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SOME REMARKS FOR λ-SPIRALLIKE FUNCTION OF COMPLEX ORDER AT THE BOUNDARY OF THE UNIT DISC

Tuğba Akyel

Abstract. We consider a different version of Schwarz Lemma for λ-spirallike function of complex order at the boundary of the unit disc D. We estimate the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ from below for λ-spirallike function $f(z)$ of complex order at the boundary of the unit disc D by taking into account the zeros of the function $f(z)-z$ which are different from zero. We also estimate the same function with the second derivatives of the function f at the points $z=0$ and $z=z_0\neq 0$. We show the sharpness of these estimates and present examples.

1. Introduction

Let $\mathcal{A}$ denote the class of functions in the form

$$f(z)=z+b_2z^2+b_3z^3+\cdots$$

which are holomorphic in the unit disc D. Let $\mathcal{M}$ denote the class of bounded holomorphic functions $p(z)$ in D, satisfying the condition $p(0)=0$ and $|p(z)|\leq |z|$ for $z\in D$. For a function belonging to the class $\mathcal{A}$, we say that $f(z)$ is a λ-spirallike function of complex order in D if and only if

$$\Re\left(\frac{1}{c\cos\lambda}\left[e^{i\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)+c\cos\lambda\right]\right)>0 \tag{1}$$

for some real λ, $|\lambda|<\frac{\pi}{2}$, $c\neq 0$, complex. We denote this class by $\mathcal{N}(\lambda)$. It has been introduced and studied by Al-Oboudi and Haidan [2].

It is easy to show that $f(z)\in\mathcal{N}(\lambda)$ if and only if there is a $p\in\mathcal{M}$ such that

$$\frac{1}{\cos\lambda}e^{i\lambda}\frac{zf'(z)}{f(z)}-i\frac{\sin\lambda}{\cos\lambda}=\frac{1+(2c-1)p(z)}{1-p(z)} \tag{2}$$

for $z\in D$ and for some λ, $|\lambda|<\frac{\pi}{2}$.

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Therefore, from (2), we take

$$p(z)=e^{i\lambda}\frac{\frac{zf'(z)}{f(z)}-1}{e^{i\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)+2c\cos\lambda}.$$

Now let us consider the following function by taking into account of the zeros, which are different from zero, of the function $f(z)-z$.

$$g(z)=\frac{p(z)}{\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

Since $g(z)\in\mathcal{M}$, from Schwarz Lemma [6], we obtain

$$g(z)=\frac{p(z)}{\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}=e^{i\lambda}\frac{b_2z+\left(2b_3-b_2^2\right)z^2+\cdots}{e^{i\lambda}\left(b_2z+\left(2b_3-b_2^2\right)z^2+\cdots\right)+2c\cos\lambda}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$\frac{g(z)}{z}=e^{i\lambda}\frac{b_2+\left(2b_3-b_2^2\right)z+\cdots}{e^{i\lambda}\left(b_2z+\left(2b_3-b_2^2\right)z^2+\cdots\right)+2c\cos\lambda}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$\left|g'(0)\right|=\frac{|b_2|}{2\,|c|\cos\lambda}\frac{1}{\prod\limits_{i=1}^{n}|z_i|}\leq 1$$

and

$$|b_2|\leq 2\,|c|\cos\lambda\prod_{i=1}^{n}|z_i|\,. \tag{3}$$

Now we shall show that the inequality (3) is sharp. Let

$$f(z)=ze^{\int_0^z\frac{2c\cos\lambda\prod\limits_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}}{e^{i\lambda}\left(1-t\prod\limits_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}dt},$$

where $z_1,z_2,\ldots,z_n$ are positive real numbers. Using logarithmic differentiation, we obtain

$$\frac{f'(z)}{f(z)}=\frac{1}{z}+\frac{2c\cos\lambda\prod\limits_{i=1}^{n}\frac{z-z_i}{1\overline{z_i}z}}{e^{i\lambda}\left(1-z\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}.$$

From the last equality, we have the following

$$m(z)=\frac{zf'(z)}{f(z)}=1+\frac{2c\cos\lambda z\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{e^{i\lambda}\left(1-z\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}$$

and also,

$$1+b_2z+\left(2b_3-b_2^2\right)z^2+\cdots=1+\frac{2c\cos\lambda z\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{e^{i\lambda}\left(1-z\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)},$$

$$b_2z+\left(2b_3-b_2^2\right)z^2+\cdots=\frac{2c\cos\lambda z\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{e^{i\lambda}\left(1-z\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)},$$

and

$$|b_2|=2|c|\cos\lambda\prod_{i=1}^{n}|z_i|.$$

We thus obtain the following lemma.

