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Some Extensions of Rings with Noetherian Spectrum

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Abstract. In this paper, we study rings with Noetherian spectrum, rings with locally Noetherian spectrum and rings with t-locally Noetherian spectrum in terms of the polynomial ring, the Serre's conjecture ring, the Nagata ring and the t-Nagata ring. In fact, we show that a commutative ring R with identity has Noetherian spectrum if and only if the Serre's conjecture ring $R[X]_U$ has Noetherian spectrum, if and only if the Nagata ring $R[X]_N$ has Noetherian spectrum. We also prove that an integral domain D has locally Noetherian spectrum if and only if the Nagata ring $D[X]_N$ has locally Noetherian spectrum. Finally, we show that an integral domain D has t-locally Noetherian spectrum if and only if the polynomial ring $D[X]$ has t-locally Noetherian spectrum, if and only if the t-Nagata ring $D[X]_{N_v}$ has (t-)locally Noetherian spectrum.

1. Introduction

1.1. Star-operations

To help readers better understanding this paper, we briefly review some definitions and notation related to star-operations. Let D be an integral domain with quotient field K and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For an element $I \in \mathbf{F}(D)$, set $I^{-1} = \{x \in K \,|\, xI \subseteq D\}$. The mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$ is called the *v-operation* on D; and the mapping on $\mathbf{F}(D)$ given by $I \mapsto I_t = \bigcup\{J_v \,|\, J$ is a nonzero finitely generated fractional subideal of $I\}$ is called the *t-operation* on D. It is easy to see that $I \subseteq I_t \subseteq I_v$ for all $I \in \mathbf{F}(D)$; and

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if an element $I \in \mathbf{F}(D)$ is finitely generated, then $I_v = I_t$. An element $I \in \mathbf{F}(D)$ is said to be a *t-ideal* of D if $I_t = I$. A *maximal t-ideal* of D means a t-ideal of D which is maximal among proper integral t-ideals of D. It is well known that a maximal t-ideal of D always exists if D is not a field. We say that D is *of finite character* (respectively, *of finite t-character*) if each nonzero nonunit in D belongs to only finitely many maximal ideals (respectively, maximal t-ideals) of D.

1.2. Rings with Noetherian Spectrum

Let R be a commutative ring with identity. Recall that an ideal I of R is *radically finite* if $\sqrt{I} = \sqrt{F}$ for some finitely generated ideal F of R; and R is said to have *Noetherian spectrum* if each ideal of R is radically finite. It is easy to see that an ideal I of R is radically finite if and only if $\sqrt{I} = \sqrt{F}$ for some finitely generated subideal of I. It was shown in [9, Corollary 2.4] that R has Noetherian spectrum if and only if every prime ideal of R is radically finite, if and only if every radical ideal of R is radically finite. Also, it is well known that a class of rings with Noetherian spectrum contains Noetherian rings and SFT rings. (Recall that R is a *Noetherian ring* if it satisfies the ascending chain condition on integral ideals of R (or equivalently, every (prime) ideal of R is finitely generated); and R is a *strong finite type ring* (SFT ring) if for each (prime) ideal I of R, there exist an integer $n \geq 1$ and a finitely generated subideal F of I such that $a^n \in F$ for all $a \in I$.) For more on rings with Noetherian spectrum, the readers can refer to [9].

We say that an integral domain D has *locally Noetherian spectrum* (respectively, *t-locally Noetherian spectrum*) if D_M has Noetherian spectrum for all maximal ideals (respectively, maximal t-ideals) M of D.

The purpose of this paper is to study rings with Noetherian spectrum, rings with locally Noetherian spectrum and rings with t-locally Noetherian spectrum in terms of the polynomial ring, the Serre's conjecture ring, the Nagata ring and the t-Nagata ring. (The concepts of the Serre's conjecture ring, the Nagata ring and the t-Nagata ring will be reviewed in Section .) More precisely, we prove the following three statements.

(1) A commutative ring R with identity has Noetherian spectrum if and only if the Serre's conjecture ring $R[X]_U$ has Noetherian spectrum, if and only if the Nagata ring $R[X]_N$ has Noetherian spectrum.

(2) An integral domain D has locally Noetherian spectrum if and only if the Nagata ring $D[X]_N$ has locally Noetherian spectrum.

(3) An integral domain D has t-locally Noetherian spectrum if and only if the polynomial ring $D[X]$ has t-locally Noetherian spectrum, if and only if the t-Nagata ring $D[X]_{N_v}$ has (t-)locally Noetherian spectrum.

