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Approximate Controllability for Semilinear Neutral Differential Systems in Hilbert Spaces

Jin-Mun Jeong* and Ah-Ran Park
Department of Applied Mathematics, Pukyong National University, Busan 48513, South Korea
e-mail : jmjeong@pknu.ac.kr *and* alanida@naver.com

Sang-Jin Son
Department of Applied Mathematics, Pukyong National University, Busan 48513, South Korea
e-mail : sangjinyeah@nate.com

Abstract. In this paper, we establish the existence of solutions and the approximate controllability for the semilinear neutral differential control system under natural assumptions such as the local Lipschitz continuity of nonlinear term. First, we deal with the regularity of solutions of the neutral control system using fractional powers of operators. We also consider the approximate controllability for the semilinear neutral control equation, with a control part in place of a forcing term, using conditions for the range of the controller without the inequality condition as in previous results.

1. Introduction

In this paper, we are concerned with the global existence of solution and the approximate controllability for the semilinear neutral system in a Hilbert space H:

(1.1)
$$\begin{cases} \dfrac{d}{dt}[x(t)+g(t,\displaystyle\int_0^t a(t,s,x(s))ds)]+Ax(t)=f(t,x(t))+k(t), & t\in(0,T],\\ x(0)=x_0. \end{cases}$$

Here, $-A$ generates an analytic semigroup in H(see [21, Theorem 3.6.1]). The nonlinear operator f is assumed to be locally Lipschitz continuous with respect to

* Corresponding Author.

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the second variable, and g is Lipschitz continuous. This kind of equation arises in heat conduction in material with memory, in population dynamics, and in control systems with hereditary feed back control governed by an integro-differential law.

In the first part of this paper, we establish the well-posedness and regularity property for (1.1). The solvability for a class of semilinear functional differential equations has been studied by many authors as seen in Section 4.3.1 of Barbu [1] and [11, 13, 17]. Our approach is to obtain the L^2-regularity under the above formulation of the semilinear neutral problem (1.1) using the contraction mapping principle (see the linear cases of [4]). Recently, the existence of solutions for mild solutions for neutral differential equations with state-dependence delay has been studied in the literature [8, 9, 10].

Next, based on the regularity for (1.1), we intend to establish the approximate controllability for the following semilinaer neutral control system with control part in place of a forcing term:

(1.2)
$$\begin{cases} \dfrac{d}{dt}[x(t) + g(t, \displaystyle\int_0^t a(t,s,x(s))ds)] + Ax(t) = f(t,x(t)) + Bu(t), \quad t \in (0,T], \\ x(0) = x_0, \end{cases}$$

namely that the reachable set of trajectories of (1.2) is a dense subset of H. Here, the controller operator B is a bounded linear operator from a Banach space of control variables into H and u is a control. This kind of equations arise naturally in physics, in biology, control engineering problem, etc.

As for the approximate controllability for semilinear control systems, we refer to [2, 3, 5, 7, 20, 23]. The controllability for neutral equations has been studied by many authors, for example, the controllability of neutral functional differential systems with unbounded delay in [5, 6, 15], neutral evolution integrodifferential systems with state dependent delay in [14, 18], impulsive neutral functional evolution integrodifferential systems with infinite delay in [19]. However, there are few literature works treating the systems with local Lipschitz continuity. As a sufficient condition for the approximate controllabilityl, Wang [24] assumed that the semigroup $S(t)$ generated by A is compact in order to guarantee the compactness of the solution mapping(see also [16]).

In this paper, we no longer require the compact property of the semigroup and the uniform boundedness of the nonlinear term, but instead we need properties fractional power of operators and conditions for the range of the controller without the inequality condition as in previous results.

The paper is organized as follows. In Section 2, the results of general linear evolution equations besides notations and assumptions are stated. In Section 3, we will obtain that the regularity for parabolic linear equations can also be applicable to (1.1) with nonlinear terms satisfying local Lipschitz continuity. The approach used here is similar to that developed in [11, 12, 16] on the general semilinear evolution equations, which is an important role to extend the theory of practical nonlinear partial differential equations. Thereafter, we investigate the approximate controllability for the problem (1.2) in Section 4. In the proofs of the main theorems, we need conditions on the range of the controller without the inequality condition as in previous results(see [16, 25]) without conditions of the compact property of a semigroup and the uniform boundedness. Finally we give a simple example to which our main result can be applied.