Lemma 1.1. *Let $f(z)\in\mathcal{N}(\lambda)$. Let $z_1,z_2,\ldots,z_n$ be zeros of the function $f(z)-z$ in D that are different from zero. Then we have*

$$|b_2|\leq 2|c|\cos\lambda\prod_{i=1}^{n}|z_i|.$$

The result is sharp with the function

$$f(z)=ze^{\int_0^z\frac{2c\cos\lambda\prod\limits_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}}{e^{i\lambda}\left(1-t\prod\limits_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}dt},$$

where $z_1,z_2,\ldots,z_n$ are positive real numbers.

One of the elementary results of Schwarz Lemma is the boundary Schwarz Lemma. It is about estimating the modulus of the derivative of the function from below at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [17]:

If w extends continuously to some boundary point b with $|b|=1$, and if $|w(b)|=1$ and $w'(b)$ exists, then

$$|w'(b)|\geq\frac{2}{1+|w'(0)|}\tag{4}$$

and

$$|w'(b)|\geq 1.$$

For $b=1$ in (4), the equality occurs for the function

$$w(z)=z\frac{z+a}{1+az},$$

where $0\leq a\leq 1$. It follows that

$$|w'(b)|\geq 1\tag{5}$$

with equality only if w is of the form $w(z)=ze^{i\theta}$, θ is real.

Vladimir N. Dubinin has continued on this line and has made a refinement on the boundary Schwarz Lemma for $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$ with a zero set $\{z_k\}$ [5]. Mercer has obtained a version of Schwarz Lemma where the images of two points are known [10]. Also, he has considered some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [11]. M. Jeong has shown some inequalities at a boundary point of the unit disc for different form of holomorphic functions and find the condition for equality [7].

In electrical and electronics engineering, it is possible to encounter applications boundary version of Schwarz lemma. As an example, the driving point impedance functions obtained as a result of boundary analysis of Schwarz lemma can be possibly used for circuit synthesis. Also, transfer functions in control theory and multi-notched filters in signals and systems can be considered as topics under the title of Schwarz lemmas applications [14–16].

For a set of references, we refer the reader to the papers [1, 4, 8, 9, 12, 13, 19] for more general results and related estimates.

For our main results, we shall need the following lemma due to Julia-Wolff [18].

Lemma 1.2 (Julia-Wolff Lemma)**.** *Let w be a holomorphic function in D, $w(0) = 0$ and $w(D) \subset D$. If, in addition, the function w has an angular limit $w(b)$ at $b \in \partial D$, $|w(b)| = 1$, then the angular derivative $w'(b)$ exists and $1 \leq |w'(b)| \leq \infty$.*

Corollary 1.3. *The holomorphic function w has a finite angular derivative $w'(b)$ if and only if w' has the finite angular limit $w'(b)$ at $b \in \partial D$.*

2. Main results

Let the holomorphic function $f(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ belong to the class of $\mathcal{N}(\lambda)$. The propose of this paper is to give a different version of Schwarz Lemma for λ-spirallike function with complex order at the boundary of D and estimate the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$.

In the following theorem, we shall estimate the above function from below with the second and third coefficient in Taylor expansion of the function $f(z)$.

