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AN EXTENSION OF THE WHITTAKER FUNCTION

Junesang Choi, Kottakkaran Sooppy Nisar, and Gauhar Rahman

Abstract. The Whittaker function and its diverse extensions have been actively investigated. Here we aim to introduce an extension of the Whittaker function by using the known extended confluent hypergeometric function $\Phi_{p,v}$ and investigate some of its formulas such as integral representations, a transformation formula, Mellin transform, and a differential formula. Some special cases of our results are also considered.

1. Introduction and preliminaries

We begin by recalling the classical beta function (see, e.g., [9, p. 8])

$$B(\mu,\,\nu)=\begin{cases}\displaystyle\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\,\mathrm{d}t & (\min\{\Re(\mu),\,\Re(\nu)\}>0)\\ \displaystyle\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)} & \left(\mu,\,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).\end{cases}\tag{1.1}$$

Here and elsewhere, let $\mathbb{N}$, $\mathbb{Z}_0^-$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$, be the sets of positive integers, non-positive integers, positive real numbers, real numbers, and complex numbers, respectively. Also put $\mathbb{R}_0^+ := \mathbb{R}^+\cup\{0\}$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

The Gauss hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1\Phi_1$ are defined by (see, e.g., [8]; see also [9, Section 1.5])

$${}_2F_1(\sigma_1,\sigma_2;\sigma_3;z)=\sum_{n=0}^{\infty}\frac{(\sigma_1)_n(\sigma_2)_n}{(\sigma_3)_n}\frac{z^n}{n!}\tag{1.2}$$

$$\Big(\sigma_1,\,\sigma_2\in\mathbb{C},\,\sigma_3\in\mathbb{C}\setminus\mathbb{Z}_0^-;\,|z|<1\Big)$$

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and

$$
{}_1F_1(\sigma_2;\sigma_3;z) = \Phi(\sigma_2;\sigma_3;z) = \sum_{n=0}^{\infty} \frac{(\sigma_2)_n}{(\sigma_3)_n} \frac{z^n}{n!} \tag{1.3}
$$
$$
\Big(\sigma_2 \in \mathbb{C},\, \sigma_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, z \in \mathbb{C}\Big),
$$

where $(\lambda)_n$ denotes the Pochhammer symbol (see, e.g., [9, Section 1.1]). The well known integral representations of the hypergeometric function and the confluent hypergeometric functions are recalled (see, e.g., [9, Section 1.5])

$$
{}_2F_1(\sigma_1,\sigma_2;\sigma_3;z) = \frac{\Gamma(\sigma_3)}{\Gamma(\sigma_2)\Gamma(\sigma_3-\sigma_2)} \int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}(1-zt)^{-\sigma_1}\, \mathrm{d}t \tag{1.4}
$$
$$
\big(\Re(\sigma_3) > \Re(\sigma_2) > 0,\, |\arg(1-z)| < \pi\big)
$$

and

$$
\Phi(\sigma_2;\sigma_3;z) = \frac{\Gamma(\sigma_3)}{\Gamma(\sigma_2)\Gamma(\sigma_3-\sigma_2)} \int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}\, \mathrm{e}^{zt}\, \mathrm{d}t \tag{1.5}
$$
$$
\big(\Re(\sigma_3) > \Re(\sigma_2) > 0\big)
$$

In last several decades, various extensions of some well-known special functions have been investigated. For example, Chaudhary et al. [1] introduced the following extended beta function

$$
B(\sigma_1,\sigma_2;p) = B_p(\sigma_1,\sigma_2) = \int_0^1 t^{\sigma_1-1}(1-t)^{\sigma_2-1}\, \mathrm{e}^{-\frac{p}{t(1-t)}}\, \mathrm{d}t \tag{1.6}
$$
$$
\big(\min\{\Re(p),\, \Re(\sigma_1),\, \Re(\sigma_2)\} > 0\big).
$$

