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LOXODROMES AND TRANSFORMATIONS IN PSEUDO-HERMITIAN GEOMETRY

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Abstract. In this paper, we prove that a diffeomorphism f on a normal almost contact 3-manifold M is a CRL-*transformation* if and only if M is an α-Sasakian manifold. Moreover, we show that a CR-loxodrome in an α-Sasakian 3-manifold is a pseudo-Hermitian magnetic curve with a strength $q = \widetilde{r}\eta(\gamma') = (r + \alpha - t)\eta(\gamma')$ for constant $\eta(\gamma')$. A non-geodesic CR-loxodrome is a non-Legendre slant helix. Next, we prove that let M be an α-Sasakian 3-manifold such that $(\nabla_Y S)X = 0$ for vector fields Y to be orthogonal to ξ, then the Ricci tensor ρ satisfies $\rho = 2\alpha^2 g$. Moreover, using the CRL-*transformation* $\widetilde{\nabla}^t$ we fine the pseudo-Hermitian curvature $\widetilde{R}$, the pseudo-Ricci tensor $\widetilde{\rho}$ and the torsion tensor field $\widetilde{\mathfrak{T}}^t(\widetilde{S}X, Y)$.

1. Introduction

Tashiro and Tachibana introduced the notion of C-loxodrome [12, p. 182] (see also [5, pp. 123–124]).

Definition. Let M be an almost contact metric manifold. An arc length parametrized curve $\gamma(s)$ in M is said to be a *C-loxodrome* if it satisfies

$$\nabla_{\gamma'}\gamma' = r\eta(\gamma')\varphi\gamma'$$

for some constant r.

Note that Tashiro and Tachibana also introduced the notion of C-loxodrome in almost contact metric manifold M equipped with an affine connection D by the ODE:

$$D_{\gamma'}\gamma' = \alpha\gamma' + r\eta(\gamma')\varphi\gamma'.$$

In this ODE system s is a general parameter.

A diffeomorphism f on an Sasakian manifold is said to be a CL-*transformation* if it carries C-loxodromes to C-loxodromes [12]. Takamatsu and Mizusawa

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studied infinitesimal CL-transformations on compact Sasakian manifolds [11]. As an analogue of Wely's conformal curvature tensor field, Koto and Nagao introduced CL-curvature tensor field for Sasakian manifolds. The CL-curvature tensor field is invariant under CL-transformations. Koto and Nagao showed that Sasakian space forms are characterized as Sasakian manifolds with vanishing CL-curvature tensor fields [7].

Now, an arc length parametrized curve γ in almost contact metric manifold M is said to be a CR-*loxodrome* if it satisfies

$$\widetilde{\nabla}^t_{\gamma'}\gamma' = \widetilde{r}\eta(\gamma')\varphi\gamma'$$

for some constant $\widetilde{r}$.

Let (N, h) be a Riemmannian manifold and $f : N \to (M, \eta, \widetilde{\nabla}^t)$ a smooth map into an almost contact metric manifold with an affine connection $\widetilde{\nabla}^t$. Then f is said to be a CRL-*transformation* if it carries C-loxodromes to CR-loxodromes.

In this paper, we study a CR-loxodrome and CRL-*transformation* in an α-Sasakian 3-manifold. In Section 3, we prove that a diffeomorphism f on a normal almost contact 3-manifold M is a CRL-*transformation* if and only if M is an α-Sasakian manifold. Moreover, we show that a CR-loxodrome in an α-Sasakian 3-manifold is a pseudo-Hermitian magnetic curve with a strength $q = \widetilde{r}\eta(\gamma') = (r+\alpha-t)\eta(\gamma')$ for constant $\eta(\gamma')$. A non-geodesic CR-loxodrome is a non-Legendre slant helix.

In Section 4 we prove that let M be an α-Sasakian 3-manifold such that $(\nabla_Y S)X = 0$ for vector fields Y to be orthogonal to ξ, then the Ricci tensor ρ satisfies $\rho = 2\alpha^2 g$. Moreover, using the CRL-*transformation* $\widetilde{\nabla}^t$ we fine the pseudo-Hermitian curvature $\widetilde{R}$, the pseudo-Ricci tensor $\widetilde{\rho}$ and the torsion tensor field $\widetilde{\mathfrak{T}}^t(\widetilde{S}X, Y)$.