Theorem 2.1. *Let $f(z) \in \mathcal{N}(\lambda)$. Let $z_1, z_2, \ldots, z_n$ be zeros of the function $f(z) - z$ in D that are different from zero. Assume that, for some $b \in \partial D$, f has angular limit $f(b)$ at b and $\frac{bf'(b)}{f(b)} = 1 - c\frac{\cos\lambda}{e^{i\lambda}}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{|c|\cos\lambda}{2}\left(1 + \sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right.$$

$$\left. + \frac{2\left(2|c|\cos\lambda\prod_{i=1}^{n}|z_i| - |b_2|\right)^2}{\left(2|c|\cos\lambda\prod_{i=1}^{n}|z_i|\right)^2 - |b_2|^2 + \prod_{i=1}^{n}|z_i|\left|\left(2b_3 - b_2^2\right)2c\cos\lambda - e^{i\lambda}b_2^2 + 2c\cos\lambda\, b_2\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right). \tag{6}$$

The equality in (6) *occurs for the function*

$$f(z)=ze^{-\int_0^z \frac{2c\cos\lambda\left(t\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}{e^{i\lambda}\left(1+t^2\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}dt},$$

where $z_1, z_2, \ldots, z_n$ *are positive real numbers.*

Proof. Let $p(z)$ be as in (2) and $z_1, z_2, \ldots, z_n$ be zeros of the function $f(z)-z$ in D that are different from zero. Also, consider the following function

$$B(z)=z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}.$$

Here $B(z)$ is holomorphic in D and $|B(z)|<1$ for $|z|<1$. By the maximum modulus principle for each $z\in D$, we have

$$|p(z)|\leq|B(z)|.$$

Consider the function

$$\varphi(z)=\frac{p(z)}{B(z)}=e^{i\lambda}\frac{b_2z+\left(2b_3-b_2^2\right)z^2+\cdots}{e^{i\lambda}\left(b_2z+\left(2b_3-b_2^2\right)z^2+\cdots\right)+2c\cos\lambda}\frac{1}{z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}$$

$$=e^{i\lambda}\frac{b_2+\left(2b_3-b_2^2\right)z+\cdots}{e^{i\lambda}\left(b_2z+\left(2b_3-b_2^2\right)z^2+\cdots\right)+2c\cos\lambda}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

$\varphi(z)$ is holomorphic in D and $|\varphi(z)|<1$ for $|z|<1$. In particular, we have

$$|\varphi(0)|=\frac{|b_2|}{2|c|\cos\lambda\prod_{i=1}^{n}|z_i|}$$

and

$$|\varphi'(0)|=\frac{\left|\left(2b_3-b_2^2\right)2c\cos\lambda-e^{i\lambda}b_2^2+2c\cos\lambda b_2\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(2|c|\cos\lambda\right)^2\prod_{i=1}^{n}|z_i|}.$$

The auxiliary function

$$s(z)=\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)}$$

is holomorphic in D, $|s(z)|<1$ for $|z|<1$ and $s(0)=0$. For $b\in\partial D$ and $\frac{bf'(b)}{f(b)}=1-c\frac{\cos\lambda}{e^{i\lambda}}$, we take $|s(b)|=1$.

From (4), we obtain

$$\frac{2}{1+|s'(0)|}\leq|s'(b)|=\frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(b)\right|}|\varphi'(b)|$$

$$\leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left(|p'(b)|-|B'(b)|\right)$$

and

$$\frac{2}{1+|s'(0)|} \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left(|p'(b)|-|B'(b)|\right). \tag{7}$$

It can be seen that

$$|s'(0)| = \frac{|\varphi'(0)|}{1-|\varphi(0)|^2}$$

and

$$|s'(0)| = \frac{\frac{\left|\left(2b_3-b_2^2\right)2c\cos\lambda - e^{i\lambda}b_2^2 + 2c\cos\lambda b_2 \sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{(2|c|\cos\lambda)^2 \prod\limits_{i=1}^{n}|z_i|}}{1-\left(\frac{|b_2|}{2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|}\right)^2}$$

$$= \prod_{i=1}^{n}|z_i| \frac{\left|\left(2b_3-b_2^2\right)2c\cos\lambda - e^{i\lambda}b_2^2 + 2c\cos\lambda b_2 \sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|\right)^2 - |b_2|^2}.$$

Also, we have

$$|B'(b)| = 1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2},\ b\in\partial D.$$