2. Regularity for Linear Equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $b(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$Re\ b(u, u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = b(u, v), \quad u,\ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. From the following inequalities

$$\omega_1 ||u||^2 \leq Re\, a(u, u) + \omega_2 |u|^2 \leq |Au|\,|u| + \omega_2 |u|^2 \leq \max\{1, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one which continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [22]).

It is also well known that $-A$ generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0, T; X)$ is the collection of all strongly measurable square integrable functions from $(0, T)$ into X and $W^{1,2}(0, T; X)$ is the set of all absolutely continuous functions on $[0, T]$ such that their derivative belongs to $L^2(0, T; X)$. $C([0, T]; X)$ will denote the set of all continuously functions from $[0, T]$ into X with the supremum norm. If X and Y are two Banach spaces, $\mathcal{L}(X, Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X, X)$ is simply written as $\mathcal{L}(X)$. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) := L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) := L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \leq M, \quad ||S(t)||_* \leq M.$$

The following Lemma is from Lemma 3.6.2 of [21].

Lemma 2.2. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^*:*

$$|S(t)x| \leq Mt^{-1/2}||x||_*, \quad ||S(t)x|| \leq Mt^{-1/2}|x|. \tag{2.5}$$

First of all, consider the following linear system

$$\begin{cases} x'(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{2.6}$$

By virtue of Theorem 3.3 of [4](or Theorem 3.1 of [11], [21]), we have the following result on the corresponding linear equation of (2.6).

Proposition 2.3. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*

(1) *For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 2.1) and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (2.6) belonging to $\mathcal{W}(T) \subset C([0,T];V)$ and satisfying*

$$||x||_{\mathcal{W}(T)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.7}$$

where C_1 is a constant depending on T.

(2) *Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (2.6) belonging to $\mathcal{W}_1(T) \subset C([0,T];H)$ and satisfying*

$$||x||_{\mathcal{W}_1(T)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

where C_1 is a constant depending on T.

Corollary 2.4. *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;D(A))} \le C_1||k||_{L^2(0,T;H)}, \tag{2.9}$$

$$||x||_{L^2(0,T;H)} \le C_2T||k||_{L^2(0,T;H)}, \tag{2.10}$$

and

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.11}$$

Proof. The assertion (2.9) is immediately obtained by (2.7). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \le M\int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M\int_0^T t\int_0^t |k(s)|^2 ds dt \le M\frac{T^2}{2}\int_0^T |k(s)|^2 ds \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2.3), (2.9), and (2.10) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

3. Semilinear Differential Equations

From now on, we establish the following results on the local solvability of the following equation;

(3.1)
$$\begin{cases} \dfrac{d}{dt}[x(t)+g(t,\displaystyle\int_0^t a(t,s,x(s))ds)]+Ax(t)=f(t,x(t))+k(t), \quad t\in(0,T] \\ x(0)=x_0, \end{cases}$$

where A is the operator mentioned in Section 2 and f is a nonlinear mapping from $[0,T]\times V$ into H which will be assumed later. It is also well known that A^α is a closed operator with its domain dense and $D(A^\alpha)\supset D(A^\beta)$ for $0<\alpha<\beta$. Due to the well known fact that $A^{-\alpha}$ is a bounded operator, we can assume that there is a constant $C_\alpha>0$ such that

$$||A^{-\alpha}||_{\mathcal{L}(H)}\le C_{-\alpha}, \quad ||A^{-\alpha}||_{\mathcal{L}(V^*,V)}\le C_{-\alpha}. \tag{3.2}$$

Lemma 3.1. *For any $T>0$, there exists a positive constant C_α such that the following inequalities hold for all $t>0$:*

$$||A^\alpha S(t)||_{\mathcal{L}(H)}\le \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(V,H)}\le \frac{C_\alpha}{t^{3\alpha/2}}. \tag{3.3}$$

Proof. The inequality (2.5) implies (3.3) by properties of fractional power of A and the definition of $W(t)$. For more details about the above lemma, we refer to [21, 17]. □

We give the following assumptions.

