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Higher Order Uniformly Convergent Numerical Scheme for Singularly Perturbed Reaction-Diffusion Problems

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Abstract. In this paper, a uniformly convergent numerical scheme is designed for solving singularly perturbed reaction-diffusion problems. The problem is converted to an equivalent weak form and then a Galerkin finite element method is used on a piecewise uniform Shishkin mesh with linear basis functions. The convergence of the developed scheme is proved and it is shown to be almost fourth order uniformly convergent in the maximum norm. To exhibit the applicability of the scheme, model examples are considered and solved for different values of a singular perturbation parameter ε and mesh elements. The proposed scheme approximates the exact solution very well.

1. Introduction

Many physical phenomena are modeled by parameter dependent differential equations. The behaviour of the solution depends on the magnitude of the parameter. A differential equation in which the highest order derivative term is multiplied by a small positive parameter $\varepsilon \in (0, 1]$ is called a singularly perturbed differential equation and the parameter ε is called singular perturbation parameter [15], [19]. Singularly perturbed problems (SPPs) arises in the modeling of fluid dy-

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namics, elasticity, quantum mechanics, reaction-diffusion process, chemical-reactor theory, plasma dynamics, meteorology, diffraction theory, aerodynamics, oceanography, modeling of semi-conductor, hydrodynamics and many other similar areas [4].

The solutions of singularly perturbed differential equation exhibit a multi-scale character; i.e. there are thin layers or regions of the domain where the solution changes rapidly or jumps suddenly, forming boundary layers, while away from the layers the solution behaves regularly or changes slowly in the outer region. As a result such problems are called boundary layer problems [5]. Due to the multi-scale character of the solution, classical numerical methods such as FDM, FEM and the collocation method are inefficient. This happens since the error estimates of numerical methods depend explicitly on the derivatives of the solution and the derivatives are unbounded as $\varepsilon \to 0$ [20]. Early numerical solutions of singularly perturbed differential equations were obtained using a standard finite difference method on a uniform mesh. In this approach, as the perturbation parameter decreases in magnitude the mesh needs to be refined sufficiently to capture the boundary layer(s). Hence, such methods are inefficient and inaccurate.

A numerical solution U is said to be ε-uniformly convergent to the exact solution u, if there exist a positive integer N_0 and positive numbers C and p, where N_0, C and p are all independent of N and ε, such that, for all $N > N_0$, $\sup_{0<\varepsilon\leq 1}\|U-u\| \leq CN^{-p}$. Here p is the order of the method and N is number of mesh elements [11], [12].

Different authors have developed several numerical methods for solving singularly perturbed reaction-diffusion problems. Some of these can be found in [4], [5], [6, 7], and [22]. The results in these papers are good in terms of the value of ε and mesh size h. The main drawback of the schemes in the above listed papers is that the methods fail to give good results as $\varepsilon \to 0$, which means that the methods are not ε-uniformly convergent. The recent papers [2], [3], [10], [13, 14], [16] and [17] contain ε-uniform numerical methods developed for solving singularly perturbed reaction-diffusion problems.

In this paper, we developed a fitted mesh finite element method with Richardson extrapolation for solving second order singularly perturbed two point boundary value problems of the reaction-diffusion type. The proposed method involves dividing the domain of the solution into a finite number of elements and using variational concepts to construct an approximate solution over the collection of elements. The main reason behind seeking an approximate solution on a collection of sub-domains is the fact that it is easier to represent a complicated function as a collection of simple polynomials [1], [18]. Most numerical methods developed so far for solving singularly perturbed reaction-diffusion problems are not parameter-uniformly convergent. So developing a higher order numerical method whose convergence does not depend on the perturbation parameter has a great importance to the scientific research area[9]. This paper deals with formulating a higher order uniformly convergent method to find numerical solution of singularly perturbed 1D reaction-diffusion problems.

Notations. In this paper, N denotes the number of mesh elements and C is a positive constant independent of ε and N. The notation $\|\cdot\|$ denotes maximum norm.

2. Description of the Problem

Consider a class of singularly perturbed reaction-diffusion problems of the form:

$$-\varepsilon \frac{d^2u(x)}{dx^2} + b(x)u(x) = f(x), \quad x \in \Omega = (0,1) \tag{2.1}$$

subjected to the boundary conditions

$$u(0) = \alpha, \quad u(1) = \gamma, \tag{2.2}$$

where $0 < \varepsilon \ll 1$ is the singular perturbation parameter and α and γ are given constants. Assume that the functions $f(x)$ and $b(x)$ are sufficiently smooth and $b(x) \geq \beta > 0$ for some constant β. As the problem is of the reaction-diffusion type, the condition $b(x) \geq \beta > 0$ ensure the existence of dual boundary layers near the two end points of the solution domain.

2.1. Properties of the exact solution

Let us denote the differential equation in (2.1)-(2.2) by the differential operator L as $L = -\varepsilon \frac{d^2}{dx^2} + b(x)$. The considered singularly perturbed problem in (2.1)-(2.2) satisfies the maximum principle.

Lemma 2.1. *Assume that $\psi(0) \geq 0$ and $\psi(1) \geq 0$. Then, $L\psi(x) \geq 0, \forall x \in \Omega$, implies that $\psi(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. Assume that there exists $x^* \in \bar{\Omega} = [0,1]$ such that $\psi(x) \in C^2(\Omega) \cup C^0(\partial\Omega)$ and $\psi(x^*) = \min_{\bar{\Omega}} \psi(x) < 0$. Thus, from the conditions given on the boundaries, it is clear that $x^* \notin \{0,1\}$ which implies that $x^* \in (0,1)$. It follows from the assumption that $\frac{d}{dx}\psi(x^*) = 0$ and $\frac{d^2}{dx^2}\psi(x^*) \geq 0$.
Consequently, $L\psi(x^*) = -\varepsilon \frac{d^2}{dx^2}\psi(x^*) + b(x)\psi(x^*) < 0$. This is a contradiction and hence we can conclude that $\psi(x) \geq 0$, $\forall x \in \bar{\Omega}$. $\square$

The next lemma gives a bound on the derivatives of the solution of the considered problem.

Lemma 2.2. *Let $u(x) \in C^2(\Omega) \cup C^0(\partial\Omega)$ be the solution of the problem in (2.1)-(2.2). Then, its derivatives satisfies the bound*

$$\left|\frac{d^k u(x)}{dx^k}\right| \leq C\big(1 + \varepsilon^{-\frac{k}{2}}(e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}})\big), \ x \in \Omega, \ k = 0,1,...,4. \tag{2.3}$$

Proof. The proof is carried out by constructing barrier functions and applying the maximum principle. The details of the proof is given in Appendix section. $\square$

Sharper bounds on the solution and its derivatives are required in the proof of the error estimates. To do that, the solution u of (2.1)-(2.2) is decomposed as $u = v + w_L + w_R$, where v denote the regular component of the solution and w_L and w_R respectively denotes the left and right singular components of the solution. This decomposition is known as Shishkin decomposition [12].

Lemma 2.3. *The derivatives of the regular component solution satisfies the bound*

$$\left|\frac{d^k v(x)}{dx^k}\right| \leq C(1+\varepsilon^{-\frac{(k-2)}{2}}),\ \forall x \in \bar{\Omega},\ k = 0,1,...,4. \tag{2.4}$$

and the derivatives of the singular components solution satisfies the bound

$$\begin{aligned}\left|\frac{d^k w_L(x)}{dx^k}\right| &\leq C\varepsilon^{\frac{-k}{2}} e^{-x\sqrt{\frac{\beta}{\varepsilon}}},\ \forall x \in \bar{\Omega},\ k = 0,1,...,4.\\ \left|\frac{d^k w_R(x)}{dx^k}\right| &\leq C\varepsilon^{\frac{-k}{2}} e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}},\ \forall x \in \bar{\Omega},\ k = 0,1,...,4.\end{aligned} \tag{2.5}$$

Proof. The detail proof is given in Appendix section. □

3. Numerical Scheme Formulation

3.1. Finite element method on Shishkin mesh

Let H_0^1 denote the set of all functions whose order 1 or less is square integrable over $\bar{\Omega} = [0,1]$ and vanishes at the end point of the domain. Using the weak formulation, we rewrite (2.1)-(2.2) as

$$B(u,v) = L(v), \tag{3.1}$$

where

$$B(u,v) = \int_0^1 [\varepsilon \frac{du(x)}{dx}\frac{dv(x)}{dx} + b(x)u(x)v(x)]dx,$$

and

$$L(v) = \int_0^1 f(x)v(x)dx,$$

so that $B(u,v)$ is a bilinear function in u and v and $L(v)$ is a linear functional of v, $\forall v \in H_0^1$. We call (3.1) is weak form (or variational form) of (2.1)-(2.2). Interested reader can refer the weak formulation of boundary value problems in [1].