Obviously $B(\sigma_1,\sigma_2;0) = B(\sigma_1,\sigma_2)$. Also, by using (1.6), Chaudhry et al. [2] introduced the extended hypergeometric function F_p and the confluent hypergeometric function Φ_p

$$
F_p(\sigma_1,\sigma_2;\sigma_3;z) = \sum_{n=0}^{\infty} \frac{B_p(\sigma_2+n,\sigma_3-\sigma_2)}{B(\sigma_2,\sigma_3-\sigma_2)} (\sigma_1)_n \frac{z^n}{n!} \tag{1.7}
$$
$$
\big(p \geq 0,\, |z| < 1,\, \Re(\sigma_3) > \Re(\sigma_2) > 0\big)
$$

and

$$
\Phi_p(\sigma_2;\sigma_3;z) = \sum_{n=0}^{\infty} \frac{B_p(\sigma_2+n,\sigma_3-\sigma_2)}{B(\sigma_2,\sigma_3-\sigma_2)} \frac{z^n}{n!} \tag{1.8}
$$
$$
\big(p \geq 0, \Re(\sigma_3) > \Re(\sigma_2) > 0\big).
$$

They [2] presented the following integral representations

$$
\begin{aligned}
F_p(\sigma_1,\sigma_2;\sigma_3;z) &= \frac{1}{B(\sigma_2,\sigma_3-\sigma_2)} \\
&\quad \times \int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}(1-zt)^{-\sigma_1} \exp\left(\frac{-p}{t(1-t)}\right) \mathrm{d}t
\end{aligned} \tag{1.9}
$$

$$\Big(p \in \mathbb{R}_0^+,\, \Re(\sigma_3) > \Re(\sigma_2) > 0,\, |\arg(1-z)| < \pi\Big)$$

and

$$\Phi_p(\sigma_2;\sigma_3;z) = \frac{1}{B(\sigma_2,\sigma_3-\sigma_2)} \times \int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}\exp\Big(zt - \frac{p}{t(1-t)}\Big)\,\mathrm{d}t \tag{1.10}$$

$$\Big(p \in \mathbb{R}_0^+,\, \Re(\sigma_3) > \Re(\sigma_2) > 0\Big).$$

Clearly, (1.7)–(1.10) when $p = 0$ reduce to (1.2)–(1.5), respectively.

Choi et al. [3] have introduced and investigated the following extended beta function

$$B(\sigma_1,\sigma_2;p,q) = B_{p,q}(\sigma_1,\sigma_2) = \int_0^1 t^{\sigma_1-1}(1-t)^{\sigma_2-1}\,\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,\mathrm{d}t \tag{1.11}$$

$$(\min\{\Re(p),\, \Re(q)\} > 0,\, \min\{\Re(\sigma_1),\, \Re(\sigma_2)\} > 0)\,.$$

Obviously, $B(\sigma_1,\sigma_2;p,p) = B(\sigma_1,\sigma_2;p)$ in (1.6) and $B(\sigma_1,\sigma_2;0,0) = B(\sigma_1,\sigma_2)$ in (1.1). They [3] have introduced the following extended (p,q)-hypergeometric function and extended (p,q)-confluent hypergeometric function defined, respectively, by

$$F_{p,q}(\sigma_1,\sigma_2;\sigma_3;z) = \sum_{n=0}^{\infty} \frac{B_{p,q}(\sigma_2+n,\sigma_3-\sigma_2)}{B(\sigma_2,\sigma_3-\sigma_2)}(\sigma_1)_n\frac{z^n}{n!} \tag{1.12}$$

$$\big(p,\, q \in \mathbb{R}_0^+,\, \Re(\sigma_3) > \Re(\sigma_2) > 0\big)$$

and

$$\Phi_{p,q}(\sigma_2;\sigma_3;z) = \sum_{n=0}^{\infty} \frac{B_{p,q}(\sigma_2+n,\sigma_3-\sigma_2)}{B(\sigma_2,\sigma_3-\sigma_2)}\frac{z^n}{n!} \tag{1.13}$$