Assumption(A). Let $a:\mathbb{R}^+\times\mathbb{R}^+\times V\to H$ be a continuous function. Then there exists a constant L_a such that

$$|a(t,s,0)|\le L_a, \quad |a(t,s,x(s))-a(t,s,y(s))|\le L_a||x(s)-y(s)||$$

Assumption(F). Let $f:[0,T]\times V\to H$ be a nonlinear mapping such that There exists a function $L:\mathbb{R}_+\to\mathbb{R}$ such that

$$|f(t,x)|\le L(r), \quad |f(t,x)-f(t,y)|\le L(r)||x-y||$$

hold for any $t\in[0,T]$, $||x||\le r$ and $||y||\le r$.

Assumption(G). Let $g : [0,T] \times H \to H$ be a nonlinear mapping such that there exist constants $\beta > 1/3$ and L_g satisfying the following conditions hold:

(i) For any $x \in H$, the mapping $g(\cdot, x)$ is strongly measurable;

(ii) There exist positive constants L_g and $\beta > 1/3$ such that

$$g(0,0) = 0, \quad |A^\beta g(t,0)| \le L_g, \quad |A^\beta g(t,x) - A^\beta g(t,\hat{x})| \le L_g |x - \hat{x}|,$$

for all $t \in [0,T]$, and x, $\hat{x} \in H$.

(iii) $\partial_i g$ is measurable in $t \in [0,T]$ for each $x \in H$ and continuous in $x \in H$ for a.e. $t \in [0,T]$, where $\partial_i g$ is the partial derivative with respect to i-th coordinate and the value $\partial_i g(t,x)$ is the Gateau derivative of $g(t,x)$ for each $i = 1,2$, and

$$|\partial_i g(t,0)| \le L_g, \quad |\partial_i g(t,x_1) - \partial_i g(t,x_2)| \le L_g |x_1 - x_2|$$

for $t \le T$ and x_1, $x_2 \in H$.

Let us rewrite $(Fx)(t) = f(t,x(t))$ for each $x \in L^2(0,T;V)$. Then by Assumption (F), there is a constant, denoted again by $L(r)$, such that

$$||Fx||_{L^2(0,T;H)} \le L(r)\sqrt{T}, \quad ||Fx_1 - Fx_2||_{L^2(0,T;H)} \le L(r)||x_1 - x_2||_{L^2(0,T;V)}$$

hold for $x_1,\ x_2 \in B_r(T) = \{x \in L^2(0,T;V) : ||x||_{L^2(0,T;V)} \le r\}$.

Lemma 3.2. *Let us assume Assumptions (F),(G) and (A) for* $0 < s \le t$, *we have*

(3.4) $$|A^\beta g(s, \int_0^s a(s,\tau,x(\tau))d\tau)| \le L_g(L_a\sqrt{t}||x||_{L^2(0,t;V)} + L_a t + 1),$$

and

(3.5)
$$|A^\beta g(s, \int_0^s a(s,\tau,x(\tau))d\tau) - A^\beta g(s, \int_0^s a(s,\tau,y(\tau))d\tau)| \le L_g L_a \sqrt{t}||x - y||_{L^2(0,t;V)}.$$