Since the considered problem exhibits two boundary layers on the left and right side, the domain $\bar{\Omega} = [0,1]$ is subdivided into boundary layer regions $(0,\tau)$ and $(1-\tau,1)$ and outer layer region $(\tau, 1-\tau)$. In this paper, we use the Shishkin piece-wise uniform mesh in order to have more mesh points in the boundary layer regions. Let N be the number of elements on the domain $\bar{\Omega}$, then we define mesh discretization as $\bar{\Omega}_\tau^N = \{x_i\}_{i=1}^{N+1}$ with $h_{i+1} = x_{i+1} - x_i$ for $i = 1, 2, 3, ..., N+1$. The sub-domains $(0,\tau)$ and $(1-\tau, 1)$ are discretized into $\frac{N}{4}$ mesh elements and the sub-domain $(\tau, 1-\tau)$ is discretized into $\frac{N}{2}$ mesh elements as

$$x_i = \begin{cases} 4\tau(\frac{i}{N}),\ i = 1, 2, ..., N/4, \\ 2(1-2\tau)(\frac{i}{N}), i = N/4+1, ..., 3N/4, \\ 4\tau(\frac{i}{N}), i = 3N/4+1, ..., N+1, \end{cases} \tag{3.2}$$

where τ is the Shishkin mesh transition parameter given by

$$\tau = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\right\}. \tag{3.3}$$

The mesh size h_i in each subdomain is given by

$$h_i = \begin{cases} \frac{4\tau}{N},\ i = 1, 2, ..., N/4, \text{ and, } i = 3N/4+1, ..., N, \\ \frac{2(1-2\tau)}{N},\ i = N/4+1, ..., 3N/4. \end{cases} \tag{3.4}$$

Using the mesh points in (3.2), we construct a set of piecewise linear basis function $\phi_i(x) \in H_0^1$ of the form;

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i} & \text{if } x_{i-1} < x < x_i, \\ \frac{x_{i+1} - x}{h_{i+1}} & \text{if } x_i < x < x_{i+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.5}$$

for $i = 1, 2, ..., N$, which are often called the hat functions.

Let us consider a typical element $\Omega_\tau^e = [x_1^e, x_2^e]$ from the domain Ω_τ^N using the above discretization. Here e denote the element number and x_1^e and x_2^e denote the left and right end of the element Ω_τ^e. Thus, the weak form of the problem over each element is given as

$$\int_{x_1^e}^{x_2^e} [\varepsilon \frac{du^e(x)}{dx}\frac{dv(x)}{dx} + b(x)u^e(x)v(x)]dx = \int_{x_1^e}^{x_2^e} f(x)v(x)dx, \tag{3.6}$$

where $u^e(x)$ is denoted for the restriction of $u(x)$ on the element $\Omega_\tau^e = [x_1^e, x_2^e]$. Representing the numerical solution by the linear combination of the basis function on each element as

$$u_N^e(x) = \sum_{j=1}^{N_e} c_j^e \phi_j^e(x), \tag{3.7}$$

where the coefficients c_j^e are unknowns to be determined, N_e is the number of nodes in Ω_τ^e and $\phi_j^e(x)$ are the basis functions on the element Ω_τ^e. Differentiating (3.7) once gives

$$\frac{du_N^e(x)}{dx} = \sum_{j=1}^{N_e} c_j^e \frac{d\phi_j^e(x)}{dx}. \tag{3.8}$$

Plugging the approximate solution in (3.7) and its derivative in (3.8) into (3.6) gives

$$\int_{x_1^e}^{x_2^e} [\sum_{j=1}^{N_e} (\varepsilon \frac{d\phi_j^e(x)}{dx}\frac{dv(x)}{dx} + b(x)\phi_j^e(x)v(x))c_j^e]dx = \int_{x_1^e}^{x_2^e} f(x)v(x)dx. \tag{3.9}$$

Applying Galerkin method i.e. taking the test function $v(x) = \phi_i^e(x)$ for $i = 1, 2$ gives

$$\int_{x_1^e}^{x_2^e} [\sum_{j=1}^{N_e} (\varepsilon \frac{d\phi_j^e(x)}{dx}\frac{d\phi_i^e(x)}{dx} + b(x)\phi_j^e(x)\phi_i^e(x))c_j^e]dx = \int_{x_1^e}^{x_2^e} f(x)\phi_i^e(x)dx, \tag{3.10}$$

where $\phi_i^e(x)$ are a piecewise linear base functions. We rewrite (3.10) as

$$\sum_{j=1}^{N_e} k_{i,j}^e c_j^e = f_i^e \text{ for } i = 1, 2, \tag{3.11}$$

where

$$k_{i,j}^e = \int_{x_1^e}^{x_2^e} \varepsilon \frac{d\phi_j^e(x)}{dx}\frac{d\phi_i^e(x)}{dx} + b(x)\phi_j^e(x)\phi_i^e(x)dx, \tag{3.12}$$

called the stiffness matrix and

$$f_i^e = \int_{x_1^e}^{x_2^e} f(x)\phi_i^e(x)dx, \tag{3.13}$$

called the load vector.

Since the base functions are linear, each element Ω_τ^e has two nodes (i.e. degree of freedom) and there are two equations per element of the form

$$\begin{aligned} k_{11}^e c_1^e + k_{12}^e c_2^e &= f_1^e, \\ k_{21}^e c_1^e + k_{22}^e c_2^e &= f_2^e. \end{aligned} \tag{3.14}$$

Here, the subscripts 1 and 2 are labels of the endpoint nodes on a typical element Ω_e^τ. These subscripts are to be relabeled upon assembling the elements so as to coincide with appropriate nodes $1, 2, 3, ..., N + 1$ in the final mesh. The equations on the entire elements assembled as follows. Since the mesh contain N elements and $N + 1$ nodes, we have $N + 1$ equations in $N + 1$ degree of freedom describing the assembled system of elements.

The linear system of equations for the entire mesh becomes

$$
\begin{bmatrix}
K_{11} & K_{12} & 0 & \dots & 0 & 0 \\
K_{21} & K_{22} & K_{23} & \dots & 0 & 0 \\
0 & K_{32} & K_{33} & \dots & 0 & 0 \\
0 & 0 & K_{43} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & 0 & 0 & \dots & K_{N,N} & K_{N,N+1} \\
0 & 0 & 0 & \dots & K_{N+1,N} & K_{N+1,N+1}
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_N \\ c_{N+1} \end{bmatrix}
=
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_N \\ F_{N+1} \end{bmatrix}
\tag{3.15}
$$

where

$$
\begin{cases}
K_{11} = k_{11}^1, K_{12} = k_{12}^1, \\
K_{21} = k_{21}^1, K_{22} = k_{22}^1 + k_{11}^2, K_{23} = k_{12}^2, \\
\vdots \\
K_{N,N-1} = k_{21}^{N-1}, K_{N,N} = k_{22}^{N-1} + k_{11}^N, \\
K_{N,N+1} = k_{12}^N, \ K_{N+1,N} = k_{21}^N, K_{N+1,N+1} = k_{22}^N,
\end{cases}
\tag{3.16}
$$

and

$$
\begin{cases}
F_1 = f_1^1, \\
F_2 = f_2^1 + f_1^2, \\
\quad \vdots \\
F_N = f_2^{N-1} + f_1^N, \\
F_{N+1} = f_2^N.
\end{cases}
\tag{3.17}
$$

