

## A NOTE ON *GCR*-LIGHTLIKE WARPED PRODUCT SUBMANIFOLDS IN INDEFINITE KAEHLER MANIFOLDS

SANGEET KUMAR AND MEGHA PRUTHI

**ABSTRACT.** We prove the non-existence of warped product *GCR*-lightlike submanifolds of the type  $K_{\perp} \times_{\lambda} K_T$  such that  $K_T$  is a holomorphic submanifold and  $K_{\perp}$  is a totally real submanifold in an indefinite Kaehler manifold  $\tilde{K}$ . Further, the existence of *GCR*-lightlike warped product submanifolds of the type  $K_T \times_{\lambda} K_{\perp}$  is obtained by establishing a characterization theorem in terms of the shape operator and the warping function in an indefinite Kaehler manifold. Consequently, we find some necessary and sufficient conditions for an isometrically immersed *GCR*-lightlike submanifold in an indefinite Kaehler manifold to be a *GCR*-lightlike warped product, in terms of the canonical structures  $f$  and  $\omega$ . Moreover, we also derive a geometric estimate for the second fundamental form of *GCR*-lightlike warped product submanifolds, in terms of the Hessian of the warping function  $\lambda$ .

### 1. Introduction

Warped product manifolds were firstly introduced in 1969 by Bishop and O’Neill [3], to construct examples of negatively curved manifolds. But, the study of warped products became more popular among researchers, when *CR*-warped product submanifolds were presented by Chen [4], in Kaehler manifolds and he derived that warped product *CR* submanifolds of the type  $K_{\perp} \times_{\lambda} K_T$  do not exist in Kaehler manifolds such that  $K_T$  and  $K_{\perp}$ , respectively, represent holomorphic submanifolds and totally real submanifolds. A detailed study on warped products focusing on manifolds with positive definite metric can be found in [5]. One may note that the concept of warped products has been successfully employed in the study of cosmological models, black holes and Einstein’s field equations (c.f., [2, 10, 16]). However, far less common are the studies where warped products are considered on manifolds with indefinite metrics. To this end, the semi-Riemannian manifolds provide a more broad framework for investigation of warped products and may result in striking applications. In

---

Received August 13, 2020; Revised April 22, 2021; Accepted May 18, 2021.

2010 *Mathematics Subject Classification.* Primary 53C50; Secondary 53C15, 53C40.

*Key words and phrases.* Indefinite Kaehler manifolds, *GCR*-lightlike submanifolds, warped product lightlike submanifolds.

this context, two classes of warped product lightlike manifolds were defined by Duggal [6]. Subsequently, Sahin [17] proposed the concept of warped product lightlike submanifolds of semi-Riemannian manifolds and obtained some basic results on this class of lightlike submanifolds. Very recently, Kumar [12–14] investigated warped product lightlike submanifolds in indefinite nearly Kaehler manifolds. Thus, the significant applications of warped product lightlike submanifolds and important geometric properties of indefinite Kaehler manifolds motivated us to analyze warped product lightlike submanifolds in indefinite Kaehler manifolds.

To this end, in the present paper, we derive the non-existence of warped product  $GCR$ -lightlike submanifolds of the type  $K_{\perp} \times_{\lambda} K_T$  provided  $K_T$  is a holomorphic submanifold and  $K_{\perp}$  is a totally real submanifold in an indefinite Kaehler manifold  $\tilde{K}$ . Then, the existence of  $GCR$ -lightlike warped product submanifolds of the type  $K_T \times_{\lambda} K_{\perp}$  is obtained by establishing a characterization theorem in terms of the shape operator and the warping function in an indefinite Kaehler manifold. Consequently, we find some necessary and sufficient conditions for an isometrically immersed  $GCR$ -lightlike submanifold in an indefinite Kaehler manifold to be a  $GCR$ -lightlike warped product, in terms of the canonical structures  $f$  and  $\omega$ . Moreover, we also derive a geometric estimate for the second fundamental form of  $GCR$ -lightlike warped product submanifolds, in terms of the Hessian of the warping function  $\lambda$ .

## 2. Preliminaries

### 2.1. Lightlike submanifolds

Assume  $(K_m, g)$  is an immersed submanifold in a semi-Riemannian manifold  $(\tilde{K}_{m+n}, \tilde{g})$  with constant index  $q$  satisfying  $m, n \geq 1$ ,  $1 \leq q \leq m+n-1$ . If the metric  $\tilde{g}$  is degenerate on  $TK$ , then  $T_x K$  and  $T_x K^{\perp}$  both become degenerate orthogonal subspaces and hence there exists a subspace  $Rad(T_x K)$  such that  $Rad(T_x K) = T_x K \cap T_x K^{\perp}$ , which is called the radical subspace with rank  $r$ ,  $1 \leq r \leq m$ . If  $Rad(TK) : x \in K \rightarrow Rad(T_x K)$  define a smooth distribution on  $K$  of rank  $r > 0$ , then  $K$  is called an  $r$ -lightlike submanifold of  $\tilde{K}$  (see [7]). Further, let  $S(TK)$  be a screen distribution in  $TK$  such that

$$(1) \quad TK = Rad(TK) \perp S(TK).$$

Similarly, let  $S(TK^{\perp})$  be a screen transversal vector bundle in  $TK^{\perp}$  such that  $TK^{\perp} = Rad(TK) \perp S(TK^{\perp})$ . On the other hand, let  $tr(TK)$  and  $ltr(TK)$  be vector bundles in  $T\tilde{K}|_K$  and  $S(TK^{\perp})^{\perp}$ , respectively, with the property that

$$(2) \quad tr(TK) = ltr(TK) \perp S(TK^{\perp})$$

and

$$(3) \quad T\tilde{K}|_K = TK \oplus tr(TK) = (Rad(TK) \oplus ltr(TK)) \perp S(TK) \perp S(TK^{\perp}).$$

Further, the Gauss and Weingarten formulae are

$$(4) \quad \tilde{\nabla}_P Q = \nabla_P Q + h(P, Q), \quad \tilde{\nabla}_P V = -A_V P + \nabla_P^t V,$$

for any  $P, Q \in \Gamma(TK)$  and  $V \in \Gamma(tr(TK))$ , where  $\tilde{\nabla}$  and  $\nabla$ , respectively, denote the Levi-Civita connection on  $\tilde{K}$  and the torsion free linear connection defined on  $K$ . In particular, one has

$$(5) \quad \tilde{\nabla}_P Q = \nabla_P Q + h^l(P, Q) + h^s(P, Q),$$

$$(6) \quad \tilde{\nabla}_P W = -A_W P + \nabla_P^s W + D^l(P, W),$$

$$(7) \quad \tilde{\nabla}_P N = -A_N P + \nabla_P^l N + D^s(P, N),$$

where  $W \in \Gamma(S(TK^\perp))$ ,  $P, Q \in \Gamma(TK)$  and  $N \in \Gamma(ltr(TK))$ . Further, employing Eqs. (5) and (6), we obtain

$$(8) \quad \tilde{g}(h^s(P, Q), W) + \tilde{g}(Q, D^l(P, W)) = g(A_W P, Q)$$

for  $P, Q \in \Gamma(TK)$  and  $W \in \Gamma(S(TK^\perp))$ .

Consider the curvature tensor  $\tilde{R}$  of  $\tilde{\nabla}$ . Then the Codazzi equation is given as

$$(9) \quad \begin{aligned} (\tilde{R}(Y_1, Y_2)Y_3)^\perp &= (\nabla_{Y_1} h^l)(Y_2, Y_3) - (\nabla_{Y_2} h^l)(Y_1, Y_3) + D^l(Y_1, h^s(Y_2, Y_3)) \\ &\quad - D^l(Y_2, h^s(Y_1, Y_3)) + (\nabla_{Y_1} h^s)(Y_2, Y_3) - (\nabla_{Y_2} h^s)(Y_1, Y_3) \\ &\quad + D^s(Y_1, h^l(Y_2, Y_3)) - D^s(Y_2, h^l(Y_1, Y_3)), \end{aligned}$$

where

$$(10) \quad (\nabla_{Y_1} h^s)(Y_2, Y_3) = \nabla_{Y_1}^s h^s(Y_2, Y_3) - h^s(\nabla_{Y_1} Y_2, Y_3) - h^s(Y_2, \nabla_{Y_1} Y_3),$$

$$(11) \quad (\nabla_{Y_1} h^l)(Y_2, Y_3) = \nabla_{Y_1}^l h^l(Y_2, Y_3) - h^l(\nabla_{Y_1} Y_2, Y_3) - h^l(Y_2, \nabla_{Y_1} Y_3),$$

for any  $Y_1, Y_2, Y_3 \in \Gamma(TK)$ .