Let us substitute the values of $|s'(0)|$, $|p'(b)|$, $|B'(b)|$ and $|\varphi(0)|$ into (7). Therefore, we get

$$\frac{2}{1+\frac{\prod\limits_{i=1}^{n}|z_i|\left|\left(2b_3-b_2^2\right)2c\cos\lambda - e^{i\lambda}b_2^2 + 2c\cos\lambda b_2 \sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|\right)^2 - |b_2|^2}}$$

$$\leq \frac{2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|+|b_2|}{2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|-|b_2|}\left(\frac{2|m'(b)|}{|c|\cos\lambda}-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right),$$

$$\frac{2\left(\left(2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|\right)^2-|b_2|^2\right)}{\left(2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|\right)^2-|b_2|^2+\prod\limits_{i=1}^{n}|z_i|\left|\left(2b_3-b_2^2\right)2c\cos\lambda - e^{i\lambda}b_2^2 + 2c\cos\lambda b_2 \sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq \frac{2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|+|b_2|}{2|c|\cos\lambda \prod\limits_{i=1}^{n}|z_i|-|b_2|}\left(\frac{2|m'(b)|}{|c|\cos\lambda}-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right),$$

$$\frac{2\left(2|c|\cos\lambda\prod_{i=1}^{n}|z_i|-|b_2|\right)^2}{\left(2|c|\cos\lambda\prod_{i=1}^{n}|z_i|\right)^2-|b_2|^2+\prod_{i=1}^{n}|z_i|\left|\left(2b_3-b_2^2\right)2c\cos\lambda-e^{i\lambda}b_2^2+2c\cos\lambda b_2\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq \frac{2\left|m'(b)\right|}{|c|\cos\lambda}-1-\sum_{i=1}^{n}\frac{1-\left|z_i\right|^2}{\left|b-z_i\right|^2}$$

and so, we get the inequality (6).

Now we shall show that the inequality (6) is sharp. Let

$$f(z)=ze^{-\int_0^z \frac{2c\cos\lambda\left(t\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}{e^{i\lambda}\left(1+t^2\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}dt}. \tag{8}$$

Differentiating (8) logarithmically, we obtain

$$\frac{f'(z)}{f(z)}=\frac{1}{z}-\frac{2c\cos\lambda\left(z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}{e^{i\lambda}\left(1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)},$$

$$\begin{aligned} m(z)=\frac{zf'(z)}{f(z)}&=1-\frac{2c\cos\lambda\left(z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}{e^{i\lambda}\left(1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}\\ &=1-\frac{2c\cos\lambda}{e^{i\lambda}}\left(1-\frac{1}{1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}\right) \end{aligned}$$

and

$$m(z)=1-\frac{2c\cos\lambda}{e^{i\lambda}}\left(1-\frac{1}{1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}\right).$$

After simple calculations, we get

$$|m'(1)|=\frac{|c|\cos\lambda}{2}\left(2+\sum_{i=1}^{n}\frac{1-z_i}{1+z_i}\right).$$

On the other hand, we obtain

$$1+b_2z+\left(2b_3-b_2^2\right)z^2+\cdots=1-\frac{2c\cos\lambda}{e^{i\lambda}}\left(1-\frac{1}{1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}\right),$$

$$b_2 z + \left(2b_3 - b_2^2\right) z^2 + \cdots = -\frac{2c\cos\lambda}{e^{i\lambda}} \left(\frac{z^2 \prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}{1 + z^2 \prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}} \right)$$

and

$$b_2 + \left(2b_3 - b_2^2\right) z + \cdots = -\frac{2c\cos\lambda}{e^{i\lambda}} \left(\frac{z \prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}{1 + z^2 \prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}} \right).$$

Passing to the limit as z tends to 0 in the last equality, we find $b_2 = 0$. Similarly, using straightforward calculations, we take $\left|2b_3 - b_2^2\right| = 2\left|c\right|\cos\lambda \prod\limits_{i=1}^{n} \left|z_i\right|$. If we regard the right-hand side of the inequality (6), for $b = -1$, we get