Applying the boundary conditions $u_N(0) = \alpha$ and $u_N(1) = \gamma$ then $N-1$ unknown nodal values c_2, c_3, c_4,..., c_N remain. The equation in (3.15) reduces to $N-1$ system of equations as

$$
\begin{bmatrix}
K_{22} & K_{23} & 0 & \dots & 0 & 0 \\
K_{32} & K_{33} & K_{34} & \dots & 0 & 0 \\
0 & K_{43} & K_{44} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & K_{N,N-1} & K_{N,N}
\end{bmatrix}
\begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_N \end{bmatrix}
=
\begin{bmatrix} F_2 - K_{21}\alpha \\ F_3 \\ F_4 \\ \vdots \\ F_N - K_{N,N+1}\gamma \end{bmatrix}
\tag{3.18}
$$

and the two auxiliary equation corresponding to nodes 1 and $N+1$ are

$$
\begin{aligned}
K_{11}\alpha + K_{12}c_1 &= F_1, \\
K_{N+1,N}c_{N+1} + K_{N+1,N+1}\gamma &= F_{N+1}.
\end{aligned}
\tag{3.19}
$$

3.2. Stability of the scheme

From (3.16) we see that the entries of the stiffness matrix in (3.18) are given by

$$K_{i,i-1} = k_{21}^{i-1} = \int_{x_{i-1}}^{x_i} \left[\varepsilon\phi_i'(x)\phi_{i-1}'(x) + b(x)\phi_i(x)\phi_{i-1}(x)\right]dx,$$

$$\begin{aligned} K_{i,i} = k_{22}^{i-1} + k_{11}^{i} = &\int_{x_{i-1}}^{x_i} \left[\varepsilon\big(\phi_i'(x)\big)^2 + b(x)(\phi_i(x))^2\right]dx \\ &+ \int_{x_i}^{x_{i+1}} \left[\varepsilon\big(\phi_i'(x)\big)^2 + b(x)(\phi_i(x))^2\right]dx, \end{aligned}$$

$$K_{i,i+1} = k_{12}^{i} = \int_{x_i}^{x_{i+1}} \left[\varepsilon\phi_i'(x)\phi_{i+1}'(x) + b(x)\phi_i(x)\phi_{i+1}(x)\right]dx.$$

On each row of the matrix in (3.18), we obtain $K_{i,i-1} < 0$, $K_{i,i+1} < 0$ and $K_{i,i} > 0$ since $\phi_i(x)$ and $\phi_{i+1}(x)$ are linear polynomials with oppositely signed slopes. In addition, for each row $i = 2, 3, ..., N$ it satisfies the diagonal dominance ([23], page 152)

$$|K_{i,i-1}|+|K_{i,i+1}| \leq |K_{i,i}|.$$

So, the reduced stiffness matrix in (3.18) satisfies the criteria of M-matrix. Hence, it is nonsingular. This guarantees the stability of the scheme. So, the unknown nodal values c_2, c_3, c_4,..., c_N are solved easily using Thomas algorithm [1].

3.3. Convergence analysis of the scheme

In this section, we prove the uniform convergence of the developed scheme using the maximum norm. The next theorem gives the ε-uniform or parameter uniform error bound of the scheme in the maximum norm.

Theorem 3.1. *Let U_i be the numerical solution and $u(x_i)$ be exact solution of the problem in* (2.1)-(2.2) *on the Shishkin mesh $\bar{\Omega}_\tau^N$. Then*

$$\sup_{0<\varepsilon\leq 1} \|U_i - u(x_i)\| \leq CN^{-2}(lnN)^2, \tag{3.20}$$

where C is a constant independent of ε and N.

Proof. The bound is obtained separately on each element $\Omega_i = [x_{i-1}, x_i]$. Note that for any function g on Ω_i, we write the approximate solution using the basis functions $\phi_i(x)$ as

$$\bar{g}(x) = \phi_{i-1}(x)g(x_{i-1}) + \phi_i(x)g(x_i),$$

so that on each Ω_i we have

$$\|\bar{g}(x)\| \leq \max_{\Omega_i} |g(x)|.$$

Using Taylor series expansion the interpolation error on Ω_i given by

$$\|\bar{g}(x_i) - g(x_i)\| \leq Ch_i^2 \max_{\Omega_i} \left|\frac{d^2 g(x_i)}{dx^2}\right| + Ch_i^4 \max_{\Omega_i} \left|\frac{d^4 g(x_i)}{dx^4}\right|, \tag{3.21}$$

giving the error in the weak form as (for detail see on [23], Theorem 7.11)

$$\begin{aligned} B(\bar{g}(x_i) - g(x_i), \bar{g}(x_i) - g(x_i)) &= \int_0^1 [\varepsilon(\bar{g}'(x_i) - g'(x_i))^2 + b(x)(\bar{g}(x_i) - g(x_i))^2]dx \\ &\leq \max\{\varepsilon, \max_{x_i \in [0,1]} b(x_i)\} \int_0^1 [(\bar{g}'(x_i) - g'(x_i))^2 + (\bar{g}(x_i) - g(x_i))^2]dx \\ &\leq Ch_i^2 \max_{\Omega_i} \left|\frac{d^2 g(x_i)}{dx^2}\right| + Ch_i^4 \max_{\Omega_i} \left|\frac{d^4 g(x_i)}{dx^4}\right|. \end{aligned}$$

Using Lemma 1.2, we obtain

$$\begin{aligned} B(U_i - u(x_i), U_i - u(x_i)) &\leq Ch_i^2 \max_{\Omega_i} \left|\frac{d^2 u(x_i)}{dx^2}\right| + Ch_i^4 \max_{\Omega_i} \left|\frac{d^4 u(x_i)}{dx^4}\right| \\ &\leq Ch_i^2(1 + \varepsilon^{-1}) + Ch_i^4(1 + \varepsilon^{-2}). \end{aligned} \tag{3.22}$$

Since the coefficient matrix is invertible from stability condition, so we have

$$\|U_i - u(x_i)\| \leq Ch_i^2(1 + \varepsilon^{-1}) + Ch_i^4(1 + \varepsilon^{-2}). \tag{3.23}$$

Similar to the decomposition of continuous solution we decompose the approximate solution as

$$U_i = V_i + W_{L,i} + W_{R,i}. \tag{3.24}$$

Using the decomposition in (3.24) and Lemma 1.3, we obtain

$$\begin{aligned} \|U_i - u(x_i)\| &\leq \|V_i - v(x_i)\| + \|W_{L,i} - w_L(x_i)\| + \|W_{R,i} - w_R(x_i)\| \\ &\leq Ch_i^2 \max_{\Omega_i} \left|\frac{d^2 v(x_i)}{dx^2}\right| + 2\max_{\Omega_i} |w_L(x_i)| + 2\max_{\Omega_i} |w_R(x_i)| \\ &\leq C[h_i^2 + e^{-x_i\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x_i)\sqrt{\frac{\beta}{\varepsilon}}}]. \end{aligned}$$

Now, the argument depends on the parameter τ. That is, whether $\tau = \frac{1}{4}$ or $\tau = 2\sqrt{\frac{\varepsilon}{\beta}} \ln N$. In case $\tau = \frac{1}{4} \leq 2\sqrt{\frac{\varepsilon}{\beta}} \ln N$ which implies $\frac{1}{\varepsilon} \leq \frac{C}{\beta}(lnN)^2$. Here since the mesh is uniform, we have $h_i = N^{-1}$ implying that $h_i^2 = N^{-2}$. Thus using (3.23), it follows that

$$\|U_i - u(x_i)\| \leq \frac{Ch_i^2}{\varepsilon} \leq CN^{-2}(lnN)^2. \tag{3.25}$$

In the second case, we have $\tau = 2\sqrt{\frac{\varepsilon}{\beta}}\ln N \leq \frac{1}{4}$ which implies $\frac{1}{\varepsilon} \geq \frac{C}{\beta}(lnN)^2$. We consider separately the inner layer regions and the outer layer region.

