theta(s)dw(s).\end{aligned}\tag{5.5}$$

It is easily shown that A is a self-adjoint operator. We also see that $A=A_+-A_-$ where

$$A_+w(t)=\int_0^t\theta^+(s)Dw(s)db(s)=\int_0^t\theta^+(s)dw(s)$$

and

$$A_-w(t)=\int_0^t\theta^-(s)Dw(s)db(s)=\int_0^t\theta^-(s)dw(s)$$

and where θ^+ and θ^- are the positive part and the negative part of θ, respectively. Also, $A_+^{1/2}$ and $A_-^{1/2}$ are given by

$$A_+^{1/2}w(t)=\int_0^t\sqrt{\theta^+}(s)dw(s)\quad\text{and}\quad A_-^{1/2}w(t)=\int_0^t\sqrt{\theta^-}(s)dw(s),$$

respectively. For a more detailed study of this decomposition, see [19, pp. 187–189]. For notational convenience, let $\vartheta_+^{1/2}=D^{-1}\sqrt{\theta^+}$ and let $\vartheta_-^{1/2}=D^{-1}\sqrt{\theta^-}$. Then it follows that

$$A_+^{1/2}w(t)=(\vartheta_+^{1/2}\odot w)(t)\quad\text{and}\quad A_-^{1/2}w(t)=(\vartheta_-^{1/2}\odot w)(t),\tag{5.6}$$

respectively.

For fixed $g\in C'_{a,b}[0,T]$, let $g_1=A_+^{1/2}g$ and $g_2=A_-^{1/2}(-g)$. Then we see that for any functions w in $C'_{a,b}[0,T]$,

$$\begin{aligned}&(A_+^{1/2}w,g_1)_{C'_{a,b}}+(A_-^{1/2}w,g_2)_{C'_{a,b}}\\&=(A_+^{1/2}w,A_+^{1/2}g)_{C'_{a,b}}-(A_-^{1/2}w,A_-^{1/2}g)_{C'_{a,b}}\\&=(A_+w,g)_{C'_{a,b}}-(A_-w,g)_{C'_{a,b}}\\&=(Aw,g)_{C'_{a,b}}.\end{aligned}\tag{5.7}$$

Using the equations (5.6), (5.3) and (5.4), we also see that

$$(A_+^{1/2}w, a)_{C'_{a,b}} = (\vartheta_+^{1/2}\odot w, a)_{C'_{a,b}} = (w\odot\vartheta_+^{1/2}, a)_{C'_{a,b}} = (w, \vartheta_+^{1/2}\odot a)_{C'_{a,b}} \tag{5.8}$$

and

$$(A_-^{1/2}w, a)_{C'_{a,b}} = (w, \vartheta_-^{1/2}\odot a)_{C'_{a,b}}. \tag{5.9}$$

Assume that F is an element of $\mathcal{F}_A^{q_0}\cap\mathcal{G}_A^{q_0}$ for some $q_0\in(0,1)$ where $\mathcal{F}_A^{q_0}\equiv\mathcal{F}_{A_+,A_-}^{q_0}$ and $\mathcal{G}_A^{q_0}\equiv\mathcal{G}_{A_+,A_-}^{q_0}$ (the classes $\mathcal{F}_{A_1,A_2}^{q_0}$ and $\mathcal{G}_{A_1,A_2}^{q_0}$ are defined in Section 4, respectively).

Applying (4.7) with $(q_1,q_2)=(1,-1)$ and $(g_1,g_2)=(A_+^{1/2}g,-A_-^{1/2}g)$, respectively, (4.8), and (4.4) with $(A_1,A_2)=(A_+,A_-)$, and using (5.7), we obtain that

$$\begin{aligned}
&E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[\delta F(x_1,x_2|A_+^{1/2}g,-A_-^{1/2}g)\big]\\
&= E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[\delta F(x_1,x_2|g_1,g_2)\big]\\
&= \int_{C'_{a,b}[0,T]}\Big[i(A_+^{1/2}w,g_1)_{C'_{a,b}} + i(A_-^{1/2}w,g_2)_{C'_{a,b}}\Big]\\
&\qquad\times\exp\bigg\{-\frac{i}{2}((A_+-A_-)w,w)_{C'_{a,b}}\bigg\}\\
&\qquad\times\exp\bigg\{i\Big[(-i)^{-1/2}(A_+^{1/2}w,a)_{C'_{a,b}}+(i)^{-1/2}(A_-^{1/2}w,a)_{C'_{a,b}}\Big]\bigg\}df(w)\\
&= \int_{C'_{a,b}[0,T]} i(Aw,g)_{C'_{a,b}}\exp\bigg\{-\frac{i}{2}(Aw,w)_{C'_{a,b}}\bigg\}\\
&\qquad\times\exp\bigg\{i\Big[(-i)^{-1/2}(A_+^{1/2}w,a)_{C'_{a,b}}+(i)^{-1/2}(A_-^{1/2}w,a)_{C'_{a,b}}\Big]\bigg\}df(w).
\end{aligned} \tag{5.10}$$