2. Main Results

We start this section with a simple result for a quotient ring of a ring with Noetherian spectrum. While the result follows from [4, Proposition 2.1] and the fact that a ring with Noetherian spectrum has S-Noetherian spectrum, we insert the proof for the sake of completeness. (For a commutative ring R with identity and a multiplicative subset S of R, recall that R has *S-Noetherian spectrum* if for each ideal I of R, there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq \sqrt{J} \subseteq \sqrt{I}$.)

Lemma 1. *Let R be a commutative ring with identity and let S be a* (*not necessarily saturated*) *multiplicative subset of R. If R has Noetherian spectrum, then R_S also has Noetherian spectrum.*

Proof. Let A be an ideal of R_S. Then $A = IR_S$ for some ideal I of R. Since R has Noetherian spectrum, $\sqrt{I} = \sqrt{F}$ for some finitely generated ideal F of R; so we obtain

$$\sqrt{A} = \sqrt{IR_S} = \sqrt{I}R_S = \sqrt{F}R_S = \sqrt{FR_S}.$$

Note that FR_S is a finitely generated ideal of R_S. Hence A is a radically finite ideal of R_S. Thus R_S has Noetherian spectrum. $\square$

Let R be a commutative ring with identity. Then we denote by $\mathrm{Max}(R)$ the set of maximal ideals of R.

We study the local-global property of rings with Noetherian spectrum.

Theorem 2. *Let D be an integral domain. Then the following statements hold.*

(1) *If D has Noetherian spectrum, then D has locally Noetherian spectrum.*

(2) *Suppose that D is of finite character. If D has locally Noetherian spectrum, then D has Noetherian spectrum.*

Proof. (1) This comes directly from Lemma 1.

(2) Let I be an ideal of D and let a be a nonzero nonunit element of I. Since D is of finite character, there exist only a finite number of maximal ideals of D containing a, say $M_1, \dots, M_n$. Fix an index $k \in \{1, \dots, n\}$. Since D has locally Noetherian spectrum, $\sqrt{ID_{M_k}} = \sqrt{F_k D_{M_k}}$ for some finitely generated subideal F_k of I. By letting $C = (a) + F_1 + \cdots + F_n$, we obtain that $\sqrt{ID_{M_k}} = \sqrt{CD_{M_k}}$. Let M' be a maximal ideal of D which is distinct from $M_1, \dots, M_n$. Then a is a unit in $D_{M'}$; so $\sqrt{ID_{M'}} = D_{M'} = \sqrt{CD_{M'}}$. Therefore we have

$$\sqrt{I}D_M = \sqrt{ID_M} = \sqrt{CD_M} = \sqrt{C}D_M$$

for all maximal ideals M of D. Hence we have

$$\sqrt{I} = \bigcap_{M \in \mathrm{Max}(D)} \sqrt{I}D_M = \bigcap_{M \in \mathrm{Max}(D)} \sqrt{C}D_M = \sqrt{C},$$

where the first and the third equalities follow from [6, Proposition 2.8(3)]. Note that C is a finitely generated ideal of D. Thus I is a radically finite ideal of D, which indicates that D has Noetherian spectrum. $\square$

Let D be an integral domain. Recall that D is an *almost Dedekind domain* if D_M is a Noetherian valuation domain for all maximal ideals M of D; and D is an *SP domain* if each proper ideal of D can be expressed as a product of radical ideals of D. It was shown that any SP domain is an almost Dedekind domain [10, Theorem 2.4]; and an almost Dedekind domain D is an SP domain if and only if for any proper finitely generated ideal I of D, $\sqrt{I}$ is a finitely generated ideal of D.

The next example shows that the converse of Theorem 2(1) does not hold in general. This also indicates that the condition "D is of finite character" in Theorem 2(2) is essential.

Example 3. Let D be an almost Dedekind domain as in [5, Example 2.2].

(1) Note that D_M is a Noetherian domain for all maximal ideals M of D; so D has locally Noetherian spectrum.

(2) Note that D is an SP domain [10, Theorem 3.4] but D is not a Noetherian domain [9, Example 2.2]. Hence D is not of finite character [3, Theorem 37.2].

(3) Let M be a nonfinitely generated maximal ideal of D. If D has Noetherian spectrum, then $M = \sqrt{F}$ for some finitely generated ideal F of D. However, M is a finitely generated ideal of D, because D is an SP domain. This is a contradiction. Thus D does not have Noetherian spectrum.