$$\big(p,\, q \in \mathbb{R}_0^+,\, \Re(\sigma_3) > \Re(\sigma_2) > 0\big)\,.$$

They [3] presented the following integral representations

$$F_{p,q}(\sigma_1,\sigma_2;\sigma_3;z) = \frac{1}{B(\sigma_2,\sigma_3-\sigma_2)} \times \int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}(1-zt)^{-\sigma_1}\exp\Big(-\frac{p}{t}-\frac{q}{1-t}\Big)\,\mathrm{d}t \tag{1.14}$$

$$\big(p,\, q \in \mathbb{R}_0^+,\, \Re(\sigma_3) > \Re(\sigma_2) > 0,\, |\arg(1-z)| < \pi\big)$$

and

$$\Phi_{p,q}(\sigma_2;\sigma_3;z) = \frac{1}{B(\sigma_2,\sigma_3-\sigma_2)} \times \int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}\exp\Big(zt-\frac{p}{t}-\frac{q}{1-t}\Big)\,\mathrm{d}t \tag{1.15}$$

$$\big(p,\, q \in \mathbb{R}_0^+,\, \Re(\sigma_3) > \Re(\sigma_2) > 0\big)\,.$$

Parmar et al. [6, Eq. (1.13)] introduced the following extended beta function

$$B_v(\sigma_1,\sigma_2;p)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{\sigma_1-\frac{3}{2}}(1-t)^{\sigma_2-\frac{3}{2}}K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\mathrm{d}t \tag{1.16}$$

for $\Re(p)>0$, where $K_v(\cdot)$ is the modified Bessel function of order v. By recalling the following identity (see, e.g., [5, Entry 10.39.2])

$$K_{\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}, \tag{1.17}$$

it is obvious that $B_0(\sigma_1,\sigma_2;p)=B(\sigma_1,\sigma_2;p)$ in (1.6). They [6] defined the following extended hypergeometric function $F_{p,v}$ and extended confluent hypergeometric function $\Phi_{p,v}$

$$F_{p,v}\Big(\sigma_1,\sigma_2;\sigma_3;z\Big)=\sum_{n=0}^{\infty}(\sigma_1)_n\frac{B_v(\sigma_2+n,\sigma_3-\sigma_2;p)}{B(\sigma_2,\sigma_3-\sigma_2)}\frac{z^n}{n!} \tag{1.18}$$

$$\Big(p,\,v\in\mathbb{R}_0^+,\,\Re(\sigma_3>\Re(\sigma_2)>0,\,|z|<1\Big)$$

and

$$\Phi_{p,v}\Big(\sigma_2;\sigma_3;z\Big)=\sum_{n=0}^{\infty}\frac{B_v(\sigma_2+n,\sigma_3-\sigma_2;p)}{B(\sigma_2,\sigma_3-\sigma_2)}\frac{z^n}{n!} \tag{1.19}$$

$$\Big(p,\,v\in\mathbb{R}_0^+,\,\Re(\sigma_3)>\Re(\sigma_2)>0\Big)$$

and presented their integral representations

$$\begin{aligned}F_{p,v}\Big(\sigma_1,\sigma_2;\sigma_3;z\Big)&=\sqrt{\frac{2p}{\pi}}\frac{1}{B(\sigma_2,\sigma_3-\sigma_2)}\\&\quad\times\int_0^1 t^{\sigma_2-\frac{3}{2}}(1-t)^{\sigma_3-\sigma_2-\frac{3}{2}}(1-zt)^{-\sigma_1}K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\mathrm{d}t\end{aligned} \tag{1.20}$$

$$\Big(p,\,v\in\mathbb{R}_0^+,\,\Re(\sigma_3)>\Re(\sigma_2)>0,\,|\arg(1-z)|<\pi\Big)$$

