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BOUNDED AND PERIODIC SOLUTIONS OF INHOMOGENEOUS LINEAR EVOLUTION EQUATIONS

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Abstract. The purpose of this paper is to prove unique existence of bounded solution and periodic solution to inhomogeneous linear evolution equations which trajectories of these solutions belong to given admissible Banach function space.

1. Introduction

The qualitative theory of linear differential equations

$$u'(t) = A(t)u(t) + f(t),$$

is always interesting and attractive topic. In finite-dimensional space, we can mention to the pioneering works of Lyapunov and Perron which characterise exponential stability and exponential dichotomy of linear differential equations, and Floquet theory to periodic linear differential equations. Most of the qualitative results to linear differential equations in finite-dimensional space are extended to Banach spaces with contributions of Daleckii, Krein, Massera, Schäffer, Bohl, Levitan, Zhikov,.... For more details about the qualitative theory of linear differential equations in Banach spaces, we refer the reader to [5,9]. In the case $A(t) \equiv A$ is the generator operator of a C_0-semigroup, the existence of bounded solution and periodic solution were proved by Kato, Naito and Shin [8].

In this paper, we consider inhomogeneous linear evolution equations

$$\begin{cases} u(t) = U(t,s)u(s) + \displaystyle\int_s^t U(t,\xi)f(\xi)d\xi \quad \text{ for } t \geq s, \\ u(s) = x \in X, \end{cases}$$

where $(U(t,s))_{t\geq s}$ is a family evolution on Banach space X (see Definition 2.2) and $f : \mathbb{R} \to X$, this is a generalized form to linear differential equations.

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Therefore, the qualitative results to linear evolution equations will also hold for linear differential equations.

The aim of the paper is to show that the inhomogeneous linear evolution equations have bounded solution and periodic solution uniquely in Banach space $\mathcal{E}$ with each $f \in \mathcal{E}$. To perform this aim, we use the concept of admissible Banach function space which was introduced in [7] and assume that the family evolution $(U(t,s))_{t\geq s}$ has exponential dichotomy on Banach space X.

Given a Banach space X and an admissible Banach function space E then we can define Banach space $\mathcal{E}$ (see Definition 2.1). The Banach space $\mathcal{E}$ is more general than bounded function space $(C_b(\mathbb{R}, X))$, e.g., the norm of a solution trajectory of evolution equations can be boundedness or belong to $L_p(\mathbb{R})$, or any an admissible Banach function space. Therefore, to study qualitative properties for evolution equations we can completely replace bounded function space by Banach space $\mathcal{E}$. This replacement can be considered the main contribution of our work.

The outline of the paper is as follows. In the second section, we recall the concept and some basic properties of admissible Banach function space and evolution family. The last section, we prove the unique existence of bounded solution and periodic solution in Banach space $\mathcal{E}$. These results are generalizations to the similar results presented in [5,12]. The unique existence of bounded solution is shown by using properties of admissible Banach function space. Using Massera's philosophy and properties of admissible Banach function space, we get the unique existence of periodic solution.

2. Admissible Banach function spaces and evolution family

2.1. Admissible Banach function spaces

Denote by $\mathcal{B}$ the Borel σ-algebra, λ the Lebesgue measure on $\mathbb{R}$ and $L_{1,\mathrm{loc}}(\mathbb{R})$ the space of real-valued locally integrable functions on $\mathbb{R}$ (modulo λ-null-functions). As already known, this space becomes a Fréchet space with the countable family of seminorms given by $p_n(f) := \int_n^{n+1} |f(t)|dt$ with $n \in \mathbb{Z}$. We can now define Banach function spaces as follows.

Definition. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ is called a *Banach function space* if:

i. E is a Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space and if $\varphi \in E$, and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, then $\psi \in E$ and $\|\psi\|_E \leq \|\varphi\|_E$.
ii. If $A \in \mathcal{B}$ has finite measure, then the characteristic function $\chi_A \in E$.
iii. $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R})$, $\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$.