Case 1: For the inner layer (boundary layer) regions. That is for $1 \leq i \leq \frac{N}{4}$ and $\frac{3N}{4} + 1 \leq i \leq N$ then

$$h_i = \frac{4\tau}{N} = 4N^{-1}\sqrt{\frac{\varepsilon}{\beta}}(lnN) \text{ implies } \frac{h_i^2}{\varepsilon} = CN^{-2}(lnN)^2.$$

Now using (3.23) it follows that

$$\begin{aligned} \|W_{L,i} - w_L(x_i)\| &\leq CN^{-2}(lnN)^2, \\ \|W_{R,i} - w_R(x_i)\| &\leq CN^{-2}(lnN)^2. \end{aligned} \tag{3.26}$$

Case 2: For the outer layer region. For $\frac{N}{4} + 1 \leq i \leq \frac{3N}{4}$ then $\tau \leq 1 - x_i$ so that $e^{-(1-x_i)\sqrt{\frac{\beta}{\varepsilon}}} \leq N^{-2}$. Similarly $e^{-x_i\sqrt{\frac{\beta}{\varepsilon}}} \leq N^{-2}$ since $\tau \leq x_i$ which implies that

$$\|V_i - v(x_i)\| \leq CN^{-2}. \tag{3.27}$$

Then combining the outer layer region bound, left and right boundary layer bounds in (3.25), (3.26) and (3.27) gives

$$\|U_i - u(x_i)\| \leq \|V_i - v(x_i)\| + \|W_{L,i} - w_L(x_i)\| + \|W_{R,i} - w_R(x_i)\| \leq CN^{-2}(lnN)^2. \tag{3.28}$$

Since the right side of (3.28) is independent of ε, taking suprimum on both sides of (3.28) we obtain the required bound.

$$\sup_{0<\varepsilon\leq 1} \|U_i - u(x_i)\| \leq CN^{-2}(lnN)^2.$$

□

3.4. Richardson extrapolation

The aim of the Richardson extrapolation is to increase the order of convergence by combining the discrete solutions calculated on different meshes. For detail analysis of Richardson extrapolation on fitted mesh one can refer [21]. To apply the technique, we solve the discrete problem (3.18) on the fine mesh element $\bar{\Omega}_\tau^{2N}$ with $2N$ mesh-intervals on piecewise Shishkin mesh having the same transition points τ. The corresponding mesh widths in $\bar{\Omega}_\tau^{2N}$ becomes

$$\hat{h}_i = \begin{cases} \frac{4\tau}{2N},\ i = 1, 2, ..., N/2, \\ \frac{2(1-2\tau)}{2N},\ i = N/2 + 1, ..., 3N/2, \\ \frac{4\tau}{2N}, i = 3N/2 + 1, ..., 2N. \end{cases} \tag{3.29}$$

Let denote U_i^N and U_i^{2N} an approximate solution on N and $2N$ number of mesh elements respectively. From (3.23) we have

$$|u(x_i) - U_i^N| \leq \frac{Ch_i^2}{\varepsilon} + \frac{Ch_i^4}{\varepsilon^2}. \tag{3.30}$$

So, we have

$$\begin{aligned} |u(x_i) - U_i^N| &= |V_i^N - v(x_i)| + |W_{L,i}^N - w_L(x_i)| + |W_{R,i}^N - w_R(x_i)| \\ &\leq CN^{-2}(lnN)^2 + CN^{-4}(lnN)^4. \end{aligned} \tag{3.31}$$

since $\frac{1}{\varepsilon} \leq \frac{C}{\beta}(lnN)^2$. In similar manner on double number of mesh elements $2N$

$$\begin{aligned} |u(x_i) - U_i^{2N}| &= |V_i^{2N} - v(x_i)| + |W_{L,i}^{2N} - w_L(x_i)| + |W_{R,i}^{2N} - w_R(x_i)| \\ &\leq C(N/2)^{-2}(ln(N/2))^2 + C(N/2)^{-4}(ln(N/2))^4. \end{aligned} \tag{3.32}$$

Combining (3.31) and (3.32) for eliminating the term $N^{-2}(lnN)^2$ gives

$$U_i^{ext} = \frac{4U_i^{2N} - U_i}{3}, \tag{3.33}$$

which is the Richardson extrapolated approximate solution. The error for the approximate solution in (3.33) becomes

$$\sup_{0<c_\varepsilon \ll 1} \| u(x_i) - U_i^{ext} \| \leq CN^{-4}(\ln N)^4. \tag{3.34}$$

4. Numerical Results and Discussion

To verify the established theoretical results in this paper, we perform an experiment using the proposed numerical scheme on the problem of the form given in (2.1)-(2.2)

Example 4.1. Consider the singularly perturbed problem

$$-\varepsilon \frac{d^2u(x)}{dx^2} + u(x) = 1 + 2\sqrt{\varepsilon}[e^{(\frac{-x}{\sqrt{\varepsilon}})} + e^{(\frac{x-1}{\sqrt{\varepsilon}})}]$$

with the boundary conditions $u(0) = 0,\ u(1) = 0$. Its exact solution is $u(x) = 1 - xe^{(\frac{x-1}{\sqrt{\varepsilon}})} - (1-x)e^{(\frac{-x}{\sqrt{\varepsilon}})}$.

Example 4.2. Consider the singularly perturbed problem

$$-\varepsilon \frac{d^2u(x)}{dx^2} + u(x) = 1$$

with the boundary conditions $u(0) = 0,\ u(1) = 0$. Its exact solution is $u(x) = 1 - e^{(\frac{-x}{\sqrt{\varepsilon}})} - e^{(\frac{-(1-x)}{\sqrt{\varepsilon}})}$.

The maximum absolute error is calculated using

$$E_\varepsilon^N = \max|U_i - u(x_i)|.$$

The ε-uniform error is calculated using

$$E^N = \max_{\varepsilon}(E_\varepsilon^N).$$

The rate of convergence is given as

$$r_\varepsilon^N = \frac{\log E_\varepsilon^N - \log E_\varepsilon^{2N}}{\log 2}.$$

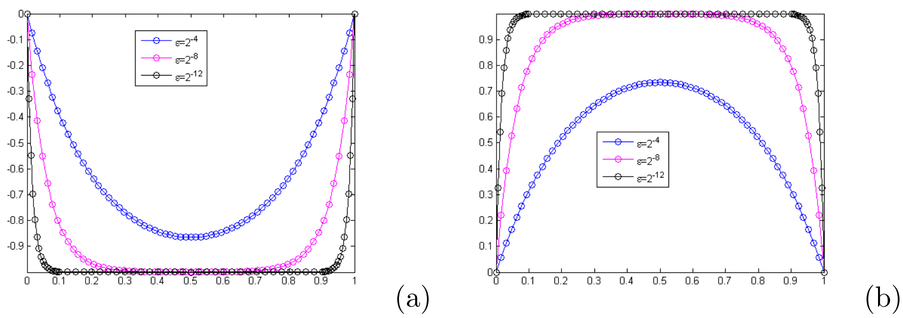


Figure 1: Solutions profile with boundary layer formation as ε goes small on (a) Example 4.1 on (b) Example 4.2.

Table 1: Maximum absolute error of Example 4.1 in inner layer(I) and outer layer(O) region before Richardson extrapolation.