Proof. From Assumptions (G), (A) and using Hölder inequality we have

$$
\begin{aligned}
&|A^\beta g(s, \int_0^s a(s,\tau,x(\tau))d\tau)| \\
&= |A^\beta g(s, \int_0^s a(s,\tau,x(\tau))d\tau) - A^\beta g(s,0)| + |A^\beta g(s,0)| \\
&\le L_g(\int_0^s |a(s,\tau,x(\tau))|d\tau + 1) \\
&\le L_g(\int_0^s |a(s,\tau,x(\tau)) - a(s,\tau,0)|d\tau + \int_0^s |a(s,\tau,0)|d\tau + 1) \\
&\le L_g(L_a \int_0^s ||x(\tau)||d\tau + L_a t + 1) \\
&\le L_g(L_a\sqrt{t}||x||_{L^2(0,T;V)} + L_a t + 1).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&|A^\beta g(s, \int_0^s a(s,\tau,x(\tau))d\tau) - A^\beta g(s, \int_0^s a(s,\tau,y(\tau))d\tau)| \\
&\le L_g(\int_0^s |a(s,\tau,x(\tau)) - a(s,\tau,y(\tau))|d\tau \\
&\le L_g L_a \int_0^s ||x(\tau) - y(\tau)||d\tau \\
&\le L_g L_a \sqrt{t}||x-y||_{L^2(0,t;V)}.
\end{aligned}
$$

□

Theorem 3.3. *Let Assumptions (F), (G) and (A) be satisfied. Assume that* $x_0 \in H$*,* $k \in L^2(0,T;V^*)$*. Then, there exists a time* $T_0 \in (0,T)$ *such that the equation (3.1) admits a solution*

$$
x \in L^2(0,T_0;V) \cap W^{1,2}(0,T_0;V^*) \subset C([0,T_0];H). \tag{3.6}
$$

Proof. For a solution of (3.1) in the wider sense, we are going to find a solution of the following integral equation

$$
\begin{aligned}
x(t) =& - g(t, \int_0^t a(t,s,x(s))ds) + S(t)x_0 + \int_0^t S(t-s)\{(Fx)(s) + k(s)\}ds \\
&+ \int_0^t AS(t-s)g(s, \int_0^s a(s,\tau,x(\tau))d\tau)ds.
\end{aligned}
\tag{3.7}
$$

To prove a local solution, we will use the successive iteration method. First, put

$$
x_0(t) = S(t)x_0 + \int_0^t S(t-s)k(s)ds
$$

and define $x_{j+1}(t)$ as

$$\begin{aligned} x_{j+1}(t) =& x_0(t) - g(t, \int_0^t a(t,s,x_j(s))ds) + \int_0^t S(t-s)(Fx_j)(s)ds \\ &+ \int_0^t AS(t-s)g(s, \int_0^s a(s,\tau,x_j(\tau))d\tau)ds. \end{aligned} \tag{3.8}$$

By virtue of Proposition 2.3, we have $x_0(\cdot) \in \mathcal{W}_1(t)$, so that

$$||x_0||_{\mathcal{W}_1(t)} \le C_1(|x_0| + ||k||_{L^2(0,t;V^*)}), \tag{3.9}$$

where C_1 is a constant in Proposition 2.3. Choose

$$r > C_1(|x_0| + ||k||_{L^2(0,t;V^*)}).$$

Putting

$$p_1(t) = \int_0^t S(t-s)(Fx_0)(s)ds,$$

by (2.11) of Corollary 2.4, we have

$$||p_1||_{L^2(0,t;V)} \le C_2\sqrt{t}||Fx_0||_{L^2(0,t;H)} \le C_2 L(r)t. \tag{3.10}$$

Let

$$p_2(s) = \int_0^s AS(s-\tau)g(\tau, \int_0^\tau a(\tau,\sigma,x_0(\sigma))d\sigma)d\tau.$$

Then From Assumption (G), (A), (3.3) and (3.4), we have

(3.11)

$$\begin{aligned} ||p_2||_{L^2(0,t;V)} &= \Big[\int_0^t \Big\| \int_0^s AS(s-\tau)g(\tau, \int_0^\tau a(\tau,\sigma,x_0(\sigma))d\sigma)d\tau\Big\|^2 ds\Big]^{\frac{1}{2}} \\ &\le \Big[\int_0^t \Big\{\int_0^s \frac{C_{1-\beta}}{(s-\tau)^{3(1-\beta)/2}} L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)d\tau\Big\}^2 ds\Big]^{\frac{1}{2}} \\ &\le C_{1-\beta}L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)\Big(\int_0^t (\int_0^s \frac{1}{(s-\tau)^{3(1-\beta)/2}}d\tau)^2 ds\Big)^{\frac{1}{2}} \\ &= \frac{2}{3\beta-1}\sqrt{\frac{1}{3\beta}}C_{1-\beta}L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)t^{\frac{3\beta}{2}} \end{aligned}$$