Given a Banach space X on the field K ($K = \mathbb{R}$ or $\mathbb{C}$), we define Banach space $\mathcal{E}$ corresponding to Banach function space E and Banach space X as follows.

Definition. Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. The set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}, X) = \{f : \mathbb{R} \to X \mid f \text{ is strongly measurable and } \|f(\cdot)\| \in E\},$$

endowed with the norm $\|f\|_{\mathcal{E}} = \|\|f(\cdot)\|\|_E$ is a Banach space. We call $\mathcal{E}$ *the Banach space corresponding to the Banach function space E and the Banach space X.*

We now introduce the notion of admissibility in the following definition.

Definition. The Banach function space E is called *admissible* if:

(i) There is a constant $M \geq 1$ such that

$$\int_a^b |\varphi(t)|dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E$$

for any compact interval $[a, b] \subset \mathbb{R}$ and for all $\varphi \in E$.

(ii) For $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) = \int_t^{t+1} \varphi(\tau)d\tau$ belongs to E.

(iii) E is T_τ^+-invariant and T_τ^--invariant for $\tau \in \mathbb{R}_+$, where T_τ^+ and T_τ^- are defined by

$$T_\tau^+\varphi(t) = \varphi(t-\tau),\ T_\tau^-\varphi(t) = \varphi(t+\tau).$$

(iv) The linear operators T_τ^+, T_τ^- are uniformly bounded, i.e., there are positive constants N_1, N_2 such that

$$\|T_\tau^+\| \leq N_1,\ \|T_\tau^-\| \leq N_2 \ \text{ for all } \ \tau \in \mathbb{R}_+.$$

Example 2.1. The Banach function spaces $L_p(\mathbb{R})$, $1 \leq p \leq \infty$ are admissible. Besides, the space

$$\mathbf{M}(\mathbb{R}) := \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(\tau)|d\tau < \infty \right\},$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(\tau)|d\tau$ and many other function spaces occurring in interpolation theory, e.g., the Lorentz spaces $L_{p,\,q}(\mathbb{R}),\ 1 < p < \infty,\ 1 \leq q \leq \infty$ are also admissible.

Remark 2.2. One can easily see that if E is an admissible Banach function space, then $E \hookrightarrow \mathbf{M}(\mathbb{R})$.

We now state some properties of admissible Banach function spaces in the following proposition.

Proposition 2.3. *Let E be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R})$ be such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where $\Lambda_1\varphi$ is defined as in Definition 2.1*(ii)*. For $\sigma > 0$, we define functions $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ by*

$$\Lambda_\sigma\varphi(t) = \int_{-\infty}^t e^{-\sigma(t-s)}\varphi(s)ds, \quad \bar{\Lambda}_\sigma\varphi(t) = \int_t^\infty e^{-\sigma(s-t)}\varphi(s)ds.$$

Then, $\Lambda_\sigma\varphi$ *and* $\bar{\Lambda}_\sigma\varphi$ *belong to* E *and satisfy the following estimates*:

$$\|\Lambda_\sigma\varphi\|_E \le \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E \quad \textit{and} \quad \|\bar{\Lambda}_\sigma\varphi\|_E \le \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E.$$

Moreover, if $\sup_{t\in\mathbb{R}}\int_t^{t+1}\varphi(\tau)\,d\tau < \infty$ *(this will be satisfied if* $\varphi \in E$ *(see Remark 2.2)), then* $\Lambda_\sigma\varphi$ *and* $\bar{\Lambda}_\sigma\varphi$ *are bounded.*

(b) *E contains exponentially decaying functions* $\psi(t) = e^{-\alpha|t|}$ *for any constant* $\alpha > 0$.

(c) *E does not contain exponentially growing functions* $f(t) = e^{\beta t}$ *for any constant* $\beta \neq 0$.

Proof. The proof of this proposition is the same as in [7, Proposition 2.6]. We present it here for the sake of completeness.