$\varepsilon \downarrow$		N=2^4	2^5	2^6	2^7	2^8	2^9
10^{-3}	I	2.0159e-02	7.0588e-03	2.5475e-03	8.6985e-04	2.5648e-04	6.4050e-05
	O	3.0700e-03	9.2902e-04	2.1628e-04	2.7825e-05	1.8248e-06	4.5649e-07
10^{-4}	I	1.8932e-02	6.6543e-03	2.3997e-03	8.1752e-04	2.6683e-04	8.4387e-05
	O	3.9895e-03	2.4233e-03	1.0726e-03	3.5717e-04	1.1995e-04	3.9066e-05
10^{-5}	I	1.8545e-02	6.5265e-03	2.3530e-03	8.0097e-04	2.6155e-04	8.2713e-05
	O	4.1588e-03	2.5426e-03	1.2422e-03	5.2323e-04	1.8505e-04	6.4679e-05
10^{-6}	I	1.8422e-02	6.4861e-03	2.3383e-03	7.9574e-04	2.5988e-04	8.2183e-05
	O	4.2122e-03	2.5745e-03	1.2593e-03	5.3739e-04	2.1131e-04	7.7506e-05
10^{-7}	I	1.8383e-02	6.4733e-03	2.3336e-03	7.9408e-04	2.5935e-04	8.2016e-05
	O	4.2292e-03	2.5846e-03	1.2647e-03	5.3998e-04	2.1258e-04	7.9837e-05
E^N		1.8365e-02	6.4674e-03	2.3314e-03	7.9332e-04	2.5911e-04	8.1938e-05

Table 2: Maximum absolute error of Example 4.1 in inner layer(I) and outer layer(O) region after Richardson extrapolation.

$\varepsilon \downarrow$		N=2^4	2^5	2^6	2^7	2^8	2^9
10^{-3}	I	7.0588e-03	8.6985e-04	6.4050e-05	4.0017e-06	2.5010e-07	1.5632e-08
	O	9.2902e-04	2.7825e-05	4.5649e-07	2.8536e-08	1.7837e-09	1.6811e-10
10^{-4}	I	6.6543e-03	8.1752e-04	8.4387e-05	7.8780e-06	6.8770e-07	5.7247e-08
	O	2.4233e-03	3.5717e-04	3.9066e-05	4.0666e-06	4.1394e-07	4.0840e-08
10^{-5}	I	6.5265e-03	8.0097e-04	8.2713e-05	7.7212e-06	6.7399e-07	5.6058e-08
	O	2.5426e-03	5.2323e-04	6.4679e-05	6.7397e-06	4.1874e-07	4.0911e-08
10^{-6}	I	6.4861e-03	7.9574e-04	8.2183e-05	7.6716e-06	6.6967e-07	5.5682e-08
	O	2.5745e-03	5.3739e-04	7.7506e-05	6.7231e-06	4.1681e-07	4.4345e-08
10^{-7}	I	6.4733e-03	7.9408e-04	8.2016e-05	7.6559e-06	6.6828e-07	5.5642e-08
	O	2.5846e-03	5.3998e-04	7.9837e-05	6.7231e-06	4.1681e-07	4.4345e-08
E^N		7.0588e-03	8.6985e-04	8.4387e-05	7.8536e-06	6.6828e-07	5.5642e-08

Table 3: Example 4.1, Comparison of maximum absolute error.

$\varepsilon \downarrow$	N= 2^4	2^5	2^6	2^7	2^8	2^9
	Proposed	Scheme				
2^{-12}	7.0588e-03	8.6985e-04	6.4050e-05	4.0017e-06	2.5010e-07	1.5632e-08
2^{-16}	6.6543e-03	8.1752e-04	8.4387e-05	7.8780e-06	6.8770e-07	5.7247e-08
2^{-20}	7.0588e-03	8.6985e-04	8.4387e-05	7.8536e-06	6.6828e-07	5.5642e-08
2^{-25}	7.0588e-03	8.6985e-04	8.4387e-05	7.8536e-06	6.6828e-07	5.5642e-08
	Result	in [8]				
2^{-12}	4.245e-03	1.957e-03	5.262e-04	1.227e-04	3.012e-05	7.496e-06
2^{-16}	1.302e-03	1.292e-03	1.061e-03	4.893e-04	1.315e-04	3.068e-05
2^{-20}	3.255e-04	3.255e-04	3.255e-04	3.231e-04	2.653e-04	1.223e-04
2^{-25}	5.754e-05	5.754e-05	5.754e-05	5.754e-05	5.754e-05	5.752e-05

Table 4: Rate of convergence of Example 4.1 before and after Richardson extrapolation.

$\varepsilon \downarrow$	N=2^4	2^5	2^6	2^7	2^8
	Before RE				
2^{-12}	1.5100	1.4719	1.5518	1.6157	1.6610
2^{-16}	1.5072	1.4749	1.5513	1.6149	1.6608
2^{-20}	1.5059	1.4758	1.5512	1.6143	1.6611
2^{-25}	1.5061	1.4760	1.5513	1.6145	1.6607
	After RE				
2^{-12}	3.0206	3.7635	4.0005	4.0000	3.9999
2^{-16}	3.0250	3.2762	3.4211	3.5180	3.5865
2^{-20}	3.0206	3.3657	3.4256	3.5548	3.5862
2^{-25}	3.0206	3.3657	3.4256	3.5548	3.5862

Table 5: Maximum absolute error of Example 4.2 in inner layer (I) and outer layer(O) region before Richardson extrapolation.

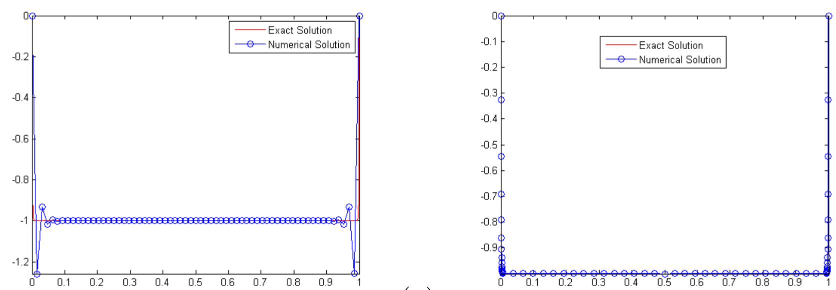
$\varepsilon \downarrow$		N=2^4	2^5	2^6	2^7	2^8	2^9
10^{-3}	I	1.8365e-02	6.4674e-03	2.3314e-03	7.9332e-04	2.3420e-04	5.8489e-05
	O	3.3885e-03	1.1233e-03	2.6567e-04	3.5484e-05	1.8514e-06	4.6313e-07
10^{-4}	I	1.8365e-02	6.4674e-03	2.3314e-03	7.9332e-04	2.5911e-04	8.1938e-05
	O	4.2368e-03	2.5689e-03	1.1405e-03	3.8306e-04	1.2970e-04	4.2701e-05
10^{-5}	I	1.8365e-02	6.4674e-03	2.3314e-03	7.9332e-04	2.5911e-04	8.1938e-05
	O	4.2370e-03	2.5893e-03	1.2671e-03	5.3499e-04	1.8961e-04	6.6528e-05
10^{-6}	I	1.8365e-02	6.4674e-03	2.3314e-03	7.9332e-04	2.5911e-04	8.1938e-05
	O	4.2370e-03	2.5893e-03	1.2672e-03	5.4118e-04	2.1300e-04	7.8195e-05
10^{-7}	I	1.8365e-02	6.4674e-03	2.3314e-03	7.9332e-04	2.5911e-04	8.1938e-05
	O	4.2370e-03	2.5893e-03	1.2672e-03	5.4118e-04	2.1311e-04	8.0063e-05
E^N		1.8365e-02	6.4674e-03	2.3314e-03	7.9332e-04	2.5911e-04	8.1938e-05

Table 6: Maximum absolute error of Example 4.2 in inner layer (I) and outer layer(O) region after Richardson extrapolation.