2. Preliminaries

2.1. Almost contact manifolds

Let M be a manifold of odd dimension $m = 2n + 1$. Then M is said to be an *almost contact manifold* if its structure group $\mathrm{GL}_m\mathbb{R}$ of the linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$. This is equivalent to existence of a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

From these conditions one can deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

Moreover, since $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n + 1)$, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all X, $Y \in \mathfrak{X}(M)$. Here $\mathfrak{X}(M) = \Gamma(TM)$ denotes the Lie algebra of all smooth vector fields on M. Such a metric is called an *associated metric* of the almost contact manifold $M = (M, \varphi, \xi, \eta)$. With respect to the associated metric g, η is metrically dual to ξ, that is

$$g(X, \xi) = \eta(X)$$

for all $X \in \mathfrak{X}(M)$. A structure (φ, ξ, η, g) on M is called an *almost contact metric structure*, and a manifold M equipped with an almost contact metric structure is said to be an *almost contact metric manifold.* A plane section Π at a point p of an almost contact metric manifold M is said to be *holomorphic* if it is invariant under φ_p. The sectional curvature function $\mathcal{H}$ of holomorphic plane sections is called the *holomorphic sectional curvature* (also called φ-sectional curvature).

Now let us consider a Riemannian product manifold $\bar{M} = (M \times \mathbb{R}, g + dt^2)$. We equip an almost complex structure J on $\bar{M}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right), \quad X \in \mathfrak{X}(M), \quad f \in C^\infty(\bar{M}).$$

Then $(\bar{M}, J)$ equipped with the product metric $\bar{g} = g + dt^2$ is an almost Hermitian manifold with Kähler form $\Omega = \Phi - 2\eta \wedge dt$.

An almost contact metric manifold M is said to be *normal* if J is integrable. In particular, a normal contact metric manifold is called a *Sasakian manifold.*

2.2. Normal almost contact manifold

For an arbitrary almost contact metric 3-manifold M, we have ([9]):

$$(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi, \tag{2.1}$$

where ∇ is the Levi-Civita connection on M. Moreover, we have

$$\mathrm{d}\eta = \eta \wedge \nabla_\xi \eta + \alpha\Phi, \quad \mathrm{d}\Phi = 2\beta\eta \wedge \Phi,$$

where α and β are the functions defined by

$$\alpha = \frac{1}{2}\mathrm{Trace}\,(\varphi\nabla\xi), \quad \beta = \frac{1}{2}\mathrm{Trace}\,(\nabla\xi) = \frac{1}{2}\mathrm{div}\,\xi. \tag{2.2}$$

Olszak [9] showed that an almost contact metric 3-manifold M is normal if and only if $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$ or, equivalently,

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \quad X \in \Gamma(TM). \tag{2.3}$$

We call the pair (α, β) the *type* of a normal almost contact metric 3-manifold M.

Using (2.1) and (2.3) we note that the covariant derivative $\nabla\varphi$ of a 3-dimensional normal almost contact metric manifold is given by

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X). \tag{2.4}$$

Moreover M satisfies

$$2\alpha\beta + \xi(\alpha) = 0.$$

Thus if α is a nonzero constant, then $\beta = 0$. In particular, a normal almost contact metric 3-manifold is said to be

- *cosymplectic* (or *coKähler*) *manifold* if $\alpha = \beta = 0$,
- *quasi-Sasakian manifold* if $\beta = 0$ and $\xi(\alpha) = 0$,
- α*-Sasakian manifold* if α is a nonzero constant and $\beta = 0$,
- β*-Kenmotsu manifold* if $\alpha = 0$ and β is a nonzero constant.

1-Sasakian manifolds and 1-Kenmotsu manifolds are simply called *Sasakian manifolds* and *Kenmotsu manifolds*, respectively. Sasakian manifolds are characterized as normal contact metric 3-manifolds.

2.3. Frenet-Serret equations

Let $\gamma : I \to M^3$ be a curve parameterized by arc-length in an almost contact metric 3-manifold M^3. We may define a Frenet frame fields (T, N, B) along γ. Then they satisfy the following

$$\begin{cases} \nabla_T T = \kappa N, \\ \nabla_T N = -\kappa T + \tau B, \\ \nabla_T B = -\tau N, \end{cases} \tag{2.5}$$

where $\kappa = |\nabla_T T|$ is the *geodesic curvature* of γ and τ its *geodesic torsion.*

A *helix* is a curve with constant geodesic curvature and geodesic torsion. In particular, curves with constant nonzero geodesic curvature and zero geodesic torsion are called (*Riemannian*) *circles*. Note that geodesics are regarded as helices with zero geodesics curvature and torsion.