(a) We have

$$\begin{aligned}\Lambda_\sigma\varphi(t) &= \sum_{k=0}^{\infty}\int_{t-(k+1)}^{t-k} e^{-\sigma(t-s)}\varphi(s)ds\\ &\le \sum_{k=0}^{\infty} e^{-\sigma k}\int_{t-(k+1)}^{t-k}\varphi(s)ds = \sum_{k=0}^{\infty} e^{-\sigma k}T_{k+1}^{+}\Lambda_1\varphi(t),\\ \bar{\Lambda}_\sigma\varphi(t) &= \sum_{k=0}^{\infty}\int_{t+k}^{t+k+1} e^{-\sigma(s-t)}\varphi(s)ds\\ &\le \sum_{k=0}^{\infty} e^{-\sigma k}\int_{t+k}^{t+k+1}\varphi(s)ds = \sum_{k=0}^{\infty} e^{-\sigma k}T_{k}^{-}\Lambda_1\varphi(t).\end{aligned}$$

On the other hand, we have the following estimates.

$$\begin{aligned}\sum_{k=0}^{\infty} e^{-\sigma k}\|T_{k+1}^{+}\Lambda_1\varphi\|_E &\le \sum_{k=0}^{\infty} e^{-\sigma k}N_1\|\Lambda_1\varphi\|_E = \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E,\\ \sum_{k=0}^{\infty} e^{-\sigma k}\|T_{k}^{-}\Lambda_1\varphi\|_E &\le \sum_{k=0}^{\infty} e^{-\sigma k}N_2\|\Lambda_1\varphi\|_E = \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E.\end{aligned}$$

Therefore, two series $\sum_{k=0}^{\infty} e^{-\sigma k}T_{k+1}^{+}\Lambda_1\varphi(t)$ and $\sum_{k=0}^{\infty} e^{-\sigma k}T_{k}^{-}\Lambda_1\varphi(t)$ are absolutely convergent in the Banach function space E. By Banach lattice property, we have $\Lambda_\sigma\varphi$, $\bar{\Lambda}_\sigma\varphi \in E$ and

$$\|\Lambda_\sigma\varphi\|_E \le \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E \quad \text{and} \quad \|\bar{\Lambda}_\sigma\varphi\|_E \le \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E.$$

Take $E = L_\infty(\mathbb{R})$, by $\sup_{t\in\mathbb{R}}\int_t^{t+1}\varphi(\tau)\,d\tau < \infty$ so $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ are bounded.

(b) Because of $\chi_{[0,1]} \in E$, $v := \Lambda_\alpha\chi_{[0,1]} + \bar{\Lambda}_\alpha\chi_{[0,1]} \in E$. We have

$$v(t) = \begin{cases} \frac{e^{-\alpha t}(e^{\alpha}-1)}{\alpha}, & t \ge 1,\\ \frac{e^{\alpha t}(1-e^{-\alpha})}{\alpha}, & t \le 0,\\ \frac{1-e^{-\alpha t}}{\alpha} + \frac{1-e^{-\alpha(1-t)}}{\alpha}, & t \in (0,1).\end{cases}$$

Therefore, $e^{\alpha|t|}v(t) \geq \frac{1-e^{-\alpha}}{\alpha}$ for all $t \in \mathbb{R}$. The Banach lattice property implies $e^{-\alpha|t|} \in E$.