**Definition** ([8]). Consider a semi-Riemannian manifold  $(\tilde{K}, \tilde{g})$ . Then a light-like submanifold  $(K, g)$  of  $\tilde{K}$  is called totally umbilical if there exists a smooth transversal curvature vector field  $H \in \Gamma(tr(TK))$  of  $K$  satisfying

$$h(Y, Z) = Hg(Y, Z)$$

for  $Y, Z \in \Gamma(TK)$ . According to Eqs. (5) and (6),  $K$  is totally umbilical if and only if there are smooth vector fields  $H^l \in \Gamma(ltr(TK))$  and  $H^s \in \Gamma(S(TK^\perp))$  with the property

$$(12) \quad h^l(Y, Z) = H^l g(Y, Z), \quad h^s(Y, Z) = H^s g(Y, Z), \quad D^l(Y, W) = 0,$$

for any  $Y, Z \in \Gamma(TK)$  and  $W \in \Gamma(S(TK^\perp))$ .

## 2.2. Indefinite Kaehler manifolds

Consider an indefinite almost Hermitian manifold  $(\tilde{K}, \tilde{J}, \tilde{g})$ . Then  $\tilde{K}$  is said to be an indefinite Kaehler manifold (c.f., [1]), if

$$(13) \quad \tilde{J}^2 = -I, \quad \tilde{g}(\tilde{J}Y, \tilde{J}Z) = \tilde{g}(Y, Z), \quad (\tilde{\nabla}_Y \tilde{J})Z = 0,$$

for  $Y, Z \in \Gamma(TK)$ , where  $\tilde{\nabla}$  denotes the Levi-Civita connection on  $\tilde{K}$ .

Moreover, an indefinite complex space form  $\tilde{K}(c)$  is an indefinite Kaehler manifold  $\tilde{K}$  with constant holomorphic curvature  $c$  and its curvature tensor  $\tilde{R}$  is given by

$$(14) \quad \begin{aligned} \tilde{R}(Y_1, Y_2)Y_3 = & \frac{c}{4} \{ \tilde{g}(Y_2, Y_3)Y_1 - \tilde{g}(Y_1, Y_3)Y_2 + \tilde{g}(\tilde{J}Y_2, Y_3)\tilde{J}Y_1 - \tilde{g}(\tilde{J}Y_1, Y_3)\tilde{J}Y_2 \\ & + 2\tilde{g}(Y_1, \tilde{J}Y_2)\tilde{J}Y_3 \} \end{aligned}$$

for  $Y_1, Y_2, Y_3$  vector fields on  $\tilde{K}$ .

## 2.3. Generalized Cauchy-Riemann (*GCR*)-lightlike submanifolds

**Definition** ([9]). A real lightlike submanifold  $(K, g, S(TK))$  of an indefinite Kaehler manifold  $(\tilde{K}, \tilde{g}, \tilde{J})$  is said to be a generalized Cauchy-Riemann (*GCR*)-lightlike submanifold, if

- (i) There exist sub-bundles  $D_1$  and  $D_2$  of  $Rad(TK)$  satisfying

$$Rad(TK) = D_1 \oplus D_2, \quad \tilde{J}(D_1) = D_1, \quad \tilde{J}(D_2) \subset S(TK).$$

- (ii) There exist sub-bundles  $D_0$  and  $D'$  of  $S(TK)$  satisfying

$$S(TK) = \{\tilde{J}D_2 \oplus D'\} \perp D_0, \quad \tilde{J}(D') = L_1 \perp L_2, \quad \tilde{J}(D_0) = D_0,$$

where  $L_1$  and  $L_2$ , respectively, denote vector subbundles of  $ltr(TK)$  and  $S(TK^\perp)$  and  $D_0$  is a non-degenerate distribution on  $K$ . Moreover, we assume that  $M_1 = \tilde{J}L_1$  and  $M_2 = \tilde{J}L_2$ .

Let  $Q, P_1$  and  $P_2$  denote the projection morphisms of  $TK$  on  $D, M_1$  and  $M_2$ , respectively. Then for  $Y \in \Gamma(TK)$ , we write

$$(15) \quad Y = QY + P_1Y + P_2Y.$$

Applying  $\tilde{J}$  on both sides of Eq. (15), we get

$$(16) \quad \tilde{J}Y = fY + \omega P_1Y + \omega P_2Y.$$

Eq. (16) can be re-written as

$$(17) \quad \tilde{J}Y = fY + \omega Y,$$

where  $fY$  and  $\omega Y$ , respectively, denote tangential and transversal parts of  $\tilde{J}Y$ . Similarly, for  $Z \in tr(TK)$ , we have

$$(18) \quad \tilde{J}Z = EZ + FZ,$$

where  $EZ$  and  $FZ$ , respectively, denote tangential and transversal parts of  $\tilde{J}Z$ .

**Lemma 2.1** ([15]). *Consider a GCR-lightlike submanifold  $K$  of an indefinite Kaehler manifold  $\tilde{K}$ . Then for  $Y_1, Y_2 \in \Gamma(TK)$ , one has*

$$(19) \quad (\nabla_{Y_1} f)Y_2 = A_{\omega Y_2}Y_1 + Eh(Y_1, Y_2)$$

and

$$(20) \quad (\nabla_{Y_1}^t \omega)Y_2 = Fh(Y_1, Y_2) - h(Y_1, fY_2),$$

where

$$(21) \quad (\nabla_{Y_1} f)Y_2 = \nabla_{Y_1} fY_2 - f\nabla_{Y_1} Y_2, \quad (\nabla_{Y_1}^t \omega)Y_2 = \nabla_{Y_1}^t \omega Y_2 - \omega \nabla_{Y_1} Y_2.$$

### 3. GCR-lightlike warped product submanifolds

In the present part of paper, we analyze warped product GCR-lightlike submanifolds of the type  $K_{\perp} \times_{\lambda} K_T$  and  $K_T \times_{\lambda} K_{\perp}$  of indefinite Kaehler manifolds, where  $K_{\perp}$  and  $K_T$ , respectively, denote totally real and holomorphic submanifolds of an indefinite Kaehler manifold  $\tilde{K}$ . Firstly, we recall a basic result for later use.

**Proposition 3.1** ([3]). *For  $Y_1, Y_2 \in \Gamma(TK_1)$  and  $U, V \in \Gamma(TK_2)$  in a warped product manifold  $K = K_1 \times_{\lambda} K_2$ , one has*

$$(22) \quad \nabla_{Y_1} Y_2 \in \Gamma(TK_1),$$

$$(23) \quad \nabla_{Y_1} V = \nabla_V Y_1 = \left( \frac{Y_1 \lambda}{\lambda} \right) V,$$

$$(24) \quad \nabla_U V = -\frac{g(U, V)}{\lambda} \nabla \lambda.$$

**Note.** In the forthcoming part of the paper, we will denote totally umbilical by **t.u.**, warped product by **w.p.** and an indefinite Kaehler manifold by  $\tilde{K}$ , unless otherwise stated.

**Theorem 3.2.** *For a t.u. GCR-lightlike submanifold  $K$  of  $\tilde{K}$ , there doesn't exist any proper w.p. GCR-lightlike submanifold  $K$  of the type  $K_{\perp} \times_{\lambda} K_T$  in  $\tilde{K}$ .*

*Proof.* From Eq. (23), for  $Y_1 \in \Gamma(TK_T)$  and  $Z_1 \in \Gamma(TK_{\perp})$ , we have

$$(25) \quad \nabla_{Y_1} Z_1 = \nabla_{Z_1} Y_1 = (Z_1 \ln \lambda) Y_1.$$

For  $Z_1, Z_2 \in \Gamma(D')$ , employing Eq. (19), we obtain  $f\nabla_{Z_1} Z_2 = -A_{\omega Z_2} Z_1 - Eh(Z_1, Z_2)$ . Further for  $Y_1 \in \Gamma(D_0)$  and using Eqs. (5) and (25), we get  $g(f\nabla_{Z_1} Z_2, Y_1) = -g(A_{\omega Z_2} Z_1, Y_1) = \tilde{g}(\tilde{\nabla}_{Z_1} \tilde{J} Z_2, Y_1) = -\tilde{g}(\tilde{J} Z_2, \tilde{\nabla}_{Z_2} Y_1) = -\tilde{g}(\tilde{J} Z_2, \nabla_{Z_1} Y_1) = 0$ . As  $D_0$  is non-degenerate, therefore  $f\nabla_{Z_1} Z_2 = 0$ , it implies that  $\nabla_{Z_1} Z_2 \in \Gamma(D')$ . Thus, the distribution  $D'$  defines a totally geodesic foliation in  $K$ .