$$\begin{aligned}
&\frac{|c|\cos\lambda}{2} \left(1 + \sum_{i=1}^{n} \frac{1-z_i^2}{\left(-1-z_i\right)^2} \right. \\
&\qquad \left. + \frac{2\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i - |b_2|\right)^2}{\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)^2 - |b_2|^2 + \prod\limits_{i=1}^{n} z_i \left| \left(2b_3 - b_2^2\right) 2c\cos\lambda - e^{i\lambda} b_2^2 + 2c\cos\lambda b_2 \sum\limits_{i=1}^{n} \frac{1-z_i^2}{z_i} \right|} \right) \\
&= \frac{|c|\cos\lambda}{2} \left(1 + \sum_{i=1}^{n} \frac{1-z_i^2}{\left(-1-z_i\right)^2} + \frac{2\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)^2}{\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)^2 + \left|\left(2b_3 - b_2^2\right) 2c\cos\lambda\right| \prod\limits_{i=1}^{n} z_i} \right) \\
&= \frac{|c|\cos\lambda}{2} \left(1 + \sum_{i=1}^{n} \frac{1-z_i^2}{\left(-1-z_i\right)^2} + \frac{2\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)^2}{\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)^2 + \left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)} \right) \\
&= \frac{|c|\cos\lambda}{2} \left(1 + \sum_{i=1}^{n} \frac{1-z_i^2}{\left(-1-z_i\right)^2} + \frac{2\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)^2}{2\left(2|c|\cos\lambda \prod\limits_{i=1}^{n} z_i\right)^2} \right) \\
&= \frac{|c|\cos\lambda}{2} \left(2 + \sum_{i=1}^{n} \frac{1-z_i}{1+z_i} \right),
\end{aligned}$$

where $z_i,\ i = 1, 2, \ldots, n$ are positive real numbers. □

Now we shall estimate the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ from below according to the second Taylor coefficients of f about $z = 0$ and $z = z_0 \neq 0$. Motivated by the results of the work presented in [3], the following result is obtained.

Theorem 2.2. *If $f(z) \in \mathcal{N}(\lambda)$ and $\frac{z_0 f'(z_0)}{f(z_0)} = 1$ for $0 < |z_0| < 1$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $\frac{bf'(b)}{f(b)} = 1 - c\frac{\cos\lambda}{e^{i\lambda}}$.*

Then we have the following inequality

(9)

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{|c|\cos\lambda}{2}\left(1+\frac{1-|z_0|^2}{|b-z_0|^2}+\frac{4|c|\cos\lambda|z_0|-\left|f''(0)\right|}{4|c|\cos\lambda|z_0|+|f''(0)|}\right.$$

$$\left.\times\left[1+\frac{8|c|^2\cos^2\lambda\,|z_0|^2+|f''(z_0)|\,|f''(0)|\left(1-|z_0|^2\right)-4|c|\cos\lambda\,|f''(z_0)|\left(1-|z_0|^2\right)-4|c|\cos\lambda\,|f''(0)|}{8|c|^2\cos^2\lambda\,|z_0|^2+|f''(z_0)|\,|f''(0)|\left(1-|z_0|^2\right)+4|c|\cos\lambda\,|f''(z_0)|\left(1-|z_0|^2\right)+4|c|\cos\lambda\,|f''(0)|}\frac{1-|z_0|^2}{|b-z_0|^2}\right]\right).$$

The inequality (9) *is sharp, with equality for each possible value of* $|f''(0)|$ *and* $|f''(z_0)|$.

Proof. Let

$$\rho(z)=\frac{z-z_0}{1-\overline{z_0}z}.$$

In addition, let $F: D\to D$ be an holomorphic function and $z_0\in D$ such that

$$\left|\frac{F(z)-F(z_0)}{1-\overline{F(z_0)}F(z)}\right| \leq \left|\frac{z-z_0}{1-\overline{z_0}z}\right| = |\rho(z)|$$

and

$$|F(z)| \leq \frac{|F(z_0)|+|\rho(z)|}{1+|F(z_0)|\,|\rho(z)|} \tag{10}$$

by Schwarz-Pick Lemma [6]. If $k: D\to D$ is a holomorphic function and $0<|z_0|<1$, letting