2.4. Canonical connections

Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Define a tensor field $A = A^t$ of type $(1, 2)$ by

$$A^t_X Y = -\frac{1}{2}\varphi(\nabla_X \varphi)Y - \frac{1}{2}\eta(Y)\nabla_X \xi - t\eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi \tag{2.6}$$

for all vector fields X and Y. Here t is a real constant. We define a linear connection $\widetilde{\nabla}^t$ on M by

$$\widetilde{\nabla}^t_X Y = \nabla_X Y + A^t_X Y. \tag{2.7}$$

We call the connection $\widetilde{\nabla}^t$ the *canonical connection* of M.

Now, we assume that M is a normal almost contact metric 3-manifold (or more generally, trans-Sasakian manifold of general dimension) of type (α, β). Then (2.6) is reduced to

$$\begin{aligned} A^t_X Y = {} & \alpha\{g(X, \varphi Y)\xi + \eta(Y)\varphi X\} \\ & + \beta\{g(X, Y)\xi - \eta(Y)X\} - t\eta(X)\varphi Y. \end{aligned} \tag{2.8}$$

The torsion tensor field $\mathfrak{T}^t$ of $\widetilde{\nabla}^t$ is given by

$$\widetilde{\mathfrak{T}}^t(X, Y) = \alpha\{2g(X, \varphi Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X\} \tag{2.9}$$

$$+\eta(X)(\beta Y - t\varphi Y) - \eta(Y)(\beta X - t\varphi X).$$

Note that the connection ∇^0 is the (φ, ξ, η)-connection introduced by Sasaki and Hatakeyama in [10]. Moreover $\widetilde{\nabla}^1$ was introduced by Cho [2].

The canonical connection $\widetilde{\nabla}^t$ on an almost contact metric manifold satisfies the following conditions:

$$\widetilde{\nabla}^t \varphi = 0, \quad \widetilde{\nabla}^t \xi = 0, \quad \widetilde{\nabla}^t \eta = 0, \quad \widetilde{\nabla}^t g = 0.$$

3. CR-loxodromes

3.1. C-loxodrome

Tashiro and Tachibana introduced the notion of C-loxodrome [12, p. 182] (see also [5, pp. 123–124], [13]). Let M be an almost contact metric manifold. An arc length parametrized curve $\gamma(s)$ in M is said to be a *C-loxodrome* if it satisfies

$$\nabla_{\gamma'}\gamma' = r\eta(\gamma')\varphi\gamma' \tag{3.1}$$

for some constant r.

Differentiating $\eta(\gamma')$ in a normal almost contact 3-manifold M, we get

$$\begin{aligned} \eta(\gamma')' &= g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi) \\ &= g(r\eta(\gamma')\varphi\gamma', \xi) + g(\gamma', -\alpha\varphi\gamma' + \beta(\gamma' - \eta(\gamma')\xi)) \\ &= \beta(1 - \eta(\gamma')^2). \end{aligned} \tag{3.2}$$

We assume that γ is a slant curve, that is $\eta(\gamma')$ is a constant, then we have $\beta = 0$ or γ is a integral curve of ξ. Hence we have:

Proposition 3.1. *Let γ be a C-loxodrome in a normal almost contact* 3-*manifold M* (*for Levi-Civita connection ∇*)*. If γ is a slant curve, then M is a quasi-Sasakian* 3-*manifold or γ integral curve of ξ.*

Moreover, since $\eta(\gamma)$ is a constant along a C-loxodrome γ in an α-Sasakian manifold M, we have

Proposition 3.2. *Let γ be a C-loxodrome in α-Sasakian manifolds M* (*for Levi-Civita connection ∇*)*. Then γ is a slant helix with*

$$\kappa = \mid r\eta(\gamma') \mid \sqrt{1 - \eta(\gamma')^2}, \quad \tau = \alpha + r\eta(\gamma')^2,$$

where $\eta(\gamma')$ is a constant. Moreover, the ratio of κ and $\tau - \alpha$ is constant.