Assume that  $h^T$  and  $A^T$ , respectively, denote the second fundamental form and the shape operator of  $K_T$  on  $K$ , therefore, we have  $g(h^T(Y_1, Y_2), Z_1) =$

$g(\nabla_{Y_1} Y_2, Z_1) = \tilde{g}(\tilde{\nabla}_{Y_1} Y_2, Z_1) = -\tilde{g}(Y_2, \tilde{\nabla}_{Y_1} Z_1) = -g(Y_2, \nabla_{Y_1} Z_1)$  for  $Y_1, Y_2 \in \Gamma(D)$  and  $Z_1 \in \Gamma(D')$ . Further using Eq. (25), we get

$$(26) \quad g(h^T(Y_1, Y_2), Z_1) = -(Z_1 \ln \lambda)g(Y_1, Y_2).$$

Consider the second fundamental form  $\hat{h}$  of  $K_T$  in  $\tilde{K}$ . Then for  $Y_1, Y_2 \in \Gamma(TK_T)$ , we attain

$$(27) \quad \hat{h}(Y_1, Y_2) = h^T(Y_1, Y_2) + h^l(Y_1, Y_2) + h^s(Y_1, Y_2).$$

Then employing Eq. (27), for  $Z_1 \in \Gamma(D')$ , we obtain

$$(28) \quad \tilde{g}(\hat{h}(Y_1, Y_2), Z_1) = g(h^T(Y_1, Y_2), Z_1) = -(Z_1 \ln \lambda)g(Y_1, Y_2).$$

As  $K_T$  is a holomorphic submanifold of  $\tilde{K}$ , it follows that

$$(29) \quad \hat{h}(Y_1, \tilde{J}Y_2) = \hat{h}(\tilde{J}Y_1, Y_2) = \tilde{J}\hat{h}(Y_1, Y_2).$$

Thus from Eqs. (28) and (29), we derive

$$(30) \quad \tilde{g}(\hat{h}(Y_1, Y_2), Z_1) = -\tilde{g}(\hat{h}(\tilde{J}Y_1, \tilde{J}Y_2), Z_1) = (Z_1 \ln \lambda)g(Y_1, Y_2).$$

Then adding Eqs. (28) and (30), we get

$$(31) \quad \tilde{g}(\hat{h}(Y_1, Y_2), Z_1) = 0.$$

Further employing Eqs. (27), (29) and (31), we obtain

$$(32) \quad \tilde{g}(h(Y_1, Y_2), \tilde{J}Z_1) = \tilde{g}(\hat{h}(Y_1, Y_2), \tilde{J}Z_1) = -\tilde{g}(\hat{h}(Y_1, \tilde{J}Y_2), Z_1) = 0.$$

Thus, we have  $\tilde{g}(h(D, D), \tilde{J}D') = 0$ , that is,  $h(D, D)$  has no component in  $\tilde{J}D'$ , which shows  $D$  defines totally geodesic foliation in  $K$ . Hence,  $K = K_\perp \times_\lambda K_T$  is *GCR*-lightlike product.  $\square$

Next, by considering *w.p.* *GCR*-lightlike submanifolds of the type  $K = K_T \times_\lambda K_\perp$  in  $\tilde{K}$ , we are going to prove the following lemma.

**Lemma 3.3.** *Consider a GCR-lightlike w.p. submanifold  $K$  of  $\tilde{K}$ . Then*

- (i)  $\tilde{g}(h(D_0, D_0), \tilde{J}D') = 0$ ,
- (ii)  $\tilde{g}(h^s(Y_1, Z_1), \tilde{J}Z_2) = -\tilde{J}Y_1(\ln \lambda)g(Z_1, Z_2)$ ,

for  $Y_1 \in \Gamma(D)$  and  $Z_1, Z_2 \in \Gamma(K_2)$ .

*Proof.* Employing Eq. (13), for  $Y_1 \in \Gamma(D_0)$  and  $Z_1 \in \Gamma(D')$ , we get  $\tilde{J}\tilde{\nabla}_{Y_1} Z_1 = \tilde{\nabla}_{Y_1} \tilde{J}Z_1$ . Then using Eq. (5), we have  $\tilde{J}\nabla_{Y_1} Z_1 + \tilde{J}h(Y_1, Z_1) = -A_{\tilde{J}Z_1} Y_1 + \nabla_{Y_1}^t \tilde{J}Z_1$ . On considering the inner product with  $\tilde{J}Y_2$  for  $Y_2 \in \Gamma(D_0)$ , we obtain  $g(\nabla_{Y_1} Z_1, Y_2) = -g(A_{\tilde{J}Z_1} Y_1, \tilde{J}Y_2)$ . Further using Eqs. (8) and (23), we derive  $\tilde{g}(h^s(Y_1, \tilde{J}Y_2), \tilde{J}Z_1) = 0$ . As  $\tilde{g}(h^l(Y_1, \tilde{J}Y_2), \tilde{J}Z_1) = 0$ . Thus  $\tilde{g}(h(Y_1, \tilde{J}Y_2), \tilde{J}Z_1) = 0$ , which proves (i).

Next, for  $Y_1 \in \Gamma(D)$  and  $Z_1, Z_2 \in \Gamma(K_2)$ , employing Eqs. (5), (13) and (23), we derive

$$\begin{aligned} \tilde{g}(h^s(Y_1, Z_1), \tilde{J}Z_2) &= \tilde{g}(\tilde{\nabla}_{Z_1} Y_1, \tilde{J}Z_2) = -\tilde{g}(\tilde{J}\tilde{\nabla}_{Z_1} Y_1, Z_2) \\ &= -\tilde{g}(\tilde{\nabla}_{Z_1} \tilde{J}Y_1, Z_2) = -g(\nabla_{Z_1} \tilde{J}Y_1, Z_2) \end{aligned}$$

$$= -\tilde{J}Y_1(\ln \lambda)g(Z_1, Z_2),$$

which proves (ii). □

In the following characterization result, we provide a relationship between the shape operator and the warping function of GCR-lightlike **w.p.** submanifolds of the type  $K_T \times_\lambda K_\perp$  in  $\tilde{K}$ .

**Theorem 3.4.** *Consider a t.u. GCR-lightlike submanifold  $K$  of  $\tilde{K}$  such that the totally real distribution  $D'$  is integrable. Then  $K$  is a locally GCR-lightlike **w.p.** submanifold if and only if*

$$(33) \quad A_{\tilde{J}Z_1}Y_1 = -(\tilde{J}Y_1)(\mu)Z_1$$

for  $Y_1 \in \Gamma(D)$ ,  $Z_1 \in \Gamma(D')$  and  $\mu$  is a  $C^\infty$  function defined on  $K$  satisfying  $Z_1\mu = 0$  for  $Z_1 \in \Gamma(D')$ .

*Proof.* Let  $K$  be a t.u. GCR-lightlike **w.p.** submanifold of the type  $K_T \times_\lambda K_\perp$ . Employing Eq. (13), for  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ , one has  $\tilde{\nabla}_{Y_1}\tilde{J}Z_1 = \tilde{J}\tilde{\nabla}_{Y_1}Z_1$ , which on using Eqs. (4) and (23) gives

$$-A_{\tilde{J}Z_1}Y_1 + \nabla_{Y_1}^t\tilde{J}Z_1 = \tilde{J}Y_1(\ln \lambda)Z_1.$$

Then equating the tangential parts, we derive  $A_{\tilde{J}Z_1}Y_1 = -\tilde{J}Y_1(\ln \lambda)Z_1$ . Since  $\mu = \ln \lambda$  is a function defined on  $K_T$ , for  $Z_1 \in \Gamma(D')$ , we attain  $Z_1(\mu) = Z_1(\ln \lambda) = 0$ .

Conversely, suppose that  $K$  is a t.u. GCR-lightlike submanifold of  $\tilde{K}$  satisfying Eq. (33). Then, Eq. (33) gives  $g(A_{\tilde{J}Z_1}Y_1, Y_2) = -g((\tilde{J}Y_1)\mu)Z_1, Y_2) = 0$  for  $Y_1, Y_2 \in \Gamma(D)$  and  $Z_1 \in \Gamma(D')$ , which on employing Eq. (8), yields  $\tilde{g}(h^s(Y_1, Y_2), \tilde{J}Z_1) = 0$ . Thus, for  $Z_1 \in \Gamma(D')$ , we get  $\tilde{g}(h^s(D, D), \tilde{J}Z_1) = 0$  and  $\tilde{g}(h^l(D, D), \tilde{J}Z_1) = 0$ . Thus

$$\tilde{g}(h(D, D), \tilde{J}Z_1) = 0,$$

which means that  $h(D, D)$  has no component in  $\tilde{J}D'$ . This implies that  $D$  defines a totally geodesic foliation in  $K$ .