(c) Assume that $f(t) = e^{\beta t} \in E$. Then, there exists a $K > 0$ such that $\|f\|_{\mathbf{M}} \leq K\|f\|_E$ (see Remark 2.2). So,

$$\frac{e^{\beta t}(e^{\beta}-1)}{\beta} \leq K\|f\|_E \quad \text{for all} \quad t \in \mathbb{R}.$$

This contradicts with $\lim_{t\to\infty} \frac{e^{\beta t}(e^{\beta}-1)}{\beta} = \infty$ or $\lim_{t\to-\infty} \frac{e^{\beta t}(e^{\beta}-1)}{\beta} = \infty$. Thus, $e^{\beta t} \notin E$. □

2.2. Evolution family

Definition. A family of bounded linear operators $(U(t,s))_{t\geq s}$ on a Banach space X is a (*strongly continuous, exponential bounded*) *evolution family* if:

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for all $t \geq r \geq s$.
(ii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$.
(iii) There are constants $K, c \geq 0$ such that $\|U(t,s)x\| \leq Ke^{c(t-s)}\|x\|$ for all $t \geq s$ and $x \in X$.

The notion of an evolution family arises naturally from the theory of non-autonomous evolution equations which are well-posed on the real line. This means that if the abstract Cauchy problem

$$\begin{cases} u'(t) = A(t)u(t), & t \geq s, \\ u(s) = x, \end{cases} \tag{1}$$

is well-posed, there exists an evolution family $(U(t,s))_{t\geq s}$ such that the solution of the Cauchy problem (1) is given by $u(t) = U(t,s)u(s)$ for every $u(s) \in D(A(s))$. For more details on the notion of evolution family, conditions for the existence of such family and applications to partial differential equations we refer the readers to Pazy [12], Nagel and Nickel [11]. The next is the property of exponential dichotomy of evolution family which we will use in later section.

Definition. An evolution family $(U(t,s))_{t\geq s}$ is said that have an *exponential dichotomy* on $\mathbb{R}$ if there exist one family of projections $(P(t))_{t\in\mathbb{R}}$ and positive constants N, β such that the following conditions are fulfilled:

(i) $P(t)U(t,s) = U(t,s)P(s)$ for $t \geq s$.
(ii) $U(t,s)|_{\mathrm{Ker}P(s)}$ is an isomorphism from $\mathrm{Ker}P(s)$ onto $\mathrm{Ker}P(t)$ for all $t \geq s$. Denote the inverse of $U(t,s)|_{\mathrm{Ker}P(s)}$ by $U(s,t)_|$, $s \leq t$.
(iii) For all $t \geq s$ and $x \in X$, the following estimates hold:

$$\|U(t,s)P(s)x\| \leq Ne^{-\beta(t-s)}\|P(s)x\|,$$
$$\|U(s,t)_|(Id - P(t))x\| \leq Ne^{-\beta(t-s)}\|(Id - P(t))x\|.$$

The projections $P(t)$, $t \in \mathbb{R}$ are called the *dichotomy projections*, and the constants N, β are *the dichotomy constants*.

Remark 2.4. The family of dichotomy projections $P(t)$ is strongly continuous and uniformly bounded on $\mathbb{R}$, the proof is the same as in [10, Lemma 4.2].

As evolution family $(U(t,s))_{t\geq s}$ on the Banach space X has an *exponential dichotomy* on $\mathbb{R}$, we can define the Green function as follows:

$$\mathcal{G}(t,s) = \begin{cases} U(t,s)P(s) & \text{for } t > s, \\ -U(t,s)_|(Id - P(s)) & \text{for } t < s. \end{cases} \tag{2}$$

Thus, we have the following estimates.

$$\|\mathcal{G}(t,s)\| \leq NHe^{-\beta|t-s|} \quad \text{for all } t \neq s,$$

where $H = \sup\{\|P(t)\|, \|Id - P(t)\| : t \in \mathbb{R}\}$. On the other hand, at each time $t \in \mathbb{R}$ then Banach space X is decomposed to direct sum of two closed subspaces

$$X = X^{\mathrm{s}}(t) \oplus X^{\mathrm{u}}(t),$$

where $X^{\mathrm{s}}(t) = P(t)X$, $X^{\mathrm{u}}(t) = (Id - P(t))X$.