Let R be a commutative ring with identity and let $R[X]$ be the polynomial ring over R. Let U be the set of monic polynomials in $R[X]$. Then U is a multiplicative subset of $R[X]$ and the quotient ring $R[X]_U$ is called the *Serre's conjecture ring* of R. For an element $f \in R[X]$, $c(f)$ denotes the *content ideal* of f, *i.e.*, the ideal of R generated by the coefficients of f. Let $N = \{f \in R[X] \,|\, c(f) = R\}$. Then it was shown that $N = R[X] \setminus \bigcup_{M \in \mathrm{Max}(R)} MR[X]$ and N is a saturated multiplicative subset of $R[X]$ consisting of regular elements of $R[X]$ [8, pages 17 and 18] (or [6, Proposition 2.1(1)]). The quotient ring $R[X]_N$ is called the *Nagata ring* of R. For more on the Nagata ring, the readers can refer to [6] and [8].

Lemma 4. *Let R be a commutative ring with identity and let $N = \{f \in R[X] \,|\, c(f) = R\}$. If I is an ideal of R, then $\sqrt{IR[X]_N} \cap R = \sqrt{I}$.*

Proof. It is clear that $\sqrt{I} \subseteq \sqrt{IR[X]_N} \cap R$; so it remains to show that $\sqrt{IR[X]_N} \cap R \subseteq \sqrt{I}$. Let $a \in \sqrt{IR[X]_N} \cap R$. Then $a^n \in IR[X]_N$ for some integer $n \geq 1$. Since N consists of regular elements in $R[X]$, $a^n g \in IR[X]$ for some $g \in N$. Therefore we obtain

$$a^n c(g) = c(a^n g) \subseteq I.$$

Since $c(g) = R$, $a^n \in I$. Hence $a \in \sqrt{I}$, which indicates that $\sqrt{IR[X]_N} \cap R \subseteq \sqrt{I}$. Thus the proof is complete. $\square$

We are now ready to study the Serre's conjecture ring and the Nagata ring of rings with Noetherian spectrum.

Theorem 5. *Let R be a commutative ring with identity, U the set of monic polynomials in $R[X]$ and $N = \{f \in R[X] \,|\, c(f) = R\}$. Then the following statements are equivalent.*

(1) *R has Noetherian spectrum.*

(2) *$R[X]$ has Noetherian spectrum.*

(3) *$R[X]_U$ has Noetherian spectrum.*

(4) *$R[X]_N$ has Noetherian spectrum.*

Proof. (1) $\Rightarrow$ (2) The result appears in [9, Theorem 2.5].

(2) $\Rightarrow$ (3) This implication follows directly from Lemma 1.

(3) $\Rightarrow$ (4) Note that N contains U; so the implication follows from Lemma 1.

(4) $\Rightarrow$ (1) Let I be an ideal of R. Then $IR[X]_N$ is an ideal of $R[X]_N$. Since $R[X]_N$ has Noetherian spectrum, there exists a finitely generated subideal A of $IR[X]$ such that $\sqrt{IR[X]_N} = \sqrt{AR[X]_N}$. Let F be a finitely generated subideal of I such that $A \subseteq FR[X]$. Then $\sqrt{IR[X]_N} = \sqrt{FR[X]_N}$. Therefore by Lemma 4, we obtain

$$\sqrt{I} = \sqrt{IR[X]_N} \cap R = \sqrt{FR[X]_N} \cap R = \sqrt{F}.$$

Hence I is a radically finite ideal of R. Thus R has Noetherian spectrum. $\square$

Lemma 6. *Let D be a quasi-local domain with unique maximal ideal M and let I be an ideal of D. Then the following assertions hold.*

(1) $\sqrt{ID[X]_{MD[X]}} \cap D = \sqrt{I}$.

(2) *I is a radically finite ideal of D if and only if $ID[X]_{MD[X]}$ is a radically finite ideal of $D[X]_{MD[X]}$.*

Proof. (1) The containment $\sqrt{I} \subseteq \sqrt{ID[X]_{MD[X]}} \cap D$ is obvious. For the reverse containment, let $a \in \sqrt{ID[X]_{MD[X]}} \cap D$. Then $a^n \in ID[X]_{MD[X]}$ for some integer $n \geq 1$; so $a^n f \in ID[X]$ for some $f \in D[X] \setminus MD[X]$. Note that $c(f) \nsubseteq M$; so $c(f) = D$ because D is a quasi-local domain. Therefore we obtain

$$a^n \in a^n c(f) = c(a^n f) \subseteq I.$$

Hence $a \in \sqrt{I}$. Thus the desired equality holds.