Next, on taking the inner product of Eq. (33) with respect to  $Z_2 \in \Gamma(D')$  and using the hypothesis along with Eqs. (5), (13) and (23), we have

$$(34) \quad \begin{aligned} g((\tilde{J}Y_1)\mu)Z_1, Z_2) &= -g(A_{\tilde{J}Z_1}Y_1, Z_2) = -\tilde{g}(\tilde{J}Z_1, \nabla_{Y_1}Z_2) \\ &= -\tilde{g}(\tilde{J}Z_1, \nabla_{Z_2}Y_1) = \tilde{g}(\tilde{\nabla}_{Z_2}\tilde{J}Z_1, Y_1) \\ &= -g(\nabla_{Z_2}Z_1, \tilde{J}Y_1), \end{aligned}$$

where  $Y_1 \in \Gamma(D)$  and  $Z_1 \in \Gamma(D')$ . Further using  $g(\nabla\phi, Y_1) = Y_1\phi$  in Eq. (34), we derive

$$(35) \quad g(\nabla_{Z_2}Z_1, \tilde{J}Y_1) = -g(\nabla\mu, \tilde{J}Y_1)g(Z_1, Z_2).$$

Considering the second fundamental form  $h'$  and the induced connection  $\nabla'$  of  $D'$  on  $K$ , for  $Y_1 \in \Gamma(D)$  and  $Z_1, Z_2 \in \Gamma(D')$ , one has

$$(36) \quad g(h'(Z_1, Z_2), \tilde{J}Y_1) = g(\nabla_{Z_2} Z_1 - \nabla'_{Z_2} Z_1, \tilde{J}Y_1) = g(\nabla_{Z_2} Z_1, \tilde{J}Y_1).$$

From Eqs. (35) and (36), we obtain

$$(37) \quad g(h'(Z_1, Z_2), \tilde{J}Y_1) = -g(\nabla\mu, \tilde{J}Y_1)g(Z_1, Z_2).$$

Then the property of non-degeneracy of  $D_0$  provides

$$(38) \quad h'(Z_1, Z_2) = -\nabla\mu g(Z_1, Z_2).$$

Hence  $D'$  becomes totally umbilical in  $K$  and clearly, by hypothesis,  $D'$  is integrable. Then using Eq. (38) and the condition  $Z_1\mu = 0$  for each  $Z_1 \in \Gamma(D')$  gives that each leaf of  $D'$  is an intrinsic sphere in  $K$ . Thus from the result of [11], which states “If the tangent bundle of a Riemannian manifold  $K$  is an orthogonal sum  $TK = K_0 \oplus K_1$  of non-trivial vector sub-bundles such that  $K_1$  is spherical and it’s orthogonal complement  $K_0$  is auto-parallel, then  $K$  is locally isometric to a  $\mathbf{w.p.} K_0 \times_\lambda K_1$ ”, we get that  $K$  is locally  $GCR$ -lightlike  $\mathbf{w.p.}$  of the type  $K_T \times_\lambda K_\perp$  in  $\tilde{K}$ , where  $\lambda = e^\mu$ .  $\square$

**Example 3.5.** Let  $K$  be an 8-dimensional submanifold in  $(R_4^{14}, \tilde{g})$  with

$$\begin{aligned} x^1 &= u^1 - u^2, \quad x^2 = u^1 + u^2, \quad x^3 = u^3, \quad x^4 = u^5, \quad x^5 = -u^5, \\ x^6 &= u^3, \quad x^7 = \sqrt{2}u^3, \quad x^8 = \sqrt{2}u^4, \quad x^9 = u^6, \quad x^{10} = u^7, \quad x^{11} = u^6 \cos u^8, \\ x^{12} &= u^7 \cos u^8, \quad x^{13} = u^6 \sin u^8, \quad x^{14} = u^7 \sin u^8, \end{aligned}$$

where  $u^8 \in R - \{\frac{n\pi}{2}, n \in \mathbb{Z}\}$  and  $g$  is of signature  $(-, +, +, +, +, +, -, +, -, -, +, +, +, +)$ . Then  $TK$  is spanned by  $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8$ , such that

$$\begin{aligned} Z_1 &= \partial x_1 + \partial x_2, \quad Z_2 = -\partial x_1 + \partial x_2, \\ Z_3 &= \partial x_3 + \partial x_6 + \sqrt{2}\partial x_7, \quad Z_4 = \sqrt{2}\partial x_8, \quad Z_5 = \partial x_4 - \partial x_5, \\ Z_6 &= \partial x_9 + \cos u^8 \partial x_{11} + \sin u^8 \partial x_{13}, \quad Z_7 = \partial x_{10} + \cos u^8 \partial x_{12} + \sin u^8 \partial x_{14}, \\ Z_8 &= -u^6 \sin u^8 \partial x_{11} - u^7 \sin u^8 \partial x_{12} + u^6 \cos u^8 \partial x_{13} + u^7 \cos u^8 \partial x_{14}. \end{aligned}$$

Clearly,  $K$  is a 3-lightlike submanifold with  $Rad(TK) = Span\{Z_1, Z_2, Z_3\}$ . As  $\tilde{J}Z_1 = Z_2$ , we have  $D_1 = Span\{Z_1, Z_2\}$  and  $\tilde{J}Z_3 = Z_4 + Z_5 \in \Gamma(S(TK))$ , obtaining that  $D_2 = Span\{Z_3\}$ . Moreover,  $\tilde{J}Z_6 = Z_7$ , thus  $D_0 = Span\{Z_6, Z_7\}$ . Further, by direct calculations,  $S(TK^\perp) = Span\{W = u^7 \sin u^8 \partial x_{11} - u^6 \sin u^8 \partial x_{12} - u^7 \cos u^8 \partial x_{13} + u^6 \cos u^8 \partial x_{14}\}$  and  $\tilde{J}Z_8 = W$ . Therefore  $L_2 = S(TK^\perp)$ . Moreover,  $ltr(TK)$  is spanned by

$$N_1 = \frac{1}{2}(-\partial x_1 + \partial x_2), \quad N_2 = \frac{1}{2}(\partial x_1 + \partial x_2), \quad N_3 = \frac{1}{4}(\partial x_3 + \partial x_6 - \sqrt{2}\partial x_7),$$

where  $Span\{N_1, N_2\}$  is invariant with respect to  $\tilde{J}$  and  $\tilde{J}N_3 = -\frac{1}{4}Z_4 + \frac{1}{4}Z_5$ . Thus,  $L_1 = Span\{N_3\}$  and  $D' = Span\{\tilde{J}N_3, \tilde{J}W\}$ . Therefore,  $K$  is a proper  $GCR$ -lightlike submanifold of  $R_4^{14}$ . Here, it is clear that  $D'$  is integrable. Now,



if the leaves of  $D$  and  $D'$  are, respectively, denoted by  $K_T$  and  $K_\perp$ , then the induced metric tensor of  $K = K_T \times_\lambda K_\perp$  is given by

$$ds^2 = 2(du_4^2 + du_5^2 + du_6^2 + du_7^2) + ((u^6)^2 + (u^7)^2)du_8^2.$$

Hence  $K$  is a **w.p.** GCR-lightlike submanifold of the type  $K_T \times_\lambda K_\perp$  in  $R_4^{14}$ , with warping function  $\lambda = \sqrt{(u^6)^2 + (u^7)^2}$ .

#### 4. GCR-lightlike warped product submanifolds and the canonical structures

In [15], Kumar et al. proved several classification theorems enabling a GCR-lightlike submanifold to be a GCR-lightlike product. Since warped products are generalizations of product manifolds, it is obvious to search for conditions reducing the GCR-lightlike submanifold of  $\tilde{K}$  to GCR-lightlike warped product. Therefore, in the present part of paper, some characterization results are established forcing the GCR-lightlike submanifold to be a GCR-lightlike warped product, in terms of canonical structures.

**Lemma 4.1.** *Assume  $K = K_T \times_\lambda K_\perp$  is a GCR-lightlike w.p. submanifold of  $\tilde{K}$ . Then*

$$\begin{aligned} (\nabla_{Z_1} f)Y_1 &= fY_1(\ln \lambda)Z_1, \\ (\nabla_{Y_2} f)Z_1 &= f(\nabla \ln \lambda)g(Y_2, Z_1), \end{aligned}$$

for  $Y_1 \in \Gamma(D)$ ,  $Y_2 \in \Gamma(TK)$  and  $Z_1 \in \Gamma(D')$ , where  $\nabla(\ln \lambda)$  denotes gradient of  $\ln \lambda$ .

*Proof.* Employing Eqs. (21) and (23), for  $Y_1 \in \Gamma(D)$  and  $Z_1 \in \Gamma(D')$ , we obtain  $(\nabla_{Z_1} f)Y_1 = \nabla_{Z_1} fY_1 = fY_1(\ln \lambda)Z_1$ .

Next, from Eq. (21) we get  $(\nabla_{Y_2} f)Z_1 = -f\nabla_{Y_2} Z_1$ , where  $Y_2 \in \Gamma(TK)$  and  $Z_1 \in \Gamma(D')$ , which implies that  $\nabla_{Y_2} fZ_1 \in \Gamma(D)$ . Then, we attain

$$\begin{aligned} g((\nabla_{Y_2} f)Z_1, Y_1) &= -g(f\nabla_{Y_2} Z_1, Y_1) = g(\nabla_{Y_2} Z_1, fY_1) \\ &= \tilde{g}(\tilde{\nabla}_{Y_2} Z_1, fY_1) = -g(Z_1, \nabla_{Y_2} fY_1) \\ (39) \qquad \qquad \qquad &= -fY_1(\ln \lambda)g(Z_1, Y_2) \end{aligned}$$

for  $Y_1 \in \Gamma(D_0)$ . Thus, the result directly follows by employing the definition of gradient for  $\lambda$  and property of non-degeneracy of  $D_0$ . □

**Theorem 4.2.** *Consider a GCR-lightlike submanifold  $K$  of  $\tilde{K}$  such that the totally real distribution  $D'$  is integrable. Then  $K$  is a locally GCR-lightlike w.p. submanifold if and only if*

$$(40) \qquad (\nabla_{Y_1} f)Y_2 = ((fY_2)\mu)PY_1 + g(PY_1, PY_2)\tilde{J}(\nabla\mu),$$

for  $Y_1, Y_2 \in \Gamma(TK)$ , where  $\mu$  is a  $C^\infty$  function defined on  $K$  satisfying  $Z_1\mu = 0$  for  $Z_1 \in \Gamma(D')$ .