We also see that for all $\rho_1>0$, $\rho_2>0$ and $h\in\mathbb{R}$

$$\begin{aligned}
&\big|\delta F(\rho_1x_1+\rho_1hg_1,\rho_2x_2+\rho_2hg_2|\rho_1g_1,\rho_2g_2)\big|\\
&\le\int_{C'_{a,b}[0,T]}\Big[\big|(A_+^{1/2}w,A_+^{1/2}(\rho_1g))_{C'_{a,b}}\big|+\big|(A_-^{1/2}w,A_-^{1/2}(-\rho_1g))_{C'_{a,b}}\big|\Big]d|f|(w)\\
&\le\rho_1\int_{C'_{a,b}[0,T]}\big|(A_+w,g)_{C'_{a,b}}\big|d|f|(w)+\rho_2\int_{C'_{a,b}[0,T]}\big|(A_-w,g)_{C'_{a,b}}\big|d|f|(w)\\
&\le\big(\rho_1\|A_+\|_o+\rho_2\|A_-\|_o\big)\|g\|_{C'_{a,b}}\int_{C'_{a,b}[0,T]}\|w\|_{C'_{a,b}}d|f|(w).
\end{aligned}$$

But the last expression above is bounded and is independent of $(x_1,x_2)\in C^2_{a,b}[0,T]$. Hence $\delta F(\rho_1x_1+\rho_1hg_1,\rho_2x_2+\rho_2hg_2|\rho_1g_1,\rho_2g_2)$ is $\mu\times\mu$-integrable in $(x_1,x_2)\in C^2_{a,b}[0,T]$ for every $\rho_1>0$ and $\rho_2>0$. Also by Theorems 4.4 and 4.3, the generalized analytic Feynman integrals $E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}[\delta F(x_1,x_2|A_+^{1/2}g,-A_-^{1/2}g)]$

and $E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}[F(x_1,x_2)]$ exist. Thus using (5.10) together with (5.8) and (5.9), (3.5), (5.8) and (5.9) with w replaced with g, it follows that

$$\begin{aligned}
&\int_{C'_{a,b}[0,T]} i(Aw,g)_{C'_{a,b}} \exp\Big\{-\frac{i}{2}(Aw,w)_{C'_{a,b}}\Big\}\\
&\quad\times \exp\Big\{i\Big[(-i)^{-1/2}(w,\vartheta_+^{1/2}\odot a)_{C'_{a,b}} + (i)^{-1/2}(w,\vartheta_-^{1/2}\odot a)_{C'_{a,b}}\Big]\Big\}df(w)\\
&= E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[\delta F(x_1,x_2|A_+^{1/2}g,-A_-^{1/2}g)\big]\\
&= -iE_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[F(x_1,x_2)\big\{(A_+^{1/2}g,x_1)^\sim + (A_-^{1/2}g,x_2)^\sim\big\}\big]\\
&\quad - \Big\{(-i)^{1/2}(A_+^{1/2}g,a)_{C'_{a,b}} - (i)^{1/2}(A_-^{1/2}g,a)_{C'_{a,b}}\Big\}E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[F(x_1,x_2)\big]\\
&= -iE_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[F(x_1,x_2)\big\{(A_+^{1/2}g,x_1)^\sim + (A_-^{1/2}g,x_2)^\sim\big\}\big]\\
&\quad - \Big\{(-i)^{1/2}(g,\vartheta_+^{1/2}\odot a)_{C'_{a,b}} - (i)^{1/2}(g,\vartheta_-^{1/2}\odot a)_{C'_{a,b}}\Big\}E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[F(x_1,x_2)\big].
\end{aligned}$$