and

$$\begin{aligned}\Phi_{p,v}\Big(\sigma_2;\sigma_3;z\Big)&=\sqrt{\frac{2p}{\pi}}\frac{1}{B(\sigma_2,\sigma_3-\sigma_2)}\\&\quad\times\int_0^1 t^{\sigma_2-\frac{3}{2}}(1-t)^{\sigma_3-\sigma_2-\frac{3}{2}}\exp(zt)K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\mathrm{d}t\end{aligned} \tag{1.21}$$

$$\Big(p,\,v\in\mathbb{R}_0^+,\,\Re(\sigma_3)>\Re(\sigma_2)>0\Big).$$

They also obtained the following transformation formula for the extended confluent hypergeometric function

$$\Phi_{p,v}(\sigma_2,\sigma_3;z)=\mathrm{e}^z\,\Phi_{p,v}\Big(\sigma_3-\sigma_2;\sigma_3;-z\Big). \tag{1.22}$$

Obviously, due to (1.17), equations (1.18)-(1.21) reduce, respectively, to (1.7)-(1.10).

Whittaker [11] introduced the so-called Whittaker function

$$M_{\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp\Big(-\frac{z}{2}\Big)\Phi\Big(\rho-\lambda+\frac{1}{2}; 2\rho+1; z\Big) \tag{1.23}$$

$$\left(\Re(\rho) > -\frac{1}{2},\ \Re(\rho\pm\lambda) > -\frac{1}{2};\ z\in\mathbb{C}\setminus(-\infty,0]\right),$$

where Φ is the confluent hypergeometric function in (1.3) and which is a modified solution of the Whittaker's equation so that formulas involving the solutions can be more symmetric (see, e.g., [12, Chapter XVI]; see also [10, p. 39]).

Nagar et al. [4] defined the following extended Whittaker function

$$M_{p,\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp\Big(-\frac{z}{2}\Big)\Phi_p\Big(\rho-\lambda+\frac{1}{2}; 2\rho+1; z\Big) \tag{1.24}$$

$$\left(p\in\mathbb{R}_0^+;\ \Re(\rho) > -\frac{1}{2};\ \Re(\rho\pm\lambda) > -\frac{1}{2};\ z\in\mathbb{C}\setminus(-\infty,0]\right),$$

where Φ_p is the extended confluent hypergeometric function in (1.8).

Rahman et al. [7] have introduced and investigated the following extended (p,q)-Whittaker function

$$M_{p,q,\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp\Big(-\frac{z}{2}\Big)\Phi_{p,q}\Big(\rho-\lambda+\frac{1}{2}; 2\rho+1; z\Big) \tag{1.25}$$

$$\left(p,\ q\in\mathbb{R}_0^+;\ \Re(\rho) > -\frac{1}{2};\ \Re(\rho\pm\lambda) > -\frac{1}{2};\ z\in\mathbb{C}\setminus(-\infty,0]\right),$$

where $\Phi_{p,q}$ is the extended (p,q)-confluent hypergeometric function in (1.13).

Here we introduce the following extended Whittaker function

$$M_{p,v,\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp\Big(-\frac{z}{2}\Big)\Phi_{p,v}\Big(\rho-\lambda+\frac{1}{2}; 2\rho+1; z\Big) \tag{1.26}$$

$$\left(p,\ v\in\mathbb{R}_0^+;\ \Re(\rho) > -\frac{1}{2};\ \Re(\rho\pm\lambda) > -\frac{1}{2};\ z\in\mathbb{C}\setminus(-\infty,0]\right),$$

where $\Phi_{p,v}$ is the extended confluent hypergeometric function in (1.19). Then we investigate certain formulas involving the extended Whittaker function (1.26) such as integral representations, a transformation formula, Mellin transform, and a differential formula. Some special cases of our results are also considered.