*Proof.* Let  $K$  be a  $GCR$ -lightlike  $\mathbf{w.p.}$  submanifold of  $\tilde{K}$ . Then, for  $Y_1, Y_2 \in \Gamma(TK)$ , we have

$$(41) \quad (\nabla_{Y_1} f)Y_2 = (\nabla_{QY_1} f)QY_2 + (\nabla_{PY_1} f)QY_2 + (\nabla_{Y_1} f)PY_2.$$

Since  $D$  defines a totally geodesic foliation in  $K$ , Eq. (19) gives

$$(42) \quad (\nabla_{QY_1} f)QY_2 = 0.$$

Further using Lemma 4.1, we get

$$(43) \quad (\nabla_{PY_1} f)QY_2 = f(QY_2)(\ln \lambda)PY_1,$$

$$(44) \quad (\nabla_{Y_1} f)PY_2 = g(Y_1, PY_2)f(\nabla \ln \lambda) = g(PY_1, PY_2)f(\nabla \ln \lambda).$$

Thus from Eqs. (41)–(44), we derive Eq. (40). As  $\mu = \ln \lambda$  is a function defined on  $K_T$ , it follows that  $Z_1(\mu) = Z_1(\ln \lambda) = 0$  for  $Z_1 \in \Gamma(D')$ .

Conversely, assume  $K$  is a  $GCR$ -lightlike submanifold of  $\tilde{K}$  satisfying Eq. (40). For  $Y_1, Y_2 \in \Gamma(D)$ , Eq. (40) yields  $(\nabla_{Y_1} T)Y_2 = 0$ . Further using Eq. (19), we get  $Eh(Y_1, Y_2) = 0$ , which shows that  $h(Y_1, Y_2)$  has no component in  $\tilde{J}D'$ , thus  $D$  defines a totally geodesic foliation in  $K$ .

Next, for  $Y_1, Y_2 \in \Gamma(D')$ , employing Eq. (40), we get

$$(45) \quad (\nabla_{Y_1} f)Y_2 = g(PY_1, PY_2)\tilde{J}\nabla\mu.$$

Further, taking the inner product of Eq. (45) with  $Y_3 \in \Gamma(D_0)$ , we derive

$$(46) \quad g((\nabla_{Y_1} f)Y_2, Y_3) = g(PY_1, PY_2)g(\tilde{J}\nabla\mu, Y_3) = -g(PY_1, PY_2)g(\nabla\mu, \tilde{J}Y_3).$$

Also for  $Y_1, Y_2 \in \Gamma(D')$  and  $Y_3 \in \Gamma(D_0)$ , from Eq. (19), we have

$$(47) \quad g((\nabla_{Y_1} f)Y_2, Y_3) = g(A_{\omega Y_2} Y_1, Y_3) = -\tilde{g}(\tilde{\nabla}_{Y_1} \tilde{J}Y_2, Y_3) = g(\nabla_{Y_1} Y_2, \tilde{J}Y_3).$$

From Eqs. (46) and (47), we obtain

$$(48) \quad g(\nabla_{Y_1} Y_2, \tilde{J}Y_3) = -g(PY_1, PY_2)g(\nabla\mu, \tilde{J}Y_3).$$

Let  $h'$  and  $\nabla'$ , respectively, denote the second fundamental form and the induced connection of  $D'$  on  $K$ . Then, for  $Y_1, Y_2 \in \Gamma(D')$  and  $Y_3 \in \Gamma(D)$ , one has

$$(49) \quad g(h'(Y_1, Y_2), \tilde{J}Y_3) = g(\nabla_{Y_1} Y_2 - \nabla'_{Y_1} Y_2, \tilde{J}Y_3) = g(\nabla_{Y_1} Y_2, \tilde{J}Y_3).$$

From Eqs. (48) and (49), we derive

$$(50) \quad g(h'(Y_1, Y_2), \tilde{J}Y_3) = -g(PY_1, PY_2)g(\nabla\mu, \tilde{J}Y_3).$$

Then, using the non-degeneracy of  $D_0$ , we get  $h'(Y_1, Y_2) = -\nabla\mu g(PY_1, PY_2)$ , which yields that the distribution  $D'$  is totally umbilical in  $K$ . By hypothesis,  $D'$  is integrable and  $Z_1\mu = 0$  for  $Z_1 \in \Gamma(D')$ , therefore each leaf of  $D'$  is an intrinsic sphere. Therefore, using a similar argument as in Theorem 3.4,  $K$  is a locally  $GCR$ -lightlike  $\mathbf{w.p.}$  of the type  $K_T \times_\lambda K_\perp$  in  $\tilde{K}$  with warping function  $\lambda = e^\mu$ .  $\square$

**Theorem 4.3.** *A GCR-lightlike submanifold  $K$  of  $\tilde{K}$  with integrable totally real distribution  $D'$  is a locally GCR-lightlike  $\mathbf{w.p.}$  submanifold if and only if*

$$(51) \quad \tilde{g}((\nabla_{Y_1}^t \omega)Y_2, \tilde{J}Z_1) = -QY_2(\mu)g(Y_1, Z_1),$$

for  $Z_1 \in \Gamma(D')$  and  $Y_1, Y_2 \in \Gamma(TK)$ , where  $\mu$  is a  $C^\infty$  function defined on  $K$  satisfying  $Z_1\mu = 0$  for  $Z_1 \in \Gamma(D')$ .

*Proof.* Let  $K$  be a GCR-lightlike  $\mathbf{w.p.}$  submanifold of  $\tilde{K}$ . Clearly, by hypothesis,  $D'$  defines a totally geodesic foliation in  $K$ . Therefore, from Eq. (21), for  $Z_1 \in \Gamma(D')$  and  $Y_1, Y_2 \in \Gamma(D)$ , we get

$$(52) \quad \tilde{g}((\nabla_{Y_1}^t \omega)Y_2, \tilde{J}Z_1) = \tilde{g}(-\omega\nabla_{Y_1}Y_2, \tilde{J}Z_1) = -g(\nabla_{Y_1}Y_2, Z_1) = 0.$$

Then, for  $Y_1, Z_1 \in \Gamma(D')$  and  $Y_2 \in \Gamma(D)$ , from Eq. (20), we obtain

$$(53) \quad \begin{aligned} \tilde{g}((\nabla_{Y_1}^t \omega)Y_2, \tilde{J}Z_1) &= -\tilde{g}(h(Y_1, fY_2), \tilde{J}Z_1) = \tilde{g}(\nabla_{Y_1}fY_2, \tilde{J}Z_1) \\ &= -QY_2(\ln \lambda)g(Y_1, Z_1). \end{aligned}$$

Next, for  $Y_1 \in \Gamma(D)$  and  $Y_2 \in \Gamma(D')$  or  $Y_1, Y_2 \in \Gamma(D')$ , using Eq. (20), we have

$$(54) \quad \tilde{g}((\nabla_{Y_1}^t \omega)Y_2, \tilde{J}Z_1) = \tilde{g}(Fh(Y_1, Y_2), \tilde{J}Z_1) = 0,$$

where  $Z_1 \in \Gamma(D')$ . Thus from Eqs. (52)–(54), we derive Eq. (51). As  $\mu = \ln \lambda$  is a function defined on  $K_T$ , we get  $Z_1(\mu) = Z_1(\ln \lambda) = 0$  for  $Z_1 \in \Gamma(D')$ .