(2) ($\Rightarrow$) Suppose that I is a radically finite ideal of D. Then there exists a finitely generated ideal F of D such that $\sqrt{I} = \sqrt{F}$. Hence we obtain

$$\sqrt{ID[X]_{MD[X]}} = \sqrt{\sqrt{I}D[X]_{MD[X]}} = \sqrt{\sqrt{F}D[X]_{MD[X]}} = \sqrt{FD[X]_{MD[X]}}.$$

Note that $FD[X]_{MD[X]}$ is a finitely generated ideal of $D[X]_{MD[X]}$. Thus $ID[X]_{MD[X]}$ is a radically finite ideal of $D[X]_{MD[X]}$.

($\Leftarrow$) Suppose that $ID[X]_{MD[X]}$ is a radically finite ideal of $D[X]_{MD[X]}$. Then there exist $f_1, \dots, f_n \in ID[X]$ such that $\sqrt{ID[X]_{MD[X]}} = \sqrt{(f_1, \dots, f_n)D[X]_{MD[X]}}$. Since $c(f_k) \subseteq I$ for all $k \in \{1, \dots, n\}$, we obtain

$$\sqrt{ID[X]_{MD[X]}} = \sqrt{(c(f_1) + \cdots + c(f_n))D[X]_{MD[X]}}.$$

Hence by (1), we obtain

$$\begin{aligned} \sqrt{I} &= \sqrt{ID[X]_{MD[X]}} \cap D \\ &= \sqrt{(c(f_1) + \cdots + c(f_n))D[X]_{MD[X]}} \cap D \\ &= \sqrt{c(f_1) + \cdots + c(f_n)}. \end{aligned}$$

Note that $c(f_1) + \cdots + c(f_n)$ is a finitely generated ideal of D. Thus I is a radically finite ideal of D. $\square$

We next investigate the Nagata ring of a ring with locally Noetherian spectrum.

Theorem 7. *Let D be an integral domain and let $N = \{f \in D[X] \mid c(f) = D\}$. Then the following statements are equivalent.*

(1) *D has locally Noetherian spectrum.*

(2) *$D[X]_N$ has locally Noetherian spectrum.*

Proof. (1) $\Rightarrow$ (2) Let Q be a maximal ideal of $D[X]_N$. Then $Q = MD[X]_N$ for some maximal ideal M of D [8, (6.17)(4)] (or [6, Proposition 2.1(2)]). Since D has locally Noetherian spectrum, D_M has Noetherian spectrum; so by Theorem 5, $D_M[X]$ has Noetherian spectrum. Hence by Lemma 1, $D_M[X]_{MD_M[X]}$ has Noetherian spectrum. Note that $(D[X]_N)_Q = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$ [1, Lemmas 1 and 2]; so $(D[X]_N)_Q$ has Noetherian spectrum. Thus $D[X]_N$ has locally Noetherian spectrum.

(2) $\Rightarrow$ (1) Let M be a maximal ideal of D. Then $MD[X]_N$ is a maximal ideal of $D[X]_N$ [8, (6.17)(4)] (or [6, Proposition 2.1(2)]). Since $D[X]_N$ has locally Noetherian spectrum, $(D[X]_N)_{MD[X]_N}$ has Noetherian spectrum. Note that $(D[X]_N)_{MD[X]_N} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$ [1, Lemmas 1 and 2]; so $D_M[X]_{MD_M[X]}$ has Noetherian spectrum.

Let I be an ideal of D_M. Then $ID_M[X]_{MD_M[X]}$ is an ideal of $D_M[X]_{MD_M[X]}$. Since $D_M[X]_{MD_M[X]}$ has Noetherian spectrum, $ID_M[X]_{MD_M[X]}$ is a radically finite ideal of $D_M[X]_{MD_M[X]}$. Since D_M is a quasi-local domain with maximal ideal MD_M, I is a radically finite ideal of D_M by Lemma 6(2). Hence D_M has Noetherian spectrum. Thus D has locally Noetherian spectrum. $\square$

Let D be an integral domain and let $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Then N_v is a saturated multiplicative subset of $D[X]$ [6, Proposition 2.1(1)] and the quotient ring $D[X]_{N_v}$ is called the *t-Nagata ring* of D.

We study Hilbert basis theorem for a ring with t-locally Noetherian spectrum and the t-Nagata ring of a ring with t-locally Noetherian spectrum.