It is remarked in passing that $M_{p,0,\lambda,\rho}(z)$=$M_{p,\lambda,\rho}(z)$ in (1.24) and $M_{0,0,\lambda,\rho}(z)$ = $M_{\lambda,\rho}(z)$ in (1.23); From (1.22), the extended (p,v)-Whittaker function (1.26) can also be expressed in the following form

$$M_{p,v,\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp\Big(\frac{z}{2}\Big)\ \Phi_{p,v}\Big(\rho+\lambda+\frac{1}{2}; 2\rho+1; -z\Big). \tag{1.27}$$

2. Formulas involving the extended Whittaker function (1.26)

Here we establish certain formulas involving the extended Whittaker function (1.26) such as integral representations, a transformation formula, Mellin transform, and a differential formula. Some special cases of our results are also considered.

Theorem 1. *Let* $p,\ v \in \mathbb{R}_0^+$, $\Re(\rho) > \Re(\rho \pm \lambda) > -\frac{1}{2}$, *and* $z \in \mathbb{C} \setminus (-\infty, 0]$. *Also let* $a,\ b \in \mathbb{R}$ *with* $b > a$. *Then each of the following integral representations holds.*

$$M_{p,v,\lambda,\rho}(z) = \frac{z^{\rho+\frac{1}{2}} \exp(-\frac{z}{2})\sqrt{2p}}{\sqrt{\pi}\, B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \times \int_0^1 t^{\rho-\lambda-1}(1-t)^{\rho+\lambda-1} \exp(zt) K_{v+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big)\, \mathtt{d}t; \tag{2.1}$$

$$M_{p,v,\lambda,\rho}(z) = \frac{z^{\rho+\frac{1}{2}} \exp(\frac{z}{2})\sqrt{2p}}{\sqrt{\pi}\, B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \times \int_0^1 u^{\rho+\lambda-1}(1-u)^{\rho-\lambda-1} \exp(-zu) K_{v+\frac{1}{2}}\Big(\frac{p}{u(1-u)}\Big)\, \mathtt{d}u; \tag{2.2}$$

$$M_{p,v,\lambda,\rho}(z) = \frac{(b-a)^{-2\rho+1} z^{\rho+\frac{1}{2}} \exp(-\frac{z}{2})\sqrt{2p}}{\sqrt{\pi}\, B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \times \int_a^b (u-a)^{\rho-\lambda-1}(b-u)^{\rho+\lambda-1} \times \exp\Big(\frac{z(u-a)}{b-a}\Big) K_{v+\frac{1}{2}}\Big(\frac{p(b-a)^2}{(u-a)(b-u)}\Big)\, \mathtt{d}u; \tag{2.3}$$

$$M_{p,v,\lambda,\rho}(z) = \frac{z^{\rho+\frac{1}{2}} \exp(-\frac{z}{2})\sqrt{2p}}{\sqrt{\pi} B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \int_0^\infty u^{\rho-\lambda-1}(1+u)^{-2\rho} \times \exp\Big(\frac{zu}{1+u}\Big) K_{v+\frac{1}{2}}\Big(\frac{p(1+u)^2}{u}\Big)\, \mathtt{d}u; \tag{2.4}$$

$$M_{p,v,\lambda,\rho}(z) = \frac{2^{-2\rho+1} z^{\rho+\frac{1}{2}} \sqrt{2p}}{\sqrt{\pi} B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \int_{-1}^1 (1+u)^{\rho-\lambda-1}(1-u)^{\rho+\lambda-1} \times \exp\Big(\frac{zu}{2}\Big) K_{v+\frac{1}{2}}\Big(\frac{4p}{(1+u)(1-u)}\Big)\, \mathtt{d}u. \tag{2.5}$$

Proof. By using the integral representation (1.21) in the definition (1.26), we obtain (2.1). Now, by setting $t = 1-u$, $t = \frac{u-a}{b-a}$, and $t = \frac{u}{1+u}$ in (2.1), we get (2.2), (2.3), and (2.4), respectively. Setting $a = -1$ and $b = 1$ in (2.3) yields (2.5). □