This result together with (4.3) yields the formula

$$\begin{aligned}
(5.11)\quad & E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\big[F(x_1,x_2)\big\{(A_+^{1/2}g,x_1)^\sim + (A_-^{1/2}g,x_2)^\sim\big\}\big]\\
&= i\Big\{(-i)^{1/2}(g,\vartheta_+^{1/2}\odot a)_{C'_{a,b}} - (i)^{1/2}(g,\vartheta_-^{1/2}\odot a)_{C'_{a,b}}\Big\}\\
&\quad\times\int_{C'_{a,b}[0,T]} \exp\Big\{-\frac{i}{2}(Aw,w)_{C'_{a,b}}\Big\}\\
&\qquad\times \exp\Big\{i\Big[(-i)^{-1/2}(w,\vartheta_+^{1/2}\odot a)_{C'_{a,b}} + (i)^{-1/2}(w,\vartheta_-^{1/2}\odot a)_{C'_{a,b}}\Big]\Big\}df(w)\\
&\quad -\int_{C'_{a,b}[0,T]} (Aw,g)_{C'_{a,b}} \exp\Big\{-\frac{i}{2}(Aw,w)_{C'_{a,b}}\Big\}\\
&\qquad\times \exp\Big\{i\Big[(-i)^{-1/2}(w,\vartheta_+^{1/2}\odot a)_{C'_{a,b}} + (i)^{-1/2}(w,\vartheta_-^{1/2}\odot a)_{C'_{a,b}}\Big]\Big\}df(w).
\end{aligned}$$

From this, we obtain more explicit formulas as follows.

Step 1. Under the special setting of ϑ in the equation (5.5) above, we first observe the following table:

TABLE 1. The results from the setting of ϑ in the equation (5.11)

ϑ	θ	θ^+	θ^-	$\sqrt{\theta^+}$	$\sqrt{\theta^-}$	$\vartheta_+^{1/2}$	$\vartheta_-^{1/2}$	A	A_+	A_-	$A_+^{1/2}$	$A_-^{1/2}$
b	1	1	0	1	0	b	0	I	I	0	I	0
$-b$	-1	0	1	0	1	0	b	$-I$	0	I	0	I

Step 2. Using the equation (5.11) above together with Table 1 we obtain the following two explicit formulas:

$$
\begin{aligned}
&E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\bigg[(g,x_1)^{\sim}\int_{C'_{a,b}[0,T]}\exp\big\{i(w,x_1)\big\}df(w)\bigg]\\
&= i(-i)^{1/2}(g,a)_{C'_{a,b}}\int_{C'_{a,b}[0,T]}\exp\bigg\{-\frac{i}{2}\|w\|_{C'_{a,b}}^2+i(-i)^{-1/2}(w,a)_{C'_{a,b}}\bigg\}df(w)\\
&\quad-\int_{C'_{a,b}[0,T]}(w,g)_{C'_{a,b}}\exp\bigg\{-\frac{i}{2}\|w\|_{C'_{a,b}}^2+i(-i)^{-1/2}(w,a)_{C'_{a,b}}\bigg\}df(w)
\end{aligned}
$$

and

$$
\begin{aligned}
&E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\bigg[(g,x_2)^{\sim}\int_{C'_{a,b}[0,T]}\exp\big\{-i(w,x_2)\big\}df(w)\bigg]\\
&= -i(i)^{1/2}(g,a)_{C'_{a,b}}\int_{C'_{a,b}[0,T]}\exp\bigg\{\frac{i}{2}\|w\|_{C'_{a,b}}^2+i(i)^{-1/2}(w,a)_{C'_{a,b}}\bigg\}df(w)\\
&\quad+\int_{C'_{a,b}[0,T]}(w,g)_{C'_{a,b}}\exp\bigg\{\frac{i}{2}\|w\|_{C'_{a,b}}^2-i(i)^{-1/2}(w,a)_{C'_{a,b}}\bigg\}df(w).
\end{aligned}
$$

Other setting. Letting $\vartheta=\sin\big(\frac{\pi b(t)}{b(T)}\big)$ in the equation (5.11) above, it also follows that

$$
\begin{aligned}
\theta(t)=D\vartheta(t)&=\frac{\pi}{b(T)}\cos\bigg(\frac{\pi b(t)}{b(T)}\bigg),\\
\sqrt{\theta^{+}}(t)&=\frac{\pi}{b(T)}\cos^{+}\bigg(\frac{\pi b(t)}{b(T)}\bigg)=\frac{\pi}{b(T)}\cos\bigg(\frac{\pi b(t)}{b(T)}\bigg)\chi_{[0,b^{-1}(\frac{b(T)}{2})]}(t),\\
\sqrt{\theta^{-}}(t)&=\frac{\pi}{b(T)}\cos^{-}\bigg(\frac{\pi b(t)}{b(T)}\bigg)=-\frac{\pi}{b(T)}\cos\bigg(\frac{\pi b(t)}{b(T)}\bigg)\chi_{[b^{-1}(\frac{b(T)}{2}),T]}(t),\\
Aw(t)&=\int_0^t\frac{\pi}{b(T)}\cos\bigg(\frac{\pi b(t)}{b(T)}\bigg)dw(t),\\
A_{+}^{1/2}w(t)&=\int_0^t\frac{\pi}{b(T)}\cos^{+}\bigg(\frac{\pi b(t)}{b(T)}\bigg)dw(t)\\
&=\int_0^t\frac{\pi}{b(T)}\cos\bigg(\frac{\pi b(t)}{b(T)}\bigg)\chi_{[0,b^{-1}(\frac{b(T)}{2})]}(t)dw(t),
\end{aligned}
$$