3.2. CR-loxodromes

In this subsection we assume that M is a normal almost contact metric 3-manifold.

Definition. An arc length parametrized curve γ in almost contact metric 3-manifold M is said to be a CR-*loxodrome* if it satisfies

$$\widetilde{\nabla}^t_{\gamma'}\gamma' = \widetilde{r}\eta(\gamma')\varphi\gamma' \tag{3.3}$$

for some constant $\widetilde{r}$.

Definition. Let (N,h) be a Riemmannian manifold and $f : N \to (M, \eta, \widetilde{\nabla}^t)$ a smooth map into an almost contact metric manifold with an affine connection $\widetilde{\nabla}^t$. Then f is said to be a CRL-*transformation* if it carries C-loxodromes to CR-loxodromes.

Using (2.8) for a C-loxodrome in a normal almost contact 3-manifold we have

$$\begin{aligned}\widetilde{\nabla}^t_{\gamma'}\gamma' &= \nabla_{\gamma'}\gamma' + (\alpha - t)\eta(\gamma')\varphi\gamma' + \beta(\xi - \eta(\gamma')\gamma') \\ &= (r + \alpha - t)\eta(\gamma')\varphi\gamma' + \beta(\xi - \eta(\gamma')\gamma'). \end{aligned} \tag{3.4}$$

From this we have:

Theorem 3.3. *A diffeomorphism f on a normal almost contact 3-manifold M is a CRL-*transformation *if and only if M is an α-Sasakian manifold.*

From now on, just think about the α-Sasakian manifold. From (3.4) in an α-Sasakian manifold M we have

$$\widetilde{\nabla}^t_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + (\alpha - t)\eta(\gamma')\varphi\gamma'. \tag{3.5}$$

Since differentiating $\eta(\gamma')$ along CR-loxodromes with respect to the canonical affine connection $\widetilde{\nabla}^t$, we have that $\eta(\gamma')$ is a constant. From (3.3) and (3.5) we have

Proposition 3.4. *Let γ be a C-loxodrom in an α-Sasakian manifold M (with respect to the Levi-Civita connection ∇). Then γ is a CR-loxodrome with respect to the canonical affine connection $\widetilde{\nabla}^t$. Moreover, γ is a slant helix with*

$$\widetilde{\kappa} = \mid \widetilde{r}\eta(\gamma') \mid \sqrt{1 - \eta(\gamma')^2}, \quad \widetilde{\tau} = \widetilde{r}\eta(\gamma')^2,$$

where $\eta(\gamma')$ is a constant and $\widetilde{r} = r + \alpha - t$. Moreover, the ratio of $\widetilde{\kappa}$ and $\widetilde{\tau}$ is constant.

3.3. Pseudo-Hermitian magnetic curves

Now, let us consider a contact magnetic curve in normal almost contact metric 3-manifolds from a pseudo-Hermitian geometrical point of view (see [8]).

Definition. A regular curve γ is said to be a *pseudo-Hermitian magnetic curve* in an almost contact metric manifold M if it satisfies the Lorentz equation with respect to the canonical affine connection:

$$\widetilde{\nabla}^t_{\gamma'}\gamma' = q\varphi\gamma'. \tag{3.6}$$

Using the equation (3.4) and (3.6), we have:

Theorem 3.5. *A CR-loxodrome in an α-Sasakian 3-manifold is a pseudo-Hermitian magnetic curve with a strength $q = \widetilde{r}\eta(\gamma') = (r + \alpha - t)\eta(\gamma')$ for constant $\eta(\gamma')$. A non-geodesic CR-loxodrome is a non-Legendre slant helix.*

From the equation (3.5) and (3.6), we get:

Proposition 3.6. *Let M be an α-Sasakian 3-manifold. Then γ is a pseudo-Hermitian magnetic curve if and only if it satisfies*

$$\nabla_{\gamma'}\gamma' = \{(t-\alpha)\cos\theta + q\}\varphi\gamma'. \tag{3.7}$$

From the above equation (3.7) and Frenet-Serret equation (2.5) we have the geodesic curvature

$$\kappa = |(t-\alpha)\cos\theta + q|\sin\theta,$$

and the normal vector field $N = \frac{\varepsilon}{\sin\theta}\varphi\gamma'$, where $\varepsilon = \frac{(t-\alpha)\cos\theta+q}{|(t-\alpha)\cos\theta+q|}$.