Theorem 2. *The following transformation formula for the extended (p, v)-Whittaker function* (1.26) *holds.*

$$M_{p,v,\lambda,\rho}(-z) = (-1)^{\rho+\frac{1}{2}} M_{p,v,-\lambda,\rho}(z) \tag{2.6}$$

$$\left(p,\, v \in \mathbb{R}_0^+;\, \Re(\rho) > -\frac{1}{2};\, \Re(\rho \pm \lambda) > -\frac{1}{2};\, z \in \mathbb{C} \setminus \mathbb{R}\right).$$

Proof. Replacing z by $-z$ in (1.26) and using (1.27) , we get the desired result. □

Theorem 3. *The following Mellin transformation holds.*

$$\begin{aligned} &\mathfrak{M}\{M_{p,v,\lambda,\rho}(z); p \to r\} \\ &= \frac{z^{\rho+\frac{1}{2}} \exp(-\frac{z}{2}) 2^{r-1} \Gamma(\frac{r-v}{2}) \Gamma(\frac{r+v+1}{2}) B(\rho + r - \lambda + \frac{1}{2}, \rho + r + \lambda + \frac{1}{2})}{\sqrt{\pi}\, B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \\ &\quad \times \Phi\Big(\rho + r - \lambda + \frac{1}{2}; 2\rho + 2r + 1; z\Big) \end{aligned} \tag{2.7}$$

$$\left(\Re(r - v) > 0,\, \Re(r + v) > -1,\, \Re(\rho + r \pm \lambda) > \frac{1}{2},\, z \in \mathbb{C} \setminus (-\infty, 0]\right).$$

Proof. Using the integral representation in (2.1) and changing the order of integrations, we get

$$\begin{aligned} &\mathfrak{M}\{M_{p,v,\lambda,\rho}(z); p \to r\} \\ &:= \int_0^\infty p^{r-1} M_{p,v,\lambda,\rho}(z)\, dp \\ &= \frac{z^{\rho+\frac{1}{2}} \exp(-\frac{z}{2}) \sqrt{2}}{\sqrt{\pi} B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \int_0^1 t^{\rho-\lambda-1} (1-t)^{\rho+\lambda-1}\, \mathtt{e}^{zt} \\ &\quad \times \left\{ \int_0^\infty p^{r-\frac{1}{2}} K_{v+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big)\, \mathtt{d}p \right\} \mathtt{d}t. \end{aligned} \tag{2.8}$$

Using a known integral formula involving K_v (see, e.g., [5, Entry 10.43.19]; see also [6]), we have

$$\begin{aligned} &\int_0^\infty p^{r-\frac{1}{2}} K_{v+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big)\, \mathtt{d}p \\ &= t^{r+\frac{1}{2}} (1-t)^{r+\frac{1}{2}} \int_0^\infty u^{r-\frac{1}{2}} K_{v+\frac{1}{2}}\Big(u\Big)\, \mathtt{d}u \\ &= t^{r+\frac{1}{2}} (1-t)^{r+\frac{1}{2}} 2^{r-\frac{3}{2}} \Gamma\left(\frac{r-v}{2}\right) \Gamma\left(\frac{r+v+1}{2}\right) \end{aligned} \tag{2.9}$$

$$(\Re(r - v) > 0,\, \Re(r + v) > -1).$$

Using (2.9) in (2.8), we obtain

$$\tag{2.10} \begin{aligned} \mathfrak{M}\{M_{p,v,\lambda,\rho}(z); p \to r\} = & \frac{z^{\rho+\frac{1}{2}} \exp(-\frac{z}{2}) 2^{r-1} \Gamma(\frac{r-v}{2}) \Gamma(\frac{r+v+1}{2})}{\sqrt{\pi} B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \\ & \times \int_0^1 t^{\rho+r-\lambda-\frac{1}{2}} (1-t)^{\rho+r+\lambda-\frac{1}{2}} \, \mathsf{e}^{zt} \, \mathsf{d}t. \end{aligned}$$