Set

$$p_3(s) = g(s, \int_0^s a(s,\tau,x_0(\tau))d\tau).$$

Then by Assumption (G), (3.2) and (3.4),

$$
\begin{aligned}
(3.12)\qquad ||p_3||_{L^2(0,t;V)} &= \Big(\int_0^t ||g(s, \int_0^s a(s,\tau,x_0(\tau))d\tau||^2 ds\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^t ||A^{-\beta}A^{\beta}g(s, \int_0^s a(s,\tau,x_0(\tau))d\tau||^2 ds\Big)^{\frac{1}{2}} \\
&\le C_{-\beta}L_g\Big(\int_0^t (L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)^2 ds\Big)^{\frac{1}{2}} \\
&\le C_{-\beta}L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)\sqrt{t}
\end{aligned}
$$

Put

$$
\begin{aligned}
(3.13)\qquad M_1 := \max\Big\{ & C_2 L(r)t, \\
& \frac{2}{3\beta-1}\sqrt{\frac{1}{3\beta}}C_{1-\beta}L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)t^{\frac{3\beta}{2}}, \\
& C_{-\beta}L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)\sqrt{t}\Big\}.
\end{aligned}
$$

Then for any t satisfying $M_1 < r$, from(3.6) and (3.7),

$$
\begin{aligned}
&||x_1||_{L^2(0,t;V)} \\
&\le r + C_2 L(r)t + \frac{2}{3\beta-1}\sqrt{\frac{1}{3\beta}}C_{1-\beta}L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)t^{\frac{3\beta}{2}} \\
&+ C_{-\beta}L_g(L_a\sqrt{t}||x_0||_{L^2(0,t;V)} + L_a t + 1)\sqrt{t} \le 4r.
\end{aligned}
$$

By induction, it can be shown that for all $j = 1, 2, \ldots$,

$$
||x_j||_{L^2(0,t;V)} \le 4r.
$$

Hence, from the equation

$$
\begin{aligned}
&x_{j+1}(t) - x_j(t) = -g(t, \int_0^t a(t,s,x_j(s))ds) + g(t, \int_0^t a(t,s,x_{j-1}(s))ds) \\
&+ \int_0^t S(t-s)\{f(t,x_j(s)) - f(t,x_{j-1}(s))\}ds \\
&+ \int_0^t AS(t-s)\Big\{g\Big(s, \int_0^s a(s,\tau,x_j(\tau))d\tau\Big)ds - g\Big(s, \int_0^s a(s,\tau,x_{j-1}(\tau))\Big)\Big\}d\tau)ds.
\end{aligned}
$$

Put

$$
(3.14)\quad M_2 := C_{-\beta}L_g L_a t + C_2 L(4r)\sqrt{t} + C_{1-\beta}L_g L_a\Big(\frac{2}{\sqrt{3}(3\beta-2)(3\beta+3)}\Big)^{\frac{1}{2}} t^{\frac{3\beta+3}{4}}.
$$

In a similar way to (3.11) and (3.12) and Assumption (F), we can observe that the inequality

$$\begin{aligned}||x_{j+1}-x_j||_{L^2(0,t;V)} &\le M_2||x_j-x_{j-1}||_{L^2(0,t;V)}\\ &\le (M_2)^j||x_1-x_0||_{L^2(0,t;V)}.\end{aligned}$$

Choose $T_0>0$ satisfying $\max\{M_1,M_2\}<1$, Then $\{x_j\}$ is strongly convergent to a function x in $L^2(0,T_0;V)$ uniformly on $0\le t\le T_0$ and so is in $W^{1,2}(0,T;V^*)$ by (iii) of Assumption (G). By letting $j\to\infty$ in (3.8) has a unique solution x in $\mathcal{W}_1(T_0)$. □

From now on, we give a norm estimation of the solution of (3.1) and establish the global existence of solutions with the aid of norm estimations by similar argument using (3.1) and (iii) of Assumption (G).