Thus the binormal vector field B is computed as

$$B = \gamma' \times N = \frac{\varepsilon}{\sin\theta}\{\xi - \cos\theta\gamma'\}. \tag{3.8}$$

Differentiating the above equation (3.8) then we have

$$\begin{aligned}\nabla_{\gamma'}B =& \nabla_{\gamma'}\frac{\varepsilon}{\sin\theta}\{\xi - \eta(\gamma')\gamma'\}\\ =& \frac{\varepsilon}{\sin\theta}\{-\alpha\varphi\gamma' - g(\kappa N, \xi) - g(\gamma', -\alpha\varphi\gamma')\gamma' - \eta(\gamma')\kappa N\}\\ =& -\frac{\varepsilon}{\sin\theta}\{\alpha + ((t-\alpha)\eta(\gamma') + q)\cos\theta\}\varphi\gamma'.\end{aligned}$$

Since the normal vector field $N = \frac{\varepsilon}{\sin\theta}\varphi\gamma'$, using the Frenet-Serret equation (2.5) we have:

Theorem 3.7. *Let γ be a pseudo-Hermitian magnetic curve in an α-Sasakian 3-manifold M. Then γ is a slant helix with*

$$\kappa = |(t-\alpha)\cos\theta + q|\sin\theta, \quad \tau = \alpha + \{(t-\alpha)\cos\theta + q\}\cos\theta.$$

Moreover, the ratio of κ and $\tau - \alpha$ is constant.

In particular, for a Sasakian 3-manifold with respect to the Tanaka-Webster connection, that is $t = -1$ and $\alpha = 1$, we have:

Corollary 3.8 (cf. [4])**.** *Let γ be a pseudo-Hermitian magnetic curve in an Sasakian 3-manifold M. Then γ is a slant helix with*

$$\kappa = |q - 2\cos\theta|\sin\theta, \quad \tau = 1 + \{q - 2\cos\theta\}\cos\theta.$$

Moreover, the ratio of κ and $\tau - 1$ is constant.

From the equation (3.7) if γ is an almost Legendre curve, then it satisfies $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$, hence we have:

Corollary 3.9. *If γ is an almost Legendre curve in an α-Sasakian 3-manifold M, then γ is a pseudo-Hermitian magnetic curve if and only if γ is a contact magnetic curve. Moreover, it has*

$$\kappa = | q |, \quad \tau = \alpha.$$

Note that Inoguchi, Munteanu and Nistor studied magnetic curves in α-Sasakian 3-manifolds with respect to the Levi-Civita connection as following:

Lemma 3.10 ([6]). *Let γ be a contact magnetic curve in an α-Sasakian 3-manifold M. Then γ is a slant helix with*

$$\kappa = |q| \sin\theta, \qquad \tau = \alpha + q\cos\theta,$$

and the ratio of κ and $\tau - \alpha$ is constant.

Remark 3.11. A contact magnetic curve has Legendre curve, but both C-loxodrome and CR-loxodrome don't have a non-geodesic Legendre curve in α-Sasakian 3-manifolds. So both C-loxodrome and CR-loxodrome have only non-Legendre slant helices in α-Sasakian 3-manifolds.

4. CRL-transformations

A normal almost contact metric manifold is said to be an α-Sasakian if $d\eta = \alpha\Phi$ for some nonzero constant α. Sasakian manifolds are regarded as 1-Sasakian manifolds.

The covariant derivatives of φ and ξ of an α-Sasakian manifold M are given by

$$(\nabla_X\varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X), \quad \nabla_X\xi = -\alpha\varphi X \tag{4.1}$$

for a nonzero constant α on M.

The Riemannian curvature R defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ in a $(2n+1)$-dimensional α-Sasakian manifold satisfies

$$R(X,Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\}. \tag{4.2}$$

Moreover, the Ricci operator S satisfies ([1])

$$\rho(X,\xi) = 2n\alpha^2\eta(X) \quad \text{and} \quad S\xi = 2n\alpha^2\xi, \tag{4.3}$$

where $\rho(X,Y) = g(SX,Y)$.