Using (1.5), we find

$$\tag{2.11} \begin{aligned} & \int_0^1 t^{\rho+r-\lambda-\frac{1}{2}} (1-t)^{\rho+r+\lambda-\frac{1}{2}} \, \mathsf{e}^{zt} \, \mathsf{d}t \\ & = \frac{\Gamma(\rho + r - \lambda + \frac{1}{2}) \, \Gamma(\rho + r + \lambda + \frac{1}{2})}{\Gamma(2\rho + 2r - 1)} \, \Phi(\rho + r - \lambda + \frac{1}{2}; 2\rho + 2r + 1; z) \\ & \qquad \left(\Re(\rho + r \pm \lambda) > \frac{1}{2} \right). \end{aligned}$$

Applying (2.11) to (2.10), we obtain the desired result. □

Theorem 4. *Let* $p, v \in \mathbb{R}_0^+$, $2\alpha > \mu$, $\Re(\delta + \rho) > -\frac{1}{2}$, *and* $\Re(\rho \pm \lambda) > -\frac{1}{2}$. *Then*

$$\tag{2.12} \begin{aligned} & \int_0^\infty x^{\delta-1} \, \mathsf{e}^{-\alpha x} \, M_{p,v,\lambda,\rho}(\mu x) \, dx = \Gamma\left(\delta + \rho + \frac{1}{2}\right) \mu^{\rho+\frac{1}{2}} \\ & \times \left(\alpha + \frac{\mu}{2}\right)^{-\delta-\rho-\frac{1}{2}} F_{p,v}\left(\delta + \rho + \frac{1}{2}, \, \rho - \lambda + \frac{1}{2}; 2\rho + 1; \frac{2\mu}{2\alpha + \mu}\right). \end{aligned}$$

Proof. Let $\mathcal{L}$ be the left side of (2.12). Using the integral representation (2.1) and changing the order of integrations, which can be verified under the conditions here, we obtain

$$\tag{2.13} \begin{aligned} \mathcal{L} = & \sqrt{\frac{2p}{\pi}} \frac{\mu^{\rho+\frac{1}{2}}}{B\left(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2}\right)} \int_0^1 t^{\rho-\lambda-1} (1-t)^{\rho+\lambda-1} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \\ & \times \left[\int_0^\infty x^{\delta+\rho-\frac{1}{2}} \exp\left\{ -\left(\alpha + \frac{\mu}{2} - \mu t\right) x \right\} \mathsf{d}x \right] \mathsf{d}t. \end{aligned}$$

Using the Euler's gamma function (see, e.g., [9, Section 1.1]), we get

$$\tag{2.14} \int_0^\infty u^{\alpha-1} \exp(-\beta \, u) \, du = \beta^{-\alpha} \, \Gamma(\alpha) \quad \left(\Re(\alpha) > 0, \, \beta \in \mathbb{R}^+ \right).$$

Applying (2.14), we have

$$\tag{2.15} \begin{aligned} & \int_0^\infty x^{\delta+\rho-\frac{1}{2}} \exp\left\{ -\left(\alpha + \frac{\mu}{2} - \mu t\right) x \right\} \mathsf{d}x \\ & = \left(\alpha + \frac{\mu}{2} - \mu t\right)^{-\left(\delta+\rho+\frac{1}{2}\right)} \Gamma\left(\delta + \rho + \frac{1}{2}\right) \\ & \qquad \left(2\alpha > \mu, \, \Re(\delta + \rho) > -\frac{1}{2} \right). \end{aligned}$$

Substituting the integral formula (2.15) for the inner integral (2.13) and using (1.20), we obtain the desired result. □