Conversely, let  $K$  be a GCR-lightlike submanifold of  $\tilde{K}$  with integrable distribution  $D'$ , satisfying Eq. (51). For any  $Y_1, Y_2 \in \Gamma(D)$  and  $Z_1 \in \Gamma(D')$ , using Eq. (51), we get  $\tilde{g}(\omega\nabla_{Y_1}Y_2, \tilde{J}Z_1) = 0$ , thus  $g(\nabla_{Y_1}Y_2, Z_1) = 0$ , which gives  $\nabla_{Y_1}Y_2 \in \Gamma(D)$ . Thus,  $D$  defines a totally geodesic foliation in  $K$ . On the other hand, for any  $Y_2 \in \Gamma(D_0)$  and  $Y_1, Z_1 \in \Gamma(D')$ , from Eq. (51), we have

$$(55) \quad \begin{aligned} -Y_2(\mu)g(Y_1, Z_1) &= \tilde{g}((\nabla_{Y_1}^t \omega)Y_2, \tilde{J}Z_1) = -\tilde{g}(\omega\nabla_{Y_1}Y_2, \tilde{J}Z_1) \\ &= -g(\nabla_{Y_1}Y_2, Z_1) = -\tilde{g}(\tilde{\nabla}_{Y_1}Y_2, Z_1) = g(Y_2, \nabla_{Y_1}Z_1). \end{aligned}$$

Then, from the definition of gradient, one has  $g(\nabla\phi, Y_2) = Y_2\phi$ , and using it in Eq. (55), we obtain

$$(56) \quad g(\nabla_{Y_1}Z_1, Y_2) = -g(\nabla\mu, Y_2)g(Y_1, Z_1).$$

Let  $h'$  and  $\nabla'$ , respectively, denote the second fundamental form and the induced connection of  $D'$  on  $K$ , then

$$(57) \quad g(h'(Y_1, Z_1), Y_2) = g(\nabla_{Y_1}Z_1 - \nabla'_{Y_1}Z_1, Y_2) = g(\nabla_{Y_1}Z_1, Y_2),$$

where  $Y_1, Z_1 \in \Gamma(D')$  and  $Y_2 \in \Gamma(D_0)$ . Now from Eqs. (56) and (57), we derive

$$(58) \quad g(h'(Y_1, Z_1), Y_2) = -g(\nabla\mu, Y_2)g(Y_1, Z_1).$$

Then, using the non-degeneracy of  $D_0$ , we have

$$(59) \quad h'(Y_1, Z_1) = -\nabla\mu g(Y_1, Z_1),$$

which implies that  $D'$  is totally umbilical in  $K$ , and by hypothesis, it is clear that  $D'$  is integrable. Further, in view of the condition that  $Z_1\mu = 0$  for

$Z_1 \in \Gamma(D')$ , each leaf of  $D'$  is an intrinsic sphere. Therefore, using similar argument as in Theorem 3.4,  $K$  is a locally *GCR*-lightlike **w.p.** submanifold of the type  $K_T \times_\lambda K_\perp$  in  $\tilde{K}$  with warping function  $\lambda = e^\mu$ . This completes the proof.  $\square$

Next, we present the following lemma.

**Lemma 4.4.** *Let  $K$  be a proper *GCR*-lightlike **w.p.** submanifold of  $\tilde{K}$ . Then*

$$(60) \quad \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = (Y_1 \ln \lambda) \|Z_1\|^2,$$

$$(61) \quad \tilde{g}(h^s(Y_1, Z_1), \tilde{J}Z_1) = -(\tilde{J}Y_1 \ln \lambda) \|Z_1\|^2,$$

for  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ .

*Proof.* Employing Eqs. (5), (13) and (23), for  $Y_1 \in \Gamma(D)$  and  $Z_1 \in \Gamma(D')$ , we obtain

$$\begin{aligned} \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= \tilde{g}(\tilde{\nabla}_{Z_1} \tilde{J}Y_1, \tilde{J}Z_1) - (\tilde{J}Y_1 \ln \lambda) \tilde{g}(Z_1, \tilde{J}Z_1) \\ &= \tilde{g}(\tilde{J} \tilde{\nabla}_{Z_1} Y_1, \tilde{J}Z_1) \\ &= g(\nabla_{Z_1} Y_1, Z_1) \\ &= (Y_1 \ln \lambda) \|Z_1\|^2, \end{aligned}$$

which proves (60). Now, using again Eqs. (5), (13) and (23),

$$\begin{aligned} \tilde{g}(h^s(Y_1, Z_1), \tilde{J}Z_1) &= \tilde{g}(\tilde{\nabla}_{Z_1} Y_1, \tilde{J}Z_1) - (Y_1 \ln \lambda) \tilde{g}(Z_1, \tilde{J}Z_1) \\ &= -\tilde{g}(Y_1, \tilde{\nabla}_{Z_1} \tilde{J}Z_1) \\ &= -\tilde{g}(Y_1, \tilde{J} \tilde{\nabla}_{Z_1} Z_1) \\ &= g(\tilde{J}Y_1, \tilde{\nabla}_{Z_1} Z_1) \\ &= -g(\nabla_{Z_1} \tilde{J}Y_1, Z_1) \\ &= -(\tilde{J}Y_1 \ln \lambda) \|Z_1\|^2, \end{aligned}$$

which proves (61).  $\square$

**Theorem 4.5.** *Let  $K$  be a proper *GCR*-lightlike **w.p.** submanifold of  $\tilde{K}$ . Then*

$$(62) \quad \|h^s(\tilde{J}Y_1, Z_1)\|^2 + \|h^s(Y_1, Z_1)\|^2 = (Y_1 \ln \lambda)^2 \|Z_1\|^2 + (\tilde{J}Y_1 \ln \lambda)^2 \|Z_1\|^2 + 2\tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1))$$

for  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ .

*Proof.* Employing Eqs. (5), (23) and (60), for  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ , we derive

$$\begin{aligned} \|h^s(\tilde{J}Y_1, Z_1)\|^2 &= \tilde{g}(h^s(\tilde{J}Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)) = \tilde{g}(\tilde{\nabla}_{Z_1} \tilde{J}Y_1, h^s(\tilde{J}Y_1, Z_1)) \\ &= \tilde{g}(\tilde{J} \tilde{\nabla}_{Z_1} Y_1, h^s(\tilde{J}Y_1, Z_1)) \\ &= \tilde{g}(\tilde{J} \nabla_{Z_1} Y_1, h^s(\tilde{J}Y_1, Z_1)) + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)) \end{aligned}$$

$$\begin{aligned}
 &= (Y_1 \ln \lambda) \tilde{g}(\tilde{J}Z_1, h^s(\tilde{J}Y_1, Z_1)) + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)) \\
 (63) \quad &= (Y_1 \ln \lambda)^2 \|Z_1\|^2 + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)).
 \end{aligned}$$

Similarly, from Eqs. (5), (23) and (61), we obtain

$$\begin{aligned}
 \|h^s(Y_1, Z_1)\|^2 &= \tilde{g}(h^s(Y_1, Z_1), h^s(Y_1, Z_1)) = \tilde{g}(\tilde{\nabla}_{Z_1} Y_1, h^s(Y_1, Z_1)) \\
 &= \tilde{g}(\tilde{J}\tilde{\nabla}_{Z_1} Y_1, \tilde{J}h^s(Y_1, Z_1)) = \tilde{g}(\tilde{\nabla}_{Z_1} \tilde{J}Y_1, \tilde{J}h^s(Y_1, Z_1)) \\
 &= \tilde{g}(\nabla_{Z_1} \tilde{J}Y_1 + h^s(Z_1, \tilde{J}Y_1), \tilde{J}h^s(Y_1, Z_1)) \\
 &= (\tilde{J}Y_1 \ln \lambda) \tilde{g}(Z_1, \tilde{J}h^s(Y_1, Z_1)) + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)) \\
 (64) \quad &= (\tilde{J}Y_1 \ln \lambda)^2 \|Z_1\|^2 + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)).
 \end{aligned}$$

Hence the result follows from Eqs. (63) and (64). □

Consider a semi-Riemannian manifold  $(\tilde{K}_m, \tilde{g})$  with a smooth function  $\lambda$  defined on  $\tilde{K}$ . Then the Hessian of  $\lambda$  is

$$(65) \quad H^\lambda(Y, Z) = YZ\lambda - (\nabla_Y Z)\lambda$$

for any  $Y, Z \in \Gamma(T\tilde{K})$ .

Next, we give a characterization result of GCR-lightlike warped products in  $\tilde{K}(c)$  involving the second fundamental form and the Hessian of the warping function  $\lambda$ .

**Theorem 4.6.** *Let  $K$  be a GCR-lightlike  $w.p.$  submanifold in  $\tilde{K}(c)$ . Then for  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ , one has*

$$\begin{aligned}
 &\|h^s(\tilde{J}Y_1, Z_1)\|^2 + \|h^s(Y_1, Z_1)\|^2 \\
 &= \{H^{\ln \lambda}(Y_1, Y_1) + H^{\ln \lambda}(\tilde{J}Y_1, \tilde{J}Y_1)\} \|Z_1\|^2 \\
 &\quad + \frac{c}{2} \{ \tilde{g}(\tilde{J}Y_1, Z_1)^2 + \tilde{g}(Y_1, Z_1)^2 + \|Y_1\|^2 \|Z_1\|^2 \} \\
 &\quad + (Y_1 \ln \lambda)^2 \|Z_1\|^2 + (\tilde{J}Y_1 \ln \lambda)^2 \|Z_1\|^2 \\
 &\quad + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^s(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1) \\
 (66) \quad &\quad + \tilde{g}(A_{h^i(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^i(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1).
 \end{aligned}$$

*Proof.* For  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ , taking in account Eq. (14), we get

$$(67) \quad \tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) = -\frac{c}{2} \{ \tilde{g}(\tilde{J}Y_1, Z_1)^2 + \tilde{g}(Y_1, Z_1)^2 + \|Y_1\|^2 \|Z_1\|^2 \}.$$