4.1. For Levi-Civita connection

We recall the curvature R of a 3-dimensional Riemannian manifold is expressed by

$$\begin{aligned} R(Y,X)Z = {} & \rho(X,Z)Y - \rho(Y,Z)X + g(X,Z)SY - g(Y,Z)SX \\ & - \frac{1}{2}r\{g(X,Z)Y - g(Y,Z)X\} \end{aligned} \tag{4.4}$$

for all vector fields X, Y, Z and r denotes the scalar curvature. Then using (2.3) and (4.4) we have:

Proposition 4.1. *For an α-Sasakian 3-manifold, we have the Ricci operator*:

$$S = -(\alpha^2 - \frac{r}{2})I + (3\alpha^2 - \frac{r}{2})\eta\otimes\xi, \tag{4.5}$$

where I denotes the identity transformation.

Differentiating (4.5) we get

$$(\nabla_Y S)X = \frac{1}{2}(Yr)(X - \eta(X)\xi) - \alpha(3\alpha^2 - \frac{r}{2})\{g(X, \varphi Y)\xi - \eta(X)\varphi Y\}$$

for any vector field X, Y on M.

Since $(\nabla_\xi S)X = 0$ for any X in an α-Sasakian 3-manifold, we consider $(\nabla_Y S)X = 0$ for vector fields Y to be orthogonal to ξ. Hence we have:

Proposition 4.2. *For an α-Sasakian 3-manifold, $(\nabla_Y S)X = 0$ for vector fields Y to be orthogonal to ξ if and only if $r = 6\alpha^2$.*

From Proposition 4.4 and Proposition 4.2 hence we have:

Theorem 4.3. *Let M be an α-Sasakian 3-manifold such that $(\nabla_Y S)X = 0$ for vector fields Y to be orthogonal to ξ. Then the Ricci tensor ρ satisfies $\rho = 2\alpha^2 g$.*

4.2. *CRL*-connection

Let M be an α-Sasakian manifold with the canonical affine connection $\widetilde{\nabla}^t$. Then the affine connection $\widetilde{\nabla}^t$ carries C-loxodromes to CR-loxodromes, so it is called a CRL-transformation. Moreover, the affine connection $\widetilde{\nabla}^t$ is called a CRL-connection, induced by Levi-Civita connection ∇.

In an α-Sasakian 3-manifold, (2.8) is reduced to

$$A^t_X Y = \alpha\{g(X, \varphi Y)\xi + \eta(Y)\varphi X\} - t\eta(X)\varphi Y. \tag{4.6}$$

From this we have:

Lemma 4.4. *In an α-Sasakian manifold M we have*

(1) $\varphi\widetilde{\nabla}^t_X Y = \varphi\nabla_X Y - \alpha\eta(Y)X + t\eta(X)Y + (\alpha - t)\eta(X)\eta(Z)\xi$,

(2) $g(\widetilde{\nabla}^t_X Y, \varphi Z) = g(\nabla_X Y, \varphi Z) + \alpha\eta(Y)g(X, Z) - t\eta(X)g(Y, Z) + (t - \alpha)\eta(X)\eta(Y)\eta(Z)$,

(3) $\eta(\widetilde{\nabla}^t_X Y) = \eta(\nabla_X Y) + \alpha g(X, \varphi Y)$.

The pseudo-Hermitian curvature $\widetilde{R}(X, Y)Z$ is defined by

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}^t_X\widetilde{\nabla}^t_Y Z - \widetilde{\nabla}^t_Y\widetilde{\nabla}^t_X Z - \widetilde{\nabla}^t_{[X,Y]}Z$$

for the affine connection $\widetilde{\nabla}^t$ in an α-Sasakian manifold. The pseudo-Ricci tensor $\widetilde{\rho}$ with respect to the affine connection $\widetilde{\nabla}^t$ in an α-Sasakian manifold M is defined by

$$\widetilde{\rho}(X, Y) = trace\ of\{V \to \widetilde{R}(V, X)Y\},$$

where X, Y are vector fields in M. Using the Lemma 4.4, we have:

Proposition 4.5. *Let M be an α-Sasakian manifold M with the canonical affine connection $\widetilde{\nabla}^t$.*