$$F(z)=\frac{k(z)-k(0)}{z\left(1-\overline{k(0)}k(z)\right)}$$

in (10), we obtain

$$\left|\frac{k(z)-k(0)}{z\left(1-\overline{k(0)}k(z)\right)}\right| \leq \frac{\left|\frac{k(z_0)-k(0)}{z_0\left(1-\overline{k(0)}k(z_0)\right)}\right|+|\rho(z)|}{1+\left|\frac{k(z_0)-k(0)}{z_0\left(1-\overline{k(0)}k(z_0)\right)}\right||\rho(z)|}$$

and

$$|k(z)| \leq \frac{|k(0)|+|z|\,\frac{|C|+|\rho(z)|}{1+|C||\rho(z)|}}{1+|k(0)|\,|z|\,\frac{|C|+|\rho(z)|}{1+|C||\rho(z)|}}, \tag{11}$$

where

$$C=\frac{k(z_0)-k(0)}{z_0\left(1-\overline{k(0)}k(z_0)\right)}.$$

Without loss of generality, we will assume that $b=1$. If we take

$$k(z)=\frac{p(z)}{z\left(\frac{z-z_0}{1-\overline{z_0}z}\right)},$$

then

$$k(0)=\frac{p'(0)}{-z_0},\ k(z_0)=\frac{p'(z_0)\left(1-|z_0|^2\right)}{z_0}$$

and

$$C=\frac{\frac{p'(z_0)\left(1-|z_0|^2\right)}{z_0}+\frac{p'(0)}{z_0}}{z_0\left(1+\frac{\overline{p'(0)}}{\overline{z_0}}\frac{p'(z_0)\left(1-|z_0|^2\right)}{z_0}\right)},$$

where $|C|\leq 1$. Let $|k(0)|=\beta$ and

$$A=\frac{\left|\frac{p'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{p'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{p'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\left|\frac{p'(0)}{z_0}\right|\right)}.$$

From (11), we get

$$|p(z)|\leq |z|\,|\rho(z)|\,\frac{\beta+|z|\,\frac{A+|\rho(z)|}{1+A|\rho(z)|}}{1+\beta\,|z|\,\frac{A+|\rho(z)|}{1+A|\rho(z)|}}$$

and

$$\frac{1-|p(z)|}{1-|z|}\geq\frac{1+\beta\,|z|\,\frac{A+|\rho(z)|}{1+A|\rho(z)|}-\beta\,|z|\,|\rho(z)|-|\rho(z)|\,|z|^2\,\frac{A+|\rho(z)|}{1+A|\rho(z)|}}{(1-|z|)\left(1+\beta\,|z|\,\frac{A+|\rho(z)|}{1+A|\rho(z)|}\right)}.$$

Let $\kappa(z)=1+\beta\,|z|\,\frac{A+|\rho(z)|}{1+A|\rho(z)|}$ and $\tau(z)=1+A\,|\rho(z)|$. Therefore, we obtain

$$\frac{1-|p(z)|}{1-|z|} \tag{12}$$

$$\geq\frac{1}{\kappa(z)\tau(z)}\left\{\frac{1-|z|^2\,|\rho(z)|^2}{1-|z|}+A\,|\rho(z)|\,\frac{1-|z|^2}{1-|z|}+\beta\,|z|\,A\frac{1-|\rho(z)|^2}{1-|z|}\right\}.$$

Since

$$\lim_{z\to 1}\kappa(z)=\lim_{z\to 1}\left(1+\beta\,|z|\,\frac{A+|\rho(z)|}{1+A\,|\rho(z)|}\right)=1+\beta,$$

$$\lim_{z\to 1}\tau(z)=\lim_{z\to 1}\left(1+A\,|\rho(z)|\right)=1+A,$$

$$\lim_{z\to 1}\frac{1-|z|^i\left|\frac{z-z_0}{1-\overline{z_0}z}\right|^j}{1-|z|}=i+j\frac{1-|z_0|^2}{|1-z_0|^2}$$

for non-negative integers i and j and

$$1-|\rho(z)|^2=1-\left|\frac{z-z_0}{1-\overline{z_0}z}\right|^2=\frac{\left(1-|z_0|^2\right)\left(1-|z|^2\right)}{|1-\overline{z_0}z|^2},$$

passing to the angular limit in (12) gives

$$|p'(1)| \geq 1 + \frac{1-|z_0|^2}{|1-z_0|^2} + \frac{1-\beta}{1+\beta}\left[1 + \frac{1-A}{1+A}\frac{1-|z_0|^2}{|1-z_0|^2}\right].$$