Theorem 5. *Let* $p,\, v \in \mathbb{R}_0^+$, $\Re(\rho) > -\frac{1}{2}$, $\Re(\rho \pm \lambda) > -\frac{1}{2}$, *and* $z \in \mathbb{C} \setminus (-\infty, 0]$. *Also let* $n \in \mathbb{N}_0$. *Then*

$$\begin{aligned}&\frac{d^n}{dz^n}\Big\{\mathsf{e}^{\frac{z}{2}}\, z^{-\rho-\frac{1}{2}} M_{p,v,\lambda,\rho}(z)\Big\}\\ &= \frac{(\rho-\lambda+\frac{1}{2})_n}{(2\rho+1)_n}\, \mathsf{e}^{\frac{z}{2}}\, z^{-\rho-\frac{n}{2}-\frac{1}{2}} M_{p,v,\lambda+\frac{n}{2},\rho+\frac{n}{2}}(z).\end{aligned} \tag{2.16}$$

Proof. Applying the following known formula (see [6])

$$\frac{d^n}{dz^n}\Big\{\Phi_{p,v}(\sigma_2;\sigma_3;z)\Big\} = \frac{(\sigma_2)_n}{(\sigma_3)_n}\Phi_{p,v}(\sigma_2+n;\sigma_3+n;z) \quad (n \in \mathbb{N}_0) \tag{2.17}$$

to (1.26), we obtain the desired result. □

3. Special cases and remarks

The results presented here, being very general, can be specialized to yield a number of relatively simple identities. We demonstrate only two examples in the following corollaries.

Setting $v = 0$ in Theorem 3, in view of (1.17) and (1.23), we obtain:

Corollary 1. *Let* $\Re(r) > 0$, $\Re(\rho + r \pm \lambda) > \frac{1}{2}$, *and* $z \in \mathbb{C} \setminus (-\infty, 0]$. *Then*

$$\begin{aligned}&\mathfrak{M}\{M_{p,\lambda,\rho}(z); p \to r\}\\ &= \frac{z^{-r}\Gamma(r)\, B(\rho+r-\lambda+\frac{1}{2}, \rho+r+\lambda+\frac{1}{2})}{B(\rho-\lambda+\frac{1}{2}, \rho+\lambda+\frac{1}{2})} M_{\lambda,\rho+r}(z).\end{aligned} \tag{3.1}$$

Setting $v = 0$ and then $p = 0$ in the result in Theorem 4, we get:

Corollary 2. *Let* $2\alpha > \mu$, $\Re(\delta + \rho) > -\frac{1}{2}$, *and* $\Re(\rho \pm \lambda) > -\frac{1}{2}$. *Then*

$$\begin{aligned}&\int_0^\infty x^{\delta-1}\, \mathsf{e}^{-\alpha x}\, M_{\lambda,\rho}(\mu x)\, dx = \Gamma\left(\delta+\rho+\frac{1}{2}\right)\mu^{\rho+\frac{1}{2}}\\ &\times \left(\alpha+\frac{\mu}{2}\right)^{-\delta-\rho-\frac{1}{2}} {}_2F_1\Big(\delta+\rho+\frac{1}{2},\, \rho-\lambda+\frac{1}{2}; 2\rho+1; \frac{2\mu}{2\alpha+\mu}\Big).\end{aligned} \tag{3.2}$$

The main results presented here when ($v = 0$) and ($v = 0$ and then $p = 0$) are reduced to yield the corresponding results in [4] and the identities for the Whittaker function (see [12]), respectively.

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Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 38066, Korea
Email address: junesang@dongguk.ac.kr

Kottakkaran Sooppy Nisar
Department of Mathematics
College of Arts and Science-Wadi Aldawaser, 11991
Prince Sattam bin Abdulaziz University
Alkharj, Kingdom of Saudi Arabia
Email address: n.sooppy@psau.edu.sa; ksnisar1@gmail.com

Gauhar Rahman
Department of Mathematics and Statistics
Hazara University
Mansehra, Khyber Pakhtunkhwa 21120, Pakistan
Email address: gauhar55uom@gmail.com