3. Bounded and periodic solutions

Given an evolution family $(U(t,s))_{t\geq s}$ on Banach space X and a function $f : \mathbb{R} \to X$, in the section we will study the following inhomogeneous linear evolution equation

$$\begin{cases} u(t) = U(t,s)u(s) + \displaystyle\int_s^t U(t,\xi)f(\xi)d\xi & \text{for } t \geq s, \\ u(s) = x \in X. \end{cases} \tag{3}$$

Note that if evolution family $(U(t,s))_{t\geq s}$ is generated by the Cauchy problem (1), then a solution $u(t)$ of Eq. (3) is called mild solution of the differential equation

$$\begin{cases} u'(t) = A(t)u(t) + f(t), & t \geq s, \\ u(s) = x. \end{cases}$$

We will now characterise solution formula of Eq. (3) in the Banach space $\mathcal{E}$ in below theorem.

Theorem 3.1. *Let evolution family $(U(t,s))_{t\geq s}$ have exponential dichotomy and function $f \in \mathcal{E}$. Then, Eq.* (3) *has a unique solution in the Banach space $\mathcal{E}$ and it has the formula as follows*:

$$u(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,\tau)f(\tau)d\tau, \quad t \in \mathbb{R}, \tag{4}$$

where $\mathcal{G}(t,\tau)$ is Green function and defined by (2)*. Moreover, this solution is bounded.*

Proof. Because of $f \in \mathcal{E}$, $\varphi(t) := \|f(t)\| \in E$. Put $y(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,\tau)f(\tau)d\tau$, using estimate of operator $\mathcal{G}(t,\tau)$ we have

$$\|y(t)\| \leq NH\int_{-\infty}^{\infty} e^{-\beta|t-\tau|}\varphi(\tau)d\tau = NH(\Lambda_\beta\varphi(t) + \bar{\Lambda}_\beta\varphi(t)).$$

So, $y(t)$ is defined for all $t \in \mathbb{R}$. Moreover, by Proposition 2.3(a) and the Banach lattice property of E we obtain $y \in \mathcal{E}$ and

$$\|y\|_{\mathcal{E}} \leq \frac{NH(N_1+N_2)\|\Lambda_1\varphi\|_E}{1-e^{-\beta}}.$$

Besides that $y(t)$ is also bounded and $\|y\|_\infty \leq \frac{NH(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-e^{-\beta}}$.

By directly computing, we easily see that y is a solution on $\mathbb{R}$ of Eq. (3). Next, we show that y is unique solution of Eq. (3) in the Banach space $\mathcal{E}$. Let u be another solution of Eq. (3) in the Banach space $\mathcal{E}$. For each fixed $s \in \mathbb{R}$ and for $t > s$, project Eq. (3) onto $X^{\mathrm{u}}(t)$ we obtain

$$(Id - P(t))u(t) = U(t,s)(Id - P(s))u(s) + \int_s^t U(t,\tau)(Id - P(\tau))f(\tau)d\tau.$$

Since $U(t,s)|_{\mathrm{Ker}P(s)}$ is an isomorphism from $\mathrm{Ker}P(s)$ onto $\mathrm{Ker}P(t)$,

$$U(s,t)_|(Id - P(t))u(t) - \int_s^t U(s,\tau)_|(Id - P(\tau))f(\tau)d\tau = (Id - P(s))u(s).$$

On the other hand, $\int_{-\infty}^{\infty} \mathcal{G}(s,\tau)f(\tau)d\tau$ converges absolutely in Banach space X. Therefore, there exists $x \in X$ such that $\lim_{t\to\infty} U(s,t)_|(Id - P(t))u(t) = x$. If $x \neq 0$, then there exists $T > s$ such that

$$\frac{\|x\|}{2} \leq NHe^{-\beta(t-s)}\|u(t)\|$$

for all $t \geq T$. This implies $e^{\beta t}\|x\| \leq 2NHe^{\beta s}\|u(t)\|$ for all $t \geq T$. Put $z(t) = e^{\beta t}\|x\|$ if $t \geq T$ and $z(t) = 0$ if otherwise. By Banach lattice property and $\|u(t)\| \in E$, we obtain $z(t) \in E$. Since $E \hookrightarrow \mathbf{M}(\mathbb{R})$,