Moreover, since

$$\frac{1-\beta}{1+\beta} = \frac{1-|k(0)|}{1+|k(0)|} = \frac{1-\left|\frac{p'(0)}{z_0}\right|}{1+\left|\frac{p'(0)}{z_0}\right|} = \frac{|z_0|-|p'(0)|}{|z_0|+|p'(0)|} = \frac{2\,|c|\cos\lambda\,|z_0|-|b_2|}{2\,|c|\cos\lambda\,|z_0|+|b_2|},$$

$$\frac{1-A}{1+A} = \frac{1-\frac{\left|\frac{p'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{p'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{p'(z_0)(1-|z_0|^2)}{z_0}\right|\left|\frac{p'(0)}{z_0}\right|\right)}}{1+\frac{\left|\frac{p'(z_0)(1-|z_0|^2)}{z_0}\right|+\left|\frac{p'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{p'(z_0)(1-|z_0|^2)}{z_0}\right|\left|\frac{p'(0)}{z_0}\right|\right)}}$$

and

$$\frac{1-A}{1+A} = \frac{|z_0|^2+|p'(z_0)|\left(1-|z_0|^2\right)|p'(0)|-|p'(z_0)|\left(1-|z_0|^2\right)-|p'(0)|}{|z_0|^2+|\phi'(z_0)|\left(1-|z_0|^2\right)|p'(0)|+|p'(z_0)|\left(1-|z_0|^2\right)+|p'(0)|},$$

we obtain

$$|p'(1)| \geq 1 + \frac{1-|z_0|^2}{|1-z_0|^2} + \frac{4\,|c|\cos\lambda\,|z_0|-|f''(0)|}{4\,|c|\cos\lambda\,|z_0|+|f''(0)|}$$
$$\times\left[1+\frac{8\,|c|^2\cos^2\lambda\,|z_0|^2+|f''(z_0)|\,|f''(0)|\left(1-|z_0|^2\right)-4\,|c|\cos\lambda\,|f''(z_0)|\left(1-|z_0|^2\right)-4\,|c|\cos\lambda\,|f''(0)|}{8\,|c|^2\cos^2\lambda\,|z_0|^2+|f''(z_0)|\,|f''(0)|\left(1-|z_0|^2\right)+4\,|c|\cos\lambda\,|f''(z_0)|\left(1-|z_0|^2\right)+4\,|c|\cos\lambda\,|f''(0)|}\frac{1-|z_0|^2}{|1-z_0|^2}\right].$$

From definition of $p(z)$, we have

$$|p'(1)| = \frac{2\,|m'(1)|}{|c|\cos\lambda}.$$

We thus take the inequality (9).

Choose arbitrary real numbers z_0, x and y such that $0 < x = |p'(0)| < |z_0|^2$, $0 < y = |p'(z_0)| < \frac{|z_0|^2}{\left(1-|z_0|^2\right)^2}$ in order to show that the inequality (9) is sharp.

Let

$$\mathtt{B} = \frac{1}{z_0^2}\frac{y\left(1-|z_0|^2\right)+x}{1+xy\frac{1-|z_0|^2}{z_0^2}}.$$

The auxiliary function

$$p(z)=z\left(\frac{z-z_0}{1-\overline{z_0}z}\right)\frac{-\frac{x}{z_0}+z\frac{\mathsf{B}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathsf{B}\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{x}{z_0}z\frac{\mathsf{B}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathsf{B}\frac{z-z_0}{1-\overline{z_0}z}}}. \tag{13}$$

From (13), with the simple calculations, we obtain

$$p'(0)=x,\ p'(z_0)=y$$

and

$$p'(1)=1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{z_0+x}{z_0-x}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+xy\left(1-|z_0|^2\right)-y\left(1-|z_0|^2\right)-x}{z_0^2+xy\left(1-|z_0|^2\right)+y\left(1-|z_0|^2\right)+x}\right).$$

Choosing suitable signs of the numbers z_0, x and y, we conclude from the last equality that the inequality (9) is sharp. □

3. Examples

The examples given below are shown to provide the conditions of Lemma 1.1 and Theorem 2.1.