$\varepsilon \downarrow$		N=2^4	2^5	2^6	2^7	2^8	2^9
10^{-3}	I	6.4674e-03	7.9332e-04	5.8489e-05	3.6546e-06	2.2841e-07	1.4276e-08
	O	1.1233e-03	3.5484e-05	4.6313e-07	2.8951e-08	1.8096e-09	1.6806e-10
10^{-4}	I	6.4674e-03	7.9332e-04	8.1938e-05	7.6487e-06	6.6766e-07	5.5580e-08
	O	2.5689e-03	3.8306e-04	4.2701e-05	4.5447e-06	4.7328e-07	4.7806e-08
10^{-5}	I	6.4674e-03	7.9332e-04	8.1938e-05	7.6487e-06	6.6766e-07	5.5531e-08
	O	2.5893e-03	5.3499e-04	6.6528e-05	8.0126e-06	9.5752e-07	1.1448e-07
10^{-6}	I	6.4674e-03	7.9332e-04	8.1938e-05	7.6487e-06	6.6766e-07	5.5515e-08
	O	2.5893e-03	5.4118e-04	7.8195e-05	9.6277e-06	1.1834e-06	1.4564e-07
10^{-7}	I	6.4674e-03	7.9332e-04	8.1938e-05	7.6487e-06	6.6765e-07	5.5590e-08
	O	2.5893e-03	5.4118e-04	8.0063e-05	1.0292e-05	1.2827e-06	1.5936e-07
E^N		2.5893e-03	5.4118e-04	8.1938e-05	1.0292e-05	1.2827e-06	1.5936e-07

Table 7: Rate of convergence of Example 4.2 before and after Richardson extrapolation.

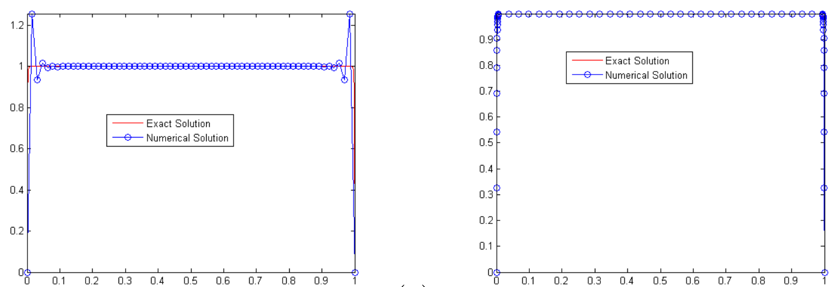
$\varepsilon\downarrow$	N=2^4	2^5	2^6	2^7	2^8
	Before RE				
10^{-3}	1.5057	1.4721	1.5550	1.6144	1.7602
10^{-4}	1.5057	1.4721	1.5550	1.6144	1.6609
10^{-5}	1.5057	1.4721	1.5550	1.6144	1.6609
10^{-6}	1.5057	1.4721	1.5550	1.6144	1.6609
10^{-7}	1.5057	1.4721	1.5550	1.6144	1.6609
	After RE				
10^{-3}	3.0272	3.7617	4.0004	4.0000	4.0000
10^{-4}	3.0272	3.2753	3.4212	3.5180	3.5865
10^{-5}	3.0272	3.2753	3.4212	3.5180	3.5877
10^{-6}	3.0272	3.2753	3.4212	3.5180	3.5877
10^{-7}	3.0272	3.2753	3.4212	3.5180	3.5877



(a) (b)

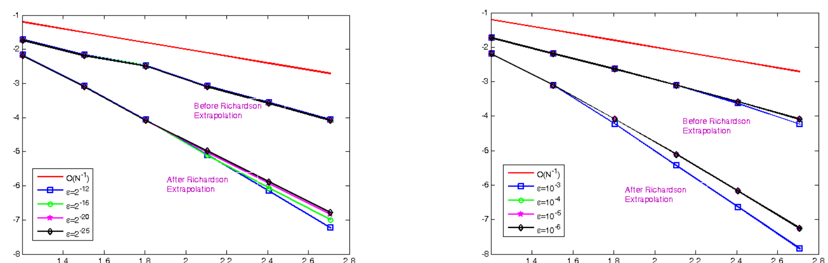
Figure 2: Numerical and Exact solutions of Example 4.1 for $\varepsilon = 2^{-20}$ and $N = 64$, (a) on uniform mesh, (b) on Shishkin mesh.

The maximum absolute error of Example 4.1 and 4.2 before Richardson extrapolation is given in Tables 1 and 5 respectively for different values of singular perturbation parameter. In these tables the result are computed for the inner boundary layers(I) and outer boundary layer(O) region separately. Similarly, in Tables 2 and 6 the maximum absolute error of Example 4.1 and 4.2 after Richardson extrapolation is given respectively. On these tables one can observe that, as ε goes small the maximum absolute error for inner layer and outer layer region becomes stable and bounded. This indicates that maximum absolute error of the scheme is independent of the perturbation parameter ε, implying that the scheme is ε-uniformly convergent. One can observe from these tables, the method with Richardson extrapolation is more accurate than that of before the Richardson extrapolation. In Table 3, we compare the result of the proposed scheme with the results in [8]. As one observes in this table the proposed scheme gives more accurate result. In Tables 4 and 7,



(a) (b)

Figure 3: Numerical and Exact solutions of Example 4.2 for $N = 64$ and $\varepsilon = 10^{-4}$, (a) on uniform mesh, (b) on Shishkin mesh.



(a) (b)

Figure 4: The graph of maximum absolute error versus the number of elements, using Log-Log scale plot on (a) Example 4.1, on (b) Example 4.2.

the rate of convergence of the proposed scheme before and after Richardson extrapolation is given. The proposed scheme before extrapolation is almost second order uniformly convergent and the scheme after the extrapolation gives almost fourth order uniform convergence. The computed result is in good agreement with the theoretical finding.

In Figure 1, the profile of the solutions is given for different values of the perturbation parameter $\varepsilon = 2^{-4}, 2^{-8}$ and 2^{-12} with boundary layer formulation with layer thickness of $O(\sqrt{\varepsilon})$ as ε goes small. In Figures 2 and 3, we plot the computed solutions of Example 4.1 and 4.2 on uniform mesh and Shishkin mesh. In Figures $2(a)$ and $3(a)$, the computed solution oscillates in the boundary layer regions while using uniform mesh, whereas in case of Shishkin mesh the exact and computed solution agree well in the boundary layer regions. In addition to that, one observes in Figures $2(b)$ and $3(b)$, sufficient number of mesh points exist in the boundary layer region for $\varepsilon = 2^{-20}$ and $N = 64$. This indicate layer resolving property of the developed scheme. In Figure 4, the Log-Log scale plot of maximum absolute error versus the number of meshes points are given for different values of the perturbation parameter ε ranging from 2^{-12} to 2^{-25}.

5. Conclusion

Uniformly convergent finite element method with Richardson extrapolation technique is presented for solving singularly perturbed reaction-diffusion problems. The stability of the scheme is discussed. Theoretically the scheme is shown ε-uniformly convergent with order of convergence almost two before Richardson extrapolation. After the extrapolation technique is applied the order of convergence of the scheme is accelerated to almost four. Model examples are considered by taking different values for ε and the results are presented in tables and graphs. The obtained result shows that the proposed method is uniformly convergent, approximate the exact solution very well and is in a good agreement with the theoretical results of the analysis. The proposed scheme also works well for variable coefficient problems.

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References

[1] E. B. Becker, G. F. Carey and J. T. Oden, Finite Elements, An Introduction: Volume I, *Printice-Hall Inc, Englewood Cliffs, New jersey*, 1981.

[2] M. Chandru, T. Prabha & V. Shanthi, *A Hybrid Difference Scheme for a Second-Order Singularly Perturbed Reaction-Diffusion Problem with Non-smooth Data*, Int. J. Appl. Comput. Math, **1**(2015), 87–100. https://doi.org/10.1007/s40819-014-0004-8

[3] C. Clavero, R. K. Bawa and S. Natesan, *A robust second-order numerical method for global solution and global normalized flux of singularly perturbed self-adjoint boundary-value problems*, Int. J. Comput. Math., **86(10-11)**(2009), 1731–1745.

[4] G. File and F. Edosa, *Higher order compact finite difference method for singularly perturbed one dimensional reaction-diffusion problems*, J. Nigerian Math. Soc., **36(3)**(2017), 491–502.