On the other hand, taking the inner product of Codazzi Eq. (9) with respect to  $\tilde{J}Z_1$ , for  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ , we obtain

$$\begin{aligned}
 &\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) \\
 &= \tilde{g}(\nabla_{Y_1}^s h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) - \tilde{g}(h^s(\nabla_{Y_1} \tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\
 &\quad - \tilde{g}(h^s(\tilde{J}Y_1, \nabla_{Y_1} Z_1), \tilde{J}Z_1) - \tilde{g}(\nabla_{\tilde{J}Y_1}^s h^s(Y_1, Z_1), \tilde{J}Z_1) \\
 &\quad + \tilde{g}(h^s(\nabla_{\tilde{J}Y_1} Y_1, Z_1), \tilde{J}Z_1) + \tilde{g}(h^s(Y_1, \nabla_{\tilde{J}Y_1} Z_1), \tilde{J}Z_1)
 \end{aligned}$$

$$(68) \quad + \tilde{g}(D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)), \tilde{J}Z_1) - \tilde{g}(D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)), \tilde{J}Z_1).$$

Next considering Eq. (6), we get

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_{Y_1}^s h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= -\tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) \\ &\quad + \tilde{g}(D^l(h^s(\tilde{J}Y_1, Z_1), Y_1), \tilde{J}Z_1) \\ &\quad + \tilde{g}(\nabla_{Y_1}^s h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1), \end{aligned}$$

which further yields

$$(69) \quad \begin{aligned} \tilde{g}(\nabla_{Y_1}^s h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= \tilde{g}(\tilde{\nabla}_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\ &\quad + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1). \end{aligned}$$

Since  $\tilde{\nabla}$  is a metric connection on  $\tilde{K}$ , we have

$$(70) \quad \begin{aligned} \tilde{g}(\tilde{\nabla}_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= Y_1 \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\ &\quad - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{\nabla}_{Y_1} \tilde{J}Z_1), \end{aligned}$$

where  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ . Further, employing Eqs. (69) and (70), we attain

$$(71) \quad \begin{aligned} \tilde{g}(\nabla_{Y_1}^s h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= Y_1 \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}\tilde{\nabla}_{Y_1} Z_1) \\ &\quad + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1). \end{aligned}$$

Then, employing Eqs. (5), (23) and (60) in Eq. (71), we derive

$$\begin{aligned} \tilde{g}(\nabla_{Y_1}^s h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= Y_1 \{(Y_1 \ln \lambda) \|Z_1\|^2\} - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}\tilde{\nabla}_{Y_1} Z_1) \\ &\quad + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) \\ &= Y_1(Y_1 \ln \lambda) \|Z_1\|^2 + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) \\ &\quad + 2(Y_1 \ln \lambda)^2 \|Z_1\|^2 - (Y_1 \ln \lambda) \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\ &\quad - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}h^s(Y_1, Z_1)) \\ &= Y_1(Y_1 \ln \lambda) \|Z_1\|^2 + 2(Y_1 \ln \lambda)^2 \|Z_1\|^2 \\ &\quad + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - (Y_1 \ln \lambda)^2 \|Z_1\|^2 \\ &\quad - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}h^s(Y_1, Z_1)) \\ &= Y_1(Y_1 \ln \lambda) \|Z_1\|^2 + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) \\ &\quad + (Y_1 \ln \lambda)^2 \|Z_1\|^2 - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}h^s(Y_1, Z_1)), \end{aligned}$$

which, on using Eq. (63), reduces to

$$(72) \quad \begin{aligned} \tilde{g}(\nabla_{Y_1}^s h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= Y_1(Y_1 \ln \lambda) \|Z_1\|^2 + 2(Y_1 \ln \lambda)^2 \|Z_1\|^2 \\ &\quad - \|h^s(\tilde{J}Y_1, Z_1)\|^2 + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1). \end{aligned}$$

Similarly, one has

$$\tilde{g}(\nabla_{\tilde{J}Y_1}^s h^s(Y_1, Z_1), \tilde{J}Z_1) = -\tilde{J}Y_1(\tilde{J}Y_1 \ln \lambda) \|Z_1\|^2 - 2(\tilde{J}Y_1 \ln \lambda)^2 \|Z_1\|^2$$

$$(73) \quad + \|h^s(Y_1, Z_1)\|^2 + \tilde{g}(A_{h^s(Y_1, Z_1)}\tilde{J}Y_1, \tilde{J}Z_1).$$

Further, from Eqs. (6), (13) and (23), we acquire

$$(74) \quad \begin{aligned} g(A_{\tilde{J}Z_1}Z_1, \tilde{J}Y_1) &= -\tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Z_1, \tilde{J}Y_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}Y_1) \\ &= -\tilde{g}(\tilde{\nabla}_{Z_1}Z_1, Y_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}Y_1) \\ &= \tilde{g}(Z_1, \tilde{\nabla}_{Z_1}Y_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}Y_1) \\ &= (Y_1 \ln \lambda)\|Z_1\|^2 + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}Y_1). \end{aligned}$$

Since  $K_T$  is totally geodesic, for  $Y_1 \in \Gamma(TK_T)$ , one has  $\nabla_{Y_1}Y_1 \in \Gamma(TK_T)$ . Then replacing  $Y_1$  by  $\nabla_{Y_1}Y_1$  in Eq. (74), we obtain

$$(75) \quad g(A_{\tilde{J}Z_1}Z_1, \tilde{J}\nabla_{Y_1}Y_1) = (\nabla_{Y_1}Y_1 \ln \lambda)\|Z_1\|^2 + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}\nabla_{Y_1}Y_1).$$

Also, one has  $h(Y_1, V) = 0$  and  $\nabla_{Y_1}V \in \Gamma(D)$  for  $Y_1, V \in \Gamma(D)$ . Then, using Eqs. (5), (6) and (13), for  $Z_1 \in \Gamma(D')$  and  $Y_1 \in \Gamma(D)$ , we get

$$(76) \quad \begin{aligned} g(A_{\tilde{J}Z_1}Z_1, \tilde{J}\nabla_{Y_1}Y_1) &= \tilde{g}(A_{\tilde{J}Z_1}Z_1, \tilde{J}\tilde{\nabla}_{Y_1}Y_1) = \tilde{g}(A_{\tilde{J}Z_1}Z_1, \tilde{\nabla}_{Y_1}\tilde{J}Y_1) \\ &= g(A_{\tilde{J}Z_1}Z_1, \nabla_{Y_1}\tilde{J}Y_1) \\ &= -\tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Z_1, \nabla_{Y_1}\tilde{J}Y_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \nabla_{Y_1}\tilde{J}Y_1) \\ &= \tilde{g}(\tilde{J}Z_1, \tilde{\nabla}_{Z_1}\nabla_{Y_1}\tilde{J}Y_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \nabla_{Y_1}\tilde{J}Y_1) \\ &= \tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1, Z_1), \tilde{J}Z_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \nabla_{Y_1}\tilde{J}Y_1). \end{aligned}$$

Then, from Eqs. (75) and (76), we attain

$$(77) \quad \tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = (\nabla_{Y_1}Y_1 \ln \lambda)\|Z_1\|^2.$$

By writing  $Y_1$  in place of  $\tilde{J}Y_1$ , Eq. (77) yields

$$(78) \quad \tilde{g}(h^s(\nabla_{\tilde{J}Y_1}Y_1, Z_1), \tilde{J}Z_1) = -(\nabla_{\tilde{J}Y_1}\tilde{J}Y_1 \ln \lambda)\|Z_1\|^2.$$

From Eqs. (23) and (60), we get

$$(79) \quad \begin{aligned} \tilde{g}(h^s(\tilde{J}Y_1, \nabla_{Y_1}Z_1), \tilde{J}Z_1) &= (Y_1 \ln \lambda)\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\ &= (Y_1 \ln \lambda)^2\|Z_1\|^2. \end{aligned}$$

In the same way, using Eqs. (23) and (61), we derive

$$(80) \quad \begin{aligned} \tilde{g}(h^s(Y_1, \nabla_{\tilde{J}Y_1}Z_1), \tilde{J}Z_1) &= (\tilde{J}Y_1 \ln \lambda)\tilde{g}(h^s(Y_1, Z_1), \tilde{J}Z_1) \\ &= -(\tilde{J}Y_1 \ln \lambda)^2\|Z_1\|^2. \end{aligned}$$

Further, considering Eqs. (5), (8) and (13), we obtain

$$\begin{aligned} &\tilde{g}(D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)), \tilde{J}Z_1) \\ &= \tilde{g}(\tilde{\nabla}_{Y_1}h^l(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) + \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)}Y_1, \tilde{J}Z_1) \\ &= -\tilde{g}(\tilde{\nabla}_{Y_1}\tilde{J}Z_1, h^l(\tilde{J}Y_1, Z_1)) + \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)}Y_1, \tilde{J}Z_1) \\ &= -\tilde{g}(\tilde{J}\tilde{\nabla}_{Y_1}Z_1, h^l(\tilde{J}Y_1, Z_1)) + \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)}Y_1, \tilde{J}Z_1) \end{aligned}$$

$$\begin{aligned}
&= -\tilde{g}(\tilde{J}\nabla_{Y_1} Z_1, h^l(\tilde{J}Y_1, Z_1)) + \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) \\
&\quad - \tilde{g}(\tilde{J}h^l(Y_1, Z_1), h^l(\tilde{J}Y_1, Z_1)) \\
(81) \quad &= \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1).
\end{aligned}$$