(1) *The pseudo-Hermitian curvature is*

$$\begin{aligned}\widetilde{R}(X, Y)Z = R(X, Y)Z &+ \alpha^2[\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}\xi \\ &+ \eta(Z)\{\eta(X)Y - \eta(Y)X\} + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X]\end{aligned}$$

$$- 2\alpha t g(X, \varphi Y)\varphi Z. \tag{4.7}$$

(2) *For an local orthonormal frame field* e_i, $i = 1, 2, \ldots, 2n+1$, *the pseudo-Ricci tensor is*

$$\widetilde{\rho} = \rho - 2\alpha t g. \tag{4.8}$$

Using Theorem 4.3 and Proposition 4.5 we have:

Theorem 4.6. *Let* M *be an* α*-Sasakian* 3*-manifold such that* $(\nabla_Y S)X = 0$ *for vector fields* Y *to be orthogonal to* ξ. *Then the pseudo-Ricci tensor is*

$$\widetilde{\rho} = 2\alpha(\alpha - t)g. \tag{4.9}$$

Remark 4.7. Replacing $Z = \xi$ in (4.7) then since $\widetilde{\nabla}^t \xi = 0$, $\widetilde{R}(X, Y)\xi = 0$. We can check the equation (2.3). We assume that $\alpha = 1$ and $t = -1$, then for the Tanaka-Webster connection $\hat{\nabla}$ in Sasakian manifolds we have ([3])

$$\begin{aligned}\hat{R}(X, Y)Z = R(X, Y)Z &+ \{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}\xi \\ &+ \eta(Z)\{\eta(X)Y - \eta(Y)X\} + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \\ &+ 2g(X, \varphi Y)\varphi Z.\end{aligned}$$

and

$$\hat{\rho} = \rho + 2g.$$

From the equations (4.5) and (4.8) we have:

Corollary 4.8. *For an* α*-Sasakian* 3*-manifold* M *with respect to the canonical affine connection* $\widetilde{\nabla}^t$, *we have the pseudo-Ricci operator*:

$$\widetilde{S} = -(\alpha^2 + 2\alpha t - \frac{r}{2})I + (3\alpha^2 - \frac{r}{2})\eta \otimes \xi, \tag{4.10}$$

where I *denotes the identity transformation.*

Differentiating the equation (4.10) we get

$$(\widetilde{\nabla}_Y \widetilde{S})X = \frac{1}{2}(Yr)(X - \eta(X)\xi).$$

Hence we have:

Proposition 4.9. *For an* α*-Sasakian* 3*-manifold* M *with respect to the canonical affine connection* $\widetilde{\nabla}^t$, $(\widetilde{\nabla}_Y \widetilde{S})X = 0$ *if and only if the scalar curvature* r *is a constant.*

4.3. Torsion tensor

From (2.9) the torsion tensor field $\widetilde{\mathfrak{T}}^t$ of $\widetilde{\nabla}^t$ is given by

$$\widetilde{\mathfrak{T}}^t(X, Y) = 2\alpha g(X, \varphi Y)\xi + (\alpha + t)\{\eta(Y)\varphi X - \eta(X)\varphi Y\}. \tag{4.11}$$

Using the pseudo-Ricci operator $\widetilde{S}$ we have

$$\widetilde{\mathfrak{T}}^t(\widetilde{S}X, Y) = 2\alpha g(\widetilde{S}X, \varphi Y)\xi + (\alpha + t)\{\eta(Y)\varphi \widetilde{S}X - \eta(\widetilde{S}X)\varphi Y\}.$$

Hence we have:

Proposition 4.10. *Let M be an α-Sasakian 3-manifold. Then the torsion tensor field $\widetilde{\mathfrak{T}}^t$ of $\widetilde{\nabla}^t$ has*

$$\widetilde{\mathfrak{T}}^t(\widetilde{S}X,Y) = \left(\frac{r}{2}-\alpha^2-2\alpha t\right)\{2\alpha g(X,\varphi Y)\xi+(\alpha+t)\eta(Y)\varphi X\} + 2\alpha(t+\alpha)(t-\alpha)\eta(X)\varphi Y. \tag{4.12}$$

In particular, (1) $\widetilde{\mathfrak{T}}^t(\widetilde{S}\xi,Y)=2\alpha(t+\alpha)(t-\alpha)\varphi Y$.

(2) $\widetilde{\mathfrak{T}}^t(\widetilde{S}X,Y)=(\frac{r}{2}-\alpha^2-2\alpha t)\{2\alpha g(X,\varphi Y)\xi+(\alpha+t)\eta(Y)\varphi X\}$ *for vector fields X to be orthogonal to ξ.*

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