[5] G. File and Y. N. Reddy, *Domain Decomposition Method for Solving Singular Perturbation Problems*, International Journal of Applied Science and Engineering, **11(4)**(2013), 433–448.

[6] F. W. Gelu, G. F. Duressa and T. A.Bullo, *Sixth-order compact finite difference method for singularly perturbed 1D reaction diffusion problems*, J. of Taibah University for Science, **11(2)**(2017), 302–308.

[7] F. W. Galu, G. F. Duressa and T. A. Bullo, *Tenth Order Compact Finite Difference Method for Solving Singularly Perturbed 1D Reaction-Diffusion Equations*, International Journal of Applied Science and Engineering, **8(3)**(2016), 15–24.

[8] M. K. Kadalbajoo and P. Arora, *B-Spline collocation method for a singularly perturbed reaction-diffusion problem using artificial viscosity*, Int. J. Comput. Methods, **6(1)**(2009), 2341.

[9] N. Kopteva and E. O'Riordan, *Shishkin meshes in the numerical solution of singularly perturbed differential equations*, Int. J. Numer. Anal. Model., **1(1)**(2004) 1–18.

[10] M. Kumar and S. C. Rao, *High order parameter-robust numerical method for singularly perturbed reaction-diffusion problems*, Appl. Math. Comput., **216(4)**(2010) 1036–1046.

[11] J. J. Miller, H. MacMullen, E. O'Riordan and G. I. Shishkin, *A second-order parameter- uniform overlapping Schwarz method for reaction-diffusion problems with boundary layers*, J. Comput. Appl. Math., **130(1-2)**(2001), 231–244.

[12] J. J. Miller, E. O'Riordan and G. I. Shishkin, Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions, *World Scientific, New Jersey*, 2012.

[13] S. Natesan, R. K. Bawa and C. A. Clavero, *Uniformly convergent compact numerical scheme for the normalized flux of singularly perturbed reaction-diffusion problems*, Int. J. Inf. Syst. Sci., **3(2)**(2007), 207–221.

[14] S. Natesan and R. K. Bawa, *Second-order numerical scheme for singularly perturbed reaction-diffusion Robin problems*, SIAM J Numer Anal, **2(3-4)**(2007), 177–192.

[15] A. H. Nayfeh, Introduction to perturbation techniques, *John Wiley & Sons, New York,* 2011.

[16] K. Phaneendra, K. M. Latha and Y. N. Reddy, *A Fitted arithmetic average three-point finite difference method for singularly Perturbed two-point boundary value Problems with dual layers*, International Journal of Applied Science and Engineering, **13(3)**(2015), 205–216.

[17] S. C. Rao and M. A. Kumar, *A uniformly convergent exponential spline difference scheme for singularly perturbed reaction-diffusion problems*, Neural Parallel Sci. Comput., **18(2)**(2010) 121–136.

[18] J. N. Reddy, An introduction to the finite element method, *McGraw-Hill, New York*, 1993.

[19] E. R. O'Malley, Singular perturbation methods for ordinary differential equations, *Springer-Verlag, New York*, 1991.

[20] S. Russell, *Sparse grid methods for singularly perturbed problems*, PhD thesis, National University of Ireland, 2016.

[21] M. K. Singh, S. Natesan, *Richardson extrapolation technique for singularly perturbed system of parabolic partial differential equations with exponential boundary layers*, Appl. Math. Comput., **333**(2018) 254–75.

[22] F. Wondimu, G. File and T. Aga, *Fourth order compact finite difference method for solving singularly perturbed 1D reaction diffusion equations with Dirichlet boundary conditions*, Momona Ethiopian J. of Science, **8(2)**(2016) 168–181.

[23] L. Zhilin, Q. Zhonghua and T. Tao, *Numerical Solutions of Differential Equations: Introduction to Finite Difference and Finite Element Methods*, Cambridge University Press, New york, 2018.

Appendix: Proof of the Lemmas

Lemma 2.2. *Let $u(x) \in C^2(\Omega) \cup C^0(\partial\Omega)$ be the solution of the problem in* (2.1)-(2.2). *Then, its derivatives satisfies the bound*

$$\left|\frac{d^k u(x)}{dx^k}\right| \leq C\big(1 + \varepsilon^{-\frac{k}{2}}(e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}})\big),\ x \in \Omega,\ k = 0, 1, ..., 4. \tag{5.1}$$

Proof. We prove first for the case $k = 0$. Consider the barrier functions $\psi^{\pm}(x) = \frac{1}{\beta}\|f\| + \max\{|\alpha|, |\gamma|\} \pm u(x)$. At $x = 0$, we have

$$\psi^{\pm}(0) = \frac{1}{\beta}\|f\| + \max\{|\alpha|, |\gamma|\} \pm u(0) \geq \frac{1}{\beta}\|f\| \geq 0, \ \text{ since } \ \max\{|\alpha|, |\gamma|\} \geq u(0).$$

At $x = 1$, we have

$$\psi^{\pm}(1) = \frac{1}{\beta}\|f\| + \max\{|\alpha|, |\gamma|\} \pm u(1) \geq \frac{1}{\beta}\|f\| \geq 0, \ \text{ since } \ \max\{|\alpha|, |\gamma|\} \geq u(1).$$

Then, for the differential operator

$$\begin{aligned}
L_\varepsilon\psi^{\pm}(x) &= -\varepsilon(\psi^{\pm}(x))'' + b(x)\psi^{\pm}(x) \\
&= \mp(\varepsilon u''(x) + \frac{b(x)}{\beta}\|f\| + b(x)\max\{|\alpha|, |\gamma|\} \pm b(x)u(x). \\
&= \pm f(x) + \frac{b(x)}{\beta}\|f\| + b(x)\max\{|\alpha|, |\gamma|\} \\
&\geq b(x)\max\{|\alpha|, |\gamma|\} \geq 0, \quad \text{since } \ \frac{b(x)}{\beta}\|f\| \geq f(x).
\end{aligned}$$

Applying the maximum principle in Lemma 2.1, it follows that $\psi^{\pm}(x) \geq 0$, for all $x \in \overline{\Omega}$. Therefore,

$$|u(x)| \leq \frac{1}{\beta}\|f\| + \max\{|\alpha|, |\gamma|\}, \ \text{ for all } \ x \in \overline{\Omega}.$$

giving that

$$|u(x)| \leq C, \ \text{ for all } \ x \in \overline{\Omega}.$$

Next, we prove for the case $k = 1$. Let $x \in \Omega$ and construct an associated neighbourhood domain $N_x = (p, p + \sqrt{\varepsilon})$, such that $x \in N_x$ and $N_x \subset \Omega$. Then, by the mean value theorem, we have $u'(q) = \frac{u(p+\sqrt{\varepsilon})-u(p)}{\sqrt{\varepsilon}}$ for some $q \in \overline{N}_x$,

$$|u'(q)| = \frac{|u(p + \sqrt{\varepsilon}) - u(p)|}{\sqrt{\varepsilon}} \leq \frac{1}{\sqrt{\varepsilon}}[|u(p + \sqrt{\varepsilon})| + |u(p)|] \leq \frac{1}{\sqrt{\varepsilon}}[||u|| + ||u||] \leq \frac{2}{\sqrt{\varepsilon}}||u|| \leq C\varepsilon^{-\frac{1}{2}}.$$

But, we have $\int_p^x u''(z)dz = u'(x) - u'(p)$ solving for $u'(x)$ gives

$$u'(x) = u'(p) - \int_p^x u''(z)dz = u'(p) + \frac{1}{\varepsilon}\int_p^x [b(z)u(z) - f(z)]dz.$$

Hence,

$$|u'(x)| \leq \frac{2}{\sqrt{\varepsilon}}\|u\| + \frac{1}{\varepsilon}\left|\int_p^x [b(z)u(z) - f(z)]dz\right|$$
$$|u'(x)| \leq \frac{2}{\sqrt{\varepsilon}}\|u\| + \frac{1}{\varepsilon}\big|b(\xi)u(\xi) - f(\xi)\big|\int_p^x dz, \quad \xi \in (p, x)$$
$$\leq \frac{2}{\sqrt{\varepsilon}}||u|| + \frac{1}{\varepsilon}(||b||||u|| + ||f||)\sqrt{\varepsilon} = C\varepsilon^{-\frac{1}{2}}$$