Example 3.1. Let us consider a function $f(z)$ given by

$$f(z)=ze^{\int_0^z \frac{2c\cos\lambda\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}}{e^{i\lambda}\left(1-t\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}dt}.$$

With logarithmic differentiation, we get

$$\frac{zf'(z)}{f(z)}=1+\frac{2c\cos\lambda z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{e^{i\lambda}\left(1-z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}.$$

By using the last equality, we obtain

$$\begin{aligned}
&\frac{1}{c\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}-(1-c)\cos\lambda-i\sin\lambda\right]\\
&=\frac{1}{c\cos\lambda}\left[e^{i\lambda}\left(1+\frac{2c\cos\lambda z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{e^{i\lambda}\left(1-z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}\right)-(1-c)\cos\lambda-i\sin\lambda\right]\\
&=\frac{1}{c\cos\lambda}\left[e^{i\lambda}+\frac{2c\cos\lambda z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{\left(1-z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}-e^{i\lambda}+c\cos\lambda\right]
\end{aligned}$$

$$= \frac{1}{c\cos\lambda}\left[\frac{2c\cos\lambda z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{\left(1-z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}+c\cos\lambda\right]$$

and, thus,

$$\frac{1}{c\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}-(1-c)\cos\lambda-i\sin\lambda\right]=\frac{1+z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1-z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}. \tag{14}$$

If we take the real part of (14), then we have

$$\Re\left(\frac{1+z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1-z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}\right)=\frac{1-\left|z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right|^2}{\left|1-z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right|^2}>0.$$

Example 3.2. Let us consider a function $f(z)$ given by

$$f(z)=ze^{-\int_0^z\frac{2c\cos\lambda\left(t\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}{e^{i\lambda}\left(1+t^2\prod_{i=1}^{n}\frac{t-z_i}{1-\overline{z_i}t}\right)}dt}.$$

By logarithmic differentiation, we get

$$\frac{zf'(z)}{f(z)}=1-\frac{2c\cos\lambda\left(z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}{e^{i\lambda}\left(1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}. \tag{15}$$

With (15), we get

$$\frac{1}{c\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}-(1-c)\cos\lambda-i\sin\lambda\right]=\frac{1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}$$

and, thus,

$$\Re\left(\frac{1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}\right)=\frac{1-\left|z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right|^2}{\left|1+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right|^2}>0.$$

The example given below has been shown to provide the conditions of Theorem 2.2.

Example 3.3. Let us consider a function $p(z)$ given by

$$p(z)=z\left(\frac{z-z_0}{1-\overline{z_0}z}\right)\frac{-\frac{x}{z_0}+z\frac{\mathtt{B}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathtt{B}\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{x}{z_0}z\frac{\mathtt{B}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathtt{B}\frac{z-z_0}{1-\overline{z_0}z}}},$$

where

$$p(z)=e^{i\lambda}\frac{m(z)-1}{e^{i\lambda}\left(m(z)-1\right)+2c\cos\lambda},\ m(z)=\frac{zf'(z)}{f(z)}.$$

Hence, we get

$$m(z)=\frac{e^{i\lambda}\left(1-p(z)\right)+2c\cos\lambda p(z)}{e^{i\lambda}\left(1-p(z)\right)}=1+\frac{2c\cos\lambda p(z)}{e^{i\lambda}\left(1-p(z)\right)}.$$

Thus,

$$\frac{1}{c\cos\lambda}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}-(1-c)\cos\lambda-i\sin\lambda\right]=\frac{1+p(z)}{1-p(z)}. \tag{16}$$

Consequently, we obtain that the real part of (16) is positive because of the following

$$\Re\left(\frac{1+p(z)}{1-p(z)}\right)=\frac{1-|p(z)|^2}{|1-p(z)|^2}>0.$$

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Tuğba Akyel
The Faculty of Engineering and Natural Sciences
Maltepe University
Maltepe, Istanbul, Turkey
Email address: tugakyel@gmail.com, tugbaakyel@maltepe.edu.tr