and

$$
\begin{aligned}
A_{-}^{1/2}w(t)&=\int_0^t\frac{\pi}{b(T)}\cos^{-}\bigg(\frac{\pi b(t)}{b(T)}\bigg)dw(t)\\
&=-\int_0^t\frac{\pi}{b(T)}\cos\bigg(\frac{\pi b(t)}{b(T)}\bigg)\chi_{[b^{-1}(\frac{b(T)}{2}),T]}(t)dw(t).
\end{aligned}
$$

Using these we can replace the equation (5.11) with the following 8 formulas

$$
\begin{aligned}
(A_+^{1/2}g, x_1)^\sim &= \frac{\pi}{b(T)}\int_0^{b^{-1}(\frac{b(T)}{2})}\cos\left(\frac{\pi b(t)}{b(T)}\right)Dg(t)dx_1(t),\\
(A_-^{1/2}g, x_2)^\sim &= -\frac{\pi}{b(T)}\int_{b^{-1}(\frac{b(T)}{2})}^{T}\cos\left(\frac{\pi b(t)}{b(T)}\right)Dg(t)dx_2(t),\\
(Aw, w)_{C'_{a,b}} &= \left(\frac{\pi}{b(T)}\right)^2\int_0^{T}\cos^2\left(\frac{\pi b(t)}{b(T)}\right)(Dw)^2(t)db(t),\\
(Aw, g)_{C'_{a,b}} &= \frac{\pi}{b(T)}\int_0^{T}\cos\left(\frac{\pi b(t)}{b(T)}\right)Dw(t)Dg(t)db(t),\\
(g, \vartheta_+^{1/2}\odot a)_{C'_{a,b}} &= \frac{\pi}{b(T)}\int_0^{b^{-1}(\frac{b(T)}{2})}\cos\left(\frac{\pi b(t)}{b(T)}\right)Dg(t)Da(t)db(t),\\
(g, \vartheta_-^{1/2}\odot a)_{C'_{a,b}} &= -\frac{\pi}{b(T)}\int_{b^{-1}(\frac{b(T)}{2})}^{T}\cos\left(\frac{\pi b(t)}{b(T)}\right)Dg(t)Da(t)db(t),\\
(w, \vartheta_+^{1/2}\odot a)_{C'_{a,b}} &= \frac{\pi}{b(T)}\int_0^{b^{-1}(\frac{b(T)}{2})}\cos\left(\frac{\pi b(t)}{b(T)}\right)Dw(t)Da(t)db(t),
\end{aligned}
$$

and

$$
(w, \vartheta_-^{1/2}\odot a)_{C'_{a,b}} = -\frac{\pi}{b(T)}\int_{b^{-1}(\frac{b(T)}{2})}^{T}\cos\left(\frac{\pi b(t)}{b(T)}\right)Dw(t)Da(t)db(t).
$$

In this example, we chose $\vartheta = \sin\left(\frac{\pi b(t)}{b(T)}\right)$ and, as presented in the equation (5.5), this function ϑ gave the operator A.

Consider the sequence $\{e_m\}$ of functions in $C'_{a,b}[0,T]$, where for each $m \in \mathbb{N}$, e_m is given by

$$
e_m(t) = \frac{\sqrt{2b(T)}}{(m-\frac{1}{2})\pi}\sin\left(\frac{(m-\frac{1}{2})\pi}{b(T)}\right), \quad t \in [0,T].
$$

One can show that the sequence $\{e_m\}$ is a complete orthonormal set in $C'_{a,b}[0,T]$ and the functions $\{e_m\}$ are the eigenvectors of the operator $B : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ given by

$$
Bw(t) = \int_0^T \min\{b(s), b(t)\}w(s)db(s), \quad s \in [0,T].
$$

It is known that the operator B is a self-adjoint positive definite trace class operator and is decomposed by $B = S^*S$ where $S : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ is the operator given by

$$
Sw(t) = \int_0^t w(s)db(s), \quad s \in [0,T].
$$

Using this, one can easily verify that for each $w \in C'_{a,b}[0,T]$,

$$\int_0^T w^2(s)db(s) = (w, Bw)_{C'_{a,b}}.$$

For more details, see [6]. Thus, under these constructions, our results in this paper can be applied to evaluate Feynman integrals associated with Fourier series approximation (see [16]) for functionals of generalized Brownian motion paths.

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