Theorem 8. *Let D be an integral domain and let $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Then the following statements are equivalent.*

(1) *D has t-locally Noetherian spectrum.*

(2) *$D[X]$ has t-locally Noetherian spectrum.*

(3) *$D[X]_{N_v}$ has locally Noetherian spectrum.*

(4) *$D[X]_{N_v}$ has t-locally Noetherian spectrum.*

Proof. (1) $\Rightarrow$ (2) Let M be a maximal t-ideal of $D[X]$.

Case 1. $M \cap D = (0)$. Let K be the quotient field of D. Then $D[X]_M$ is a quotient ring of $K[X]$; so $D[X]_M$ is a principal ideal domain. Hence $D[X]_M$ has Noetherian spectrum.

Case 2. $M \cap D \neq (0)$. Let $P = M \cap D$. Then $M = PD[X]$ and P is a maximal t-ideal of D [2, Proposition 2.2]. Since D has t-locally Noetherian spectrum, D_P has Noetherian spectrum; so by Theorem 5, $D_P[X]$ has Noetherian spectrum. Hence by Lemma 1, $D_P[X]_{PD_P[X]}$ has Noetherian spectrum. Note that $D[X]_M = D_P[X]_{PD_P[X]}$ [1, Lemma 2]; so $D[X]_M$ has Noetherian spectrum.

In either case, $D[X]_M$ has Noetherian spectrum. Thus $D[X]$ has t-locally Noetherian spectrum.

(2) $\Rightarrow$ (3) Let Q be a maximal ideal of $D[X]_{N_v}$. Then $Q = MD[X]_{N_v}$ for some maximal t-ideal M of D [6, Proposition 2.1(2)]. Note that $(D[X]_{N_v})_Q = (D[X]_{N_v})_{MD[X]_{N_v}} = D[X]_{MD[X]}$ [1, Lemma 1] and $MD[X]$ is a maximal t-ideal of $D[X]$ [2, Proposition 2.2]. Since $D[X]$ has t-locally Noetherian spectrum, $D[X]_{MD[X]}$ has Noetherian spectrum. Hence $(D[X]_{N_v})_Q$ has Noetherian spectrum. Thus $D[X]_{N_v}$ has locally Noetherian spectrum.

(3) $\Rightarrow$ (1) Let M be a maximal t-ideal of D. Then $MD[X]_{N_v}$ is a maximal ideal of $D[X]_{N_v}$ [6, Proposition 2.1(2)]. Since $D[X]_{N_v}$ has locally Noetherian spectrum, $(D[X]_{N_v})_{MD[X]_{N_v}}$ has Noetherian spectrum. Note that $(D[X]_{N_v})_{MD[X]_{N_v}} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$ [1, Lemmas 1 and 2]; so $D_M[X]_{MD_M[X]}$ has Noetherian spectrum.

Let I be an ideal of D_M. Then $ID_M[X]_{MD_M[X]}$ is an ideal of $D_M[X]_{MD_M[X]}$. Since $D_M[X]_{MD_M[X]}$ has Noetherian spectrum, $ID_M[X]_{MD_M[X]}$ is a radically finite ideal of $D_M[X]_{MD_M[X]}$. Since D_M is a quasi-local domain with maximal ideal MD_M, Lemma 6(2) forces I to be a radically finite ideal of D_M. Hence D_M has Noetherian spectrum. Thus D has t-locally Noetherian spectrum.

(3) $\Leftrightarrow$ (4) This equivalence follows directly from the fact that the set of maximal t-ideals of $D[X]_{N_v}$ is precisely the same as the set of maximal ideals of $D[X]_{N_v}$ (cf. [6, Propositions 2.1(2) and 2.2(3)]). □

Corollary 9. (cf. Theorems 2(2) and 5) *Let D be an integral domain. Let $N = \{f \in D[X] \mid c(f) = D\}$ and let $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Then the following assertions hold.*

(1) *D has locally Noetherian spectrum and is of finite character if and only if $D[X]_N$ has locally Noetherian spectrum and is of finite character.*

(2) *D has t-locally Noetherian spectrum and is of finite t-character if and only if $D[X]_{N_v}$ has locally Noetherian spectrum and is of finite character.*

Proof. (1) Note that D is of finite character if and only if $D[X]_N$ is of finite character [7, Lemma 8(1)]. Thus the equivalence is an immediate consequence of Theorem 7.

(2) Note that D is of finite t-character if and only if $D[X]_{N_v}$ is of finite character [7, Lemma 8(2)]. Thus the equivalence comes directly from Theorem 8. $\square$

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