$$\sup_{t\geq T}\int_t^{t+1} e^{\beta\tau}\|x\|d\tau < \infty.$$

However, this contradicts with $\sup_{t\geq T}\int_t^{t+1} e^{\beta\tau}\|x\|d\tau = \infty$. So,

$$\lim_{t\to\infty} U(s,t)_|(Id - P(t))u(t) = 0.$$

Thus,

$$(Id - P(s))u(s) = -\int_s^\infty U(s,\tau)_|(Id - P(\tau))f(\tau)d\tau.$$

For $t < s$, Eq. (3) is rewritten as follows

$$u(s) = U(s,t)u(t) + \int_t^s U(s,\tau)f(\tau)d\tau.$$

Project the above equation onto $X^s(s)$ we obtain

$$P(s)u(s) = U(s,t)P(t)u(t) + \int_t^s U(s,\tau)P(\tau)f(\tau)d\tau.$$

By the similar argument as above, we have $\lim_{t\to-\infty} U(s,t)P(t)u(t) = 0$. Therefore,

$$P(s)u(s) = \int_{-\infty}^s U(s,\tau)P(\tau)f(\tau)d\tau.$$

So, for each fixed $s \in \mathbb{R}$ then we have

$$P(s)u(s) = \int_{-\infty}^s U(s,\tau)P(\tau)f(\tau)d\tau,$$
$$(Id - P(s))u(s) = -\int_s^\infty U(s,\tau)_|(Id - P(\tau))f(\tau)d\tau.$$

Thus, $u(s) = y(s)$ for all $s \in \mathbb{R}$. □

So, for each $f \in \mathcal{E}$ then Eq. (3) has a unique solution in Banach space $\mathcal{E}$ and this solution is bounded on $\mathbb{R}$. A natural question poses: if $f \in \mathcal{E}$ is periodic, then does Eq. (3) have a unique periodic solution in Banach space $\mathcal{E}$? Of course, the periodicity of evolutionary family $(U(t,s))_{t\geq s}$ is imperative when studying this question, i.e., there exists $T > 0$ such that $U(t+T, s+T) = U(t,s)$ for all $t \geq s$. Since the solution in Theorem 3.1 is bounded, we will answer the above question using Massera's method.

Theorem 3.2. *Let evolution family $(U(t,s))_{t\geq s}$ have exponential dichotomy. Assume that evolution family $(U(t,s))_{t\geq s}$ and function $f \in \mathcal{E}$ are periodic with the same period T, and one of the following conditions holds*:

(i) $1 \notin \sigma(U(T,0))$;

(ii) *X is a reflexive Banach space*;

(iii) *X is the dual to some separable Banach space Z, i.e., $X = Z^*$, and $U(T,0)^*Z \subset Z$ with $U(T,0)^*$ is an adjoint operator of $U(T,0)$.*

Then, Eq. (3) *has a unique periodic solution in the Banach space $\mathcal{E}$ and its formula defined by* (4).

Proof. Firstly, we prove that Eq. (3) has a periodic solution with period T on $[0,\infty)$. We have

$$u(t+T) - u(t) = U(t,0)(u(T) - u(0))$$

for all $t \geq 0$. Therefore, the solution $u(t)$ of Eq. (3) is periodic on $[0,+\infty)$ if and only if $u(T) = u(0)$.