Therefore,

$$|u'(x)| \leq C(1 + \varepsilon^{-\frac{1}{2}}(e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}})),$$

since $(e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}})$ is bounded. Substituting the bounds $u(x)$, $u'(x)$ in the differential equation, we obtain the bounds for $|u^{(k)}(x)| \leq C(1 + \varepsilon^{-\frac{k}{2}}(e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}))$, for $= 2, 3, 4$. □

Lemma 2.3. *The derivatives of the regular component solution satisfies the bound*

$$\left|\frac{d^k v(x)}{dx^k}\right| \leq C(1 + \varepsilon^{-\frac{(k-2)}{2}}), \ \forall x \in \bar{\Omega}, \ k = 0, 1, ..., 4. \tag{5.2}$$

and the derivatives of the singular components solution satisfies the bound

$$\left|\frac{d^k w_L(x)}{dx^k}\right| \leq C\varepsilon^{\frac{-k}{2}} e^{-x\sqrt{\frac{\beta}{\varepsilon}}}, \ \forall x \in \bar{\Omega}, \ k = 0, 1, ..., 4. \tag{5.3}$$
$$\left|\frac{d^k w_R(x)}{dx^k}\right| \leq C\varepsilon^{\frac{-k}{2}} e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}, \ \forall x \in \bar{\Omega}, \ k = 0, 1, ..., 4.$$

Proof. In the proof of error estimates, sharper bounds on the solution and its derivatives are required. To find these, the solution u is decomposed in to a regular component v and a singular component w as follows.

$$u(x) = v(x) + w(x).$$

Here, $v(x) = v_0(x) + \varepsilon v_1(x)$, where $v_0(x)$ is the solution of the reduced problem, $w(x)$ is the solution of the homogeneous equation

$$\begin{cases} Lw(x) = 0, \\ w(0) = u_0 - v_0(0), \ \ w(1) = u_1 - v_0(1), \end{cases}$$

consequently, $v_1(x)$ satisfies

$$\begin{cases} Lv_1(x) = v_0''(x), \\ v_1(0) = 0, \ \ v_1 = 0. \end{cases}$$

Since,

$$v_1(x) = \varepsilon^{-1}(v(x) - v_0(x)) = \varepsilon^{-1}(u(x) - w(x) - v_0(x))$$

giving that

$$\begin{aligned}Lv_1(x) &= \varepsilon^{-1}(Lu(x) - Lw(x) - Lv_0(x))\\ &= \varepsilon^{-1}(Lu(x) - Lv_0(x))\\ &= \varepsilon^{-1}(f(x) - Lv_0(x))\\ &= \varepsilon^{-1}(f(x) - (-\varepsilon v_0''(x) + bv_0(x)))\\ &= \varepsilon^{-1}(\varepsilon v_0''(x) + (f(x) - b(x)v_0(x))) = v_0''(x).\end{aligned}$$

Hence,

$$Lv_1(x) = v_0''(x).$$

Since $v_1(x) = \varepsilon^{-1}(u(x) - v_0(x) - w(x))$, we have on the boundary points

$$\begin{aligned}v_1(0) &= \varepsilon^{-1}(u(0) - v_0(0) - w(0))\\ &= \varepsilon^{-1}((u(0) - v_0(0)) - w(0)) = 0,\end{aligned}$$

and

$$\begin{aligned}v_1(1) &= \varepsilon^{-1}(u(1) - v_0(1) - w(1))\\ &= \varepsilon^{-1}((u(1) - v_0(1)) - w(1)) = 0.\end{aligned}$$

Thus, because of the bound on $v_0''(x)$, $v_1(x)$ is the solution of a problem similar to (2.1)-(2.2). This implies that, for $0 \le k \le 4$,

$$|v_1^{(k)}(x)| \le C(1 + \varepsilon^{-\frac{k}{2}}).$$

Noting that, $v = v_0 + \varepsilon v_1$, so that the regular component v satisfies,

$$|v^{(k)}(x)| \le C(1 + \varepsilon^{-\frac{(k-2)}{2}}), \quad \text{for all } x \in \Omega.$$

Next, the singular component w of the solution is also bounded as shown below. Decompose the singular component into left layer and right layer as

$$w(x) = w_l(x) + w_R(x)$$

where the boundary layer functions, w_L and $w_R(x)$ are defined as the solution of the problems

$$\begin{cases} Lw_L(x) = 0, \\ w_L(0) = w(0), \ \ w_L(1) = 0. \end{cases} \text{and} \begin{cases} Lw_R(x) = 0, \\ w_R(0) = 0, \ \ w_R(1) = w(1) \end{cases}$$

Now, we define the barrier functions, $\psi^{\pm}(x) = Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm w_L(x)$, where the constant C is chosen sufficiently large that the inequalities $\psi^{\pm}(0) \ge 0$ and $\psi^{\pm}(1) \ge 0$ holds. Thus,

$$\begin{aligned}L_\varepsilon \psi^{\pm}(x) &= -\varepsilon(\psi^{\pm}(x))'' + b(x)\psi^{\pm}(x)\\ &= -\varepsilon[\frac{C\beta}{\varepsilon}e^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm w_L''(x)] + b(x)[Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm w_L(x)]\\ &= C(b(x) - \beta)e^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm (-\varepsilon w_L''(x) + b(x)w_L(x))\\ &= C(b(x) - \beta)e^{-x\sqrt{\frac{\beta}{\varepsilon}}} \ge 0, \ \text{since} \ \ b(x) \ge \beta.\end{aligned}$$

which gives $\psi^{\pm}(x) \geq 0, x \in \bar{\Omega}$, by the maximum principle it follows that

$$|w_L(x)| \leq Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}}, \quad \text{for all}, x \in \bar{\Omega}.$$

Using similar procedure for the right boundary layer, we obtain

$$|w_R(x)| \leq Ce^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}, \quad \text{for all}, x \in \bar{\Omega},$$

which implies that the boundary layer solution is bounded. To bound the first derivative w_L', we use the same technique as in the proof of Lemma 2.2.
For each $x \in \overline{N}_x = (y, y+\sqrt{\varepsilon})$, such that $|w_L'(y)| \leq 2\varepsilon^{-\frac{1}{2}}||w_L||$. Hence,

$$\begin{aligned}
w_L'(x) &= w_L'(y) - \int_y^x w_L''(z)dz = w_L'(y) - \varepsilon^{-1}\int_y^x b(z)w_L(z)dz \\
|w_L'(x)| &\leq ||w_L'(y)|| + \varepsilon^{-1}||bw_L||\int_y^x dz \\
&\leq 2\varepsilon^{-\frac{1}{2}}||w_L|| + \varepsilon^{-\frac{1}{2}}||bw_L|| \leq C\varepsilon^{-\frac{1}{2}}||w_L||.
\end{aligned}$$

But $||w_L|| = \sup_{x\in N_x}|w_L(x)| \leq Ce^{-y\sqrt{\frac{\beta}{\varepsilon}}}$, because $w_L(x)$ is monotonically decreasing.

$$\begin{aligned}
||w_L|| \leq Ce^{-y\sqrt{\frac{\beta}{\varepsilon}}} &= Ce^{(x-y\sqrt{\frac{\beta}{\varepsilon}}}e^{-x\sqrt{\frac{\beta}{\varepsilon}}} \\
&= Ce^{\frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\sqrt{\beta}}e^{-x\sqrt{\frac{\beta}{\varepsilon}}} = Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}}, \text{ since } x-y \leq \sqrt{\varepsilon}
\end{aligned}$$

Therefore,

$$|w_L'(x)| \leq C\varepsilon^{\frac{-1}{2}}e^{-x\sqrt{\frac{\beta}{\varepsilon}}}.$$

To obtain the bounds for higher derivatives, we simply differentiate it. □