Similarly,

$$(82) \quad \tilde{g}(D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)), \tilde{J}Z_1) = \tilde{g}(A_{h^l(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1).$$

Now, employing Eqs. (72), (73), (77), (78), (79), (80), (81) and (82) in Eq. (68), we derive

$$\begin{aligned}
&\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) \\
&= \{Y_1(Y_1 \ln \lambda) - \nabla_{Y_1} Y_1 \ln \lambda + \tilde{J}Y_1(\tilde{J}Y_1 \ln \lambda)\} \|Z_1\|^2 \\
&\quad - \nabla_{\tilde{J}Y_1} \tilde{J}Y_1 \ln \lambda \|Z_1\|^2 + (Y_1 \ln \lambda)^2 \|Z_1\|^2 + (\tilde{J}Y_1 \ln \lambda)^2 \|Z_1\|^2 \\
&\quad - \|h^s(\tilde{J}Y_1, Z_1)\|^2 - \|h^s(Y_1, Z_1)\|^2 + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) \\
&\quad - \tilde{g}(A_{h^s(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1) + \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) \\
(83) \quad &- \tilde{g}(A_{h^l(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1).
\end{aligned}$$

Next, using Eq. (65) in Eq. (83), we obtain

$$\begin{aligned}
&\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) \\
&= \{H^{\ln \lambda}(Y_1, Y_1) + H^{\ln \lambda}(\tilde{J}Y_1, \tilde{J}Y_1)\} \|Z_1\|^2 + (Y_1 \ln \lambda)^2 \|Z_1\|^2 \\
&\quad + (\tilde{J}Y_1 \ln \lambda)^2 \|Z_1\|^2 - \|h^s(\tilde{J}Y_1, Z_1)\|^2 - \|h^s(Y_1, Z_1)\|^2 \\
&\quad + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^s(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1) \\
(84) \quad &+ \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^l(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1).
\end{aligned}$$

Further, from Eqs. (67) and (84), we derive

$$\begin{aligned}
&- \frac{c}{2} \{\tilde{g}(\tilde{J}Y_1, Z_1)^2 + \tilde{g}(Y_1, Z_1)^2 + \|Y_1\|^2 \|Z_1\|^2\} \\
&= H^{\ln \lambda}(Y_1, Y_1) \|Z_1\|^2 + H^{\ln \lambda}(\tilde{J}Y_1, \tilde{J}Y_1) \|Z_1\|^2 \\
&\quad + \{(Y_1 \ln \lambda)^2 + (\tilde{J}Y_1 \ln \lambda)^2\} \|Z_1\|^2 \\
&\quad - \|h^s(\tilde{J}Y_1, Z_1)\|^2 - \|h^s(Y_1, Z_1)\|^2 \\
&\quad + \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^s(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1) \\
(85) \quad &+ \tilde{g}(A_{h^l(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^l(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\|h^s(\tilde{J}Y_1, Z_1)\|^2 + \|h^s(Y_1, Z_1)\|^2 \\
&= \{H^{\ln \lambda}(Y_1, Y_1) + H^{\ln \lambda}(\tilde{J}Y_1, \tilde{J}Y_1)\} \|Z_1\|^2 \\
&\quad + \frac{c}{2} \{\tilde{g}(\tilde{J}Y_1, Z_1)^2 + \tilde{g}(Y_1, Z_1)^2 + \|Y_1\|^2 \|Z_1\|^2\}
\end{aligned}$$



$$\begin{aligned}
 &+ \{(Y_1 \ln \lambda)^2 + (\tilde{J}Y_1 \ln \lambda)^2\} \|Z_1\|^2 \\
 &+ \tilde{g}(A_{h^s(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^s(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1) \\
 &+ \tilde{g}(A_{h^t(\tilde{J}Y_1, Z_1)} Y_1, \tilde{J}Z_1) - \tilde{g}(A_{h^t(Y_1, Z_1)} \tilde{J}Y_1, \tilde{J}Z_1).
 \end{aligned}$$

Hence the desirable outcome is accomplished. □

**Corollary 4.7.** *Let  $K = K_T \times_\lambda K_\perp$  be a GCR-lightlike  $w. p.$  submanifold in  $\tilde{K}(c)$ . Then*

$$\begin{aligned}
 \|h^s(\tilde{J}Y_1, Z_1)\|^2 + \|h^s(Y_1, Z_1)\|^2 &= H^{\ln \lambda}(Y_1, Y_1) + H^{\ln \lambda}(\tilde{J}Y_1, \tilde{J}Y_1) + \frac{c}{2} \\
 &+ (Y_1 \ln \lambda)^2 + (\tilde{J}Y_1 \ln \lambda)^2
 \end{aligned}$$

for  $Z \in \Gamma(K_2)$  and  $Y \in \Gamma(D_0)$ .

*Proof.* Particularly, for unit vectors  $Z_1 \in \Gamma(K_2)$  and  $Y_1 \in \Gamma(D_0)$ , the result directly follows from Eq. (66). □

**Acknowledgement.** Sangeet Kumar wishes to thank Science and Engineering Research Board, GoI, New Delhi for the financial funding vide File No. ECR/2017/000786.

### References

- [1] M. Barros and A. Romero, *Indefinite Kähler manifolds*, Math. Ann. **261** (1982), no. 1, 55–62. <https://doi.org/10.1007/BF01456410>
- [2] J. K. Beem, P. E. Ehrlich, and T. G. Powell, *Warped product manifolds in relativity*, in Selected Studies: Physics-Astrophysics, Mathematics, History of Science, 41–56, North-Holland, Amsterdam, 1982.
- [3] R. L. Bishop and B. O’Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49. <https://doi.org/10.2307/1995057>
- [4] B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. **133** (2001), no. 3, 177–195. <https://doi.org/10.1007/s006050170019>
- [5] ———, *Differential Geometry of Warped Product Manifolds and Submanifolds*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. <https://doi.org/10.1142/10419>
- [6] K. L. Duggal, *Constant scalar curvature and warped product globally null manifolds*, J. Geom. Phys. **4** (2002), 327–340.
- [7] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and its Applications, 364, Kluwer Academic Publishers Group, Dordrecht, 1996. <https://doi.org/10.1007/978-94-017-2089-2>
- [8] K. L. Duggal and D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math. J. **26** (2003), no. 1, 49–68. <https://doi.org/10.2996/kmj/1050496648>
- [9] K. L. Duggal and B. Sahin, *Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds*, Acta Math. Hungar. **112** (2006), no. 1-2, 107–130. <https://doi.org/10.1007/s10474-006-0068-y>
- [10] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, London, 1973.
- [11] S. Hiepko, *Eine innere Kennzeichnung der verzerrten Produkte*, Math. Ann. **241** (1979), no. 3, 209–215. <https://doi.org/10.1007/BF01421206>

- [12] S. Kumar, *Geometry of warped product lightlike submanifolds of indefinite nearly Kaehler manifolds*, J. Geom. **109** (2018), no. 1, Paper No. 21, 18 pp. <https://doi.org/10.1007/s00022-018-0425-3>
- [13] ———, *Warped product semi-transversal lightlike submanifolds of indefinite nearly Kaehler manifolds*, Differ. Geom. Dyn. Syst. **20** (2018), 106–118.
- [14] ———, *Some characterization theorems on GCR-lightlike warped product submanifolds of indefinite nearly Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys. **17** (2020), no. 3, 2050039, 17 pp. <https://doi.org/10.1142/S0219887820500395>
- [15] R. Kumar, S. Kumar, and R. K. Nagaich, *GCR-lightlike product of indefinite Kaehler manifolds*, ISRN Geometry **2011** (2011), Article ID 531281, 13 pp.
- [16] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.
- [17] B. Şahin, *Warped product lightlike submanifolds*, Sarajevo J. Math. **14** (2005), no. 2, 251–260.

SANGEET KUMAR  
DEPARTMENT OF MATHEMATICS  
SRI GURU TEG BAHADUR KHALSA COLLEGE  
SRI ANANDPUR SAHIB - 140118  
RUPNAGAR, PUNJAB, INDIA  
Email address: [sp7maths@gmail.com](mailto:sp7maths@gmail.com)

MEGHA PRUTHI  
DEPARTMENT OF MATHEMATICS  
SRI GURU TEG BAHADUR KHALSA COLLEGE  
SRI ANANDPUR SAHIB - 140118  
RUPNAGAR, PUNJAB, INDIA  
Email address: [meghapruthi4194@gmail.com](mailto:meghapruthi4194@gmail.com)