If (i) holds, then

$$u(0) = (I - U(T,0))^{-1} \int_0^T U(T,\tau)f(\tau)d\tau.$$

Otherwise, since (ii) and (iii) have the same way of proof, we only prove the existence of $u(0)$ in the case X being reflexive Banach space. To perform this work, we define Poincaré mapping $P : X \to X$ as follows:

$$Px = U(T,0)x + \int_0^T U(T,\tau)f(\tau)d\tau, \quad x \in X.$$

Denoting $y(t)$ is a bounded solution on $\mathbb{R}$ of Eq. (3) corresponding to $f \in \mathcal{E}$. By chain property of solution, we have $P^n y(0) = y(nT)$ for all $n \geq 1$. Take Cesàro average of sequence $\{P^n y(0)\}$,

$$P_n := \frac{1}{n}\sum_{m=1}^{n} P^m y(0).$$

Therefore,

$$P_{n+1} - P_n = \frac{1}{n}\big(P^{n+1}y(0) - P_{n+1}\big).$$

Because $y(t)$ is bounded, two sequences $\{P^n y(0)\}$ and $\{P_n\}$ are also bounded. Thus,

$$\lim_{n\to\infty}(P_{n+1} - P_n) = 0. \tag{5}$$

Since X is reflexive, there exist a subsequence $\{P_{n_k}\}$ and $y \in X$ such that $P_{n_k} \overset{\text{weak}}{\longrightarrow} y$, i.e.,

$$\langle P_{n_k} - y, g\rangle \to 0 \quad \text{for all } \ g \in X^*,$$

where $\langle \cdot, \cdot \rangle$ is dual pair. On the other hand, we have

$$PP_n = P_n - \frac{1}{n}Py(0) + \frac{1}{n}P^{n+1}y(0) = \frac{n+1}{n}P_{n+1} - \frac{1}{n}Py(0).$$

Thus,

$$\lim_{n\to\infty}(PP_n - P_{n+1}) = 0.$$

For $g \in X^*$, we also have

$$\langle PP_{n_k} - Py, g\rangle = \langle U(T,0)(P_{n_k} - y), g\rangle = \langle P_{n_k} - y, U(T,0)^* g\rangle \to 0,$$

where $U(T,0)^*$ is an adjoint operator of $U(T,0)$. So,

$$\langle P_{n_k+1} - Py, g\rangle = \langle P_{n_k+1} - PP_{n_k}, g\rangle + \langle PP_{n_k} - Py, g\rangle \to 0.$$

Combining with (5), we obtain

$$\langle y - Py, g\rangle = \lim_{k\to\infty}\big[\,\langle y - P_{n_k}, g\rangle + \langle P_{n_k} - P_{n_k+1}, g\rangle + \langle P_{n_k+1} - Py, g\rangle\,\big] = 0$$

for all $g \in X^*$. Therefore, $Py = y$. So, Eq. (3) has a periodic solution with period T on $[0,\infty)$ when one of the above three conditions holds, and this solution is defined by the formula

$$u(t) = U(t,0)y + \int_0^t U(t,\tau)f(\tau)d\tau \quad \text{for all} \quad t \geq 0.$$

Let $v(t)$ be the periodic extension with period T of the function $u(t)$ on the line. Next, we show that $v(t)$ is a periodic solution on $\mathbb{R}$ of Eq. (3). Indeed, if $s \geq 0$, then

$$v(t) = u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau = U(t,s)v(s) + \int_s^t U(t,\tau)f(\tau)d\tau$$

for all $t \geq s$. If $s < 0$, then there exists $k \in \mathbb{N}$ such that $s + kT \geq 0$, for $t \geq s$ we have

$$\begin{aligned} v(t) = v(t+kT) &= u(t+kT) \\ &= U(t+kT, s+kT)u(s+kT) + \int_{s+kT}^{t+kT} U(t+kT,\tau)f(\tau)d\tau \\ &= U(t,s)v(s) + \int_s^t U(t+kT,\tau+kT)f(\tau+kT)d\tau \\ &= U(t,s)v(s) + \int_s^t U(t,\tau)f(\tau)d\tau. \end{aligned}$$

So, Eq. (3) has a periodic solution with period T on $\mathbb{R}$. Because $v(t)$ is bounded on $\mathbb{R}$, we obtain

$$\lim_{t\to\infty} U(s,t)_|(Id - P(t))v(t) = 0 = \lim_{t\to-\infty} U(s,t)P(t)v(t),$$

with similar discussion as in the proof of Theorem 3.1. Therefore, v is defined by the formula (4). So, $v \in \mathcal{E}$ and is unique periodic solution. □

We now give an example to illustrate the theoretical results.

Example 3.3. Consider the following differential equation:

$$(6) \quad \begin{cases} \frac{\partial}{\partial t}u(t,x) = a(t)\big(\frac{\partial^2}{\partial x^2}u(t,x) + ru(t,x)\big) + x^2\cos t, & \text{for } t \geq s,\ x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \geq s, \\ u(s,x) = u_0(x), & s \in \mathbb{R},\ x \in [0,1]. \end{cases}$$

Assume that $r > 0$ and $r \neq n^2\pi^2$ for all $n \in \mathbb{N}$, $a : \mathbb{R} \to \mathbb{R}_+$ is periodic with the period 2π such that $a \in \mathbf{M}(\mathbb{R})$ and $m := \inf_{t\in\mathbb{R}} \int_t^{t+1} a(\tau)d\tau > 0$.

Let $X = L^2[0,1], E = \mathbf{M}(\mathbb{R})$ and define

$$\begin{aligned} A(t)u = a(t)Au \ \text{ in which } \ Au = u'' + ru, \\ D(A(t)) = D(A) = \{u \in H^2[0,1] : u(0) = u(1) = 0\}. \end{aligned}$$

We know that A generates an analytic semigroup $(T(t))_{t\geq 0}$ (see [6, Chap. II]). Therefore, the family of operators $A(t)$ also generate an evolution family which have the following form

$$U(t,s) = T\Big(\int_s^t a(\tau)d\tau\Big), \quad t \geq s.$$

By the spectral mapping theorem for analytic semigroups and the spectrum of A is the set

$$\sigma(A) = \{-\pi^2 + r, -2^2\pi^2 + r, \ldots, -n^2\pi^2 + r, \ldots\},$$

the analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic. Thus, the evolution family $(U(t,s))_{t\geq s}$ has exponential dichotomy.

The equation (6) is now rewritten as follows:

$$\begin{cases} \frac{d}{dt}u(t,\cdot) = A(t)u(t,\cdot) + f(t) & \text{for } t \geq s, \\ u(s,\cdot) = u_0(\cdot) \in X, \quad s \in \mathbb{R}, \end{cases}$$

where $f : \mathbb{R} \to X$ is defined by

$$f(t)(x) = x^2 \cos t.$$

Put $v(t) = u(t,\cdot)$ for $t \in \mathbb{R}$. Then, the mild solution of above Cauchy problem is solution of evolution equation

$$v(t) = U(t,s)u_0 + \int_s^t U(t,\tau)f(\tau)d\tau, \quad t \geq s. \tag{7}$$

Since a is a periodic function, $U(t,s) = T\left(\int_s^t a(\tau)d\tau\right)$ is also periodic with the period 2π. On the other hand, the mapping $f : \mathbb{R} \to X$ is periodic with the period 2π and

$$\varphi(t) = \|f(t)\|_2 = \frac{1}{\sqrt{5}}|\cos t| \in \mathbf{M}(\mathbb{R}).$$

According to Theorem 3.2, the equation (7) has unique periodic solution.

From the above example we see that the model in this paper is generalized form for some concrete models. Therefore, the quantitative properties obtained in the paper are general. So, a concrete model will has these properties if it satisfies the assumptions in the paper. But in reality, to prove the exponential dichotomy of an evolution family corresponding to a particular model arising from physics and mechanics, e.g. [2–4], is not easy and must be studied carefully. For instance, Beniani et al. proved the well-posedness of the linear coupled Lamé system in the paper [1], this means that this system generates an evolution family. A natural question arises: under what conditions does it have exponential dichotomy? The same problems pose for linearized part of the models in the articles [2–4]. These are interesting open works in the near future.

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