Theorem 3.4. *Under the Assumptions (A), (F) and (G), there exists a unique solution x of (3.1) such that*

$$x\in\mathcal{W}_1(T)\equiv L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H),\quad T>0.$$

for any $x_0\in H$, $k\in L^2(0,T;V^*)$. Moreover, there exists a constant C_3 such that

$$||x||_{\mathcal{W}_1}\le C_3(1+|x_0|+||k||_{L^2(0,T;V^*)}),\tag{3.15}$$

where C_3 is a constant depending on T.

Proof. Let x be a solution of (3.1) on $[0,T_0]$, $T_0>0$ satisfies $\max\{M_1,M_2\}<1$. Here, M_1 and M_2 be constants in (3.13) and (3.14), respectively. Then by virtue of Theorem 3.1, the solution x is represented as

$$\begin{aligned}x(t)=&x_0(t)-g(t,\int_0^t a(t,s,x(s))ds)+\int_0^t S(t-s)(Fx)(s)ds\\ &+\int_0^t AS(t-s)g(s,\int_0^s a(s,\tau,x(\tau))d\tau)ds.\end{aligned}$$

where

$$x_0(t)=S(t)x_0+\int_0^t S(t-s)k(s)ds$$

By (3.9), we have $x_0(\cdot)\in\mathcal{W}_1(T_0)$, so that

$$||x_0||_{\mathcal{W}_1(T_0)}\le C_1(|x_0|+||k||_{L^2(0,T_0;V^*)}),$$

where C_1 is a constant in Proposition 2.3. Moreover, from (3.9)-(3.12), it follows that

$$||x||_{\mathcal{W}_1(T_0)}\le C_1(|x_0|+||k||_{L^2(0,T_0;V^*)})+\max\{M_1,M_2\}||x||_{\mathcal{W}_1(T_0)}.\tag{3.16}$$

Thus, Moreover, there exists a constant C_3 such that

$$||x||_{\mathcal{W}_1(T_0)} \le C_3(1 + |x_0| + ||k||_{L^2(0,T_0;V^*)}).$$

Now from

$$\begin{aligned}&|S(T_0)x_0 + \int_0^{T_0} S(T_0 - s)\{(Fx)(s) + k(s)\}ds|\\ &\le M|x_0| + MT_0L(r) + M\sqrt{T_0}||k||_{L^2(0,T_0;H)},\end{aligned}$$

$$\big| - g(t, \int_0^t a(t,s,x(s))ds)\big| \le L_g(L_a\sqrt{t}||x||_{L^2(0,t;V)} + L_a t + 1),$$

and

$$\begin{aligned}&\big|\int_0^t AS(t-s)g(s, \int_0^s a(s,\tau,x(\tau))d\tau)ds\big|\\ &\le \int_0^t \big|\frac{C_{1-\beta}}{(t-s)^{(1-\beta)/2}} L_g(L_a\sqrt{t}||x||_{L^2(0,t;V)} + L_a t + 1)\big|ds\\ &= 2(\beta+1)^{-1}t^{(\beta+1)/2}C_{1-\beta}L_g(L_a\sqrt{t}||x||_{L^2(0,t;V)} + L_a t + 1).\end{aligned}$$

it follows that

$$\begin{aligned}&|x(T_0)| \le M|x_0| + MT_0L(r) + M\sqrt{T_0}||k||_{L^2(0,T_0;H)}\\ &+ L_g(L_a\sqrt{t}||x||_{L^2(0,t;V)} + L_a t + 1)\\ &+ 2((\beta+1)^{-1}t^(\beta+1)/2C_{1-\beta}L_g(L_a\sqrt{t}||x||_{L^2(0,t;V)} + L_a t + 1) < \infty.\end{aligned}$$

Hence, we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$ and obtain an analogous estimate to (3.16). Since the condition (3.13), (3.14) is independent of initial values, the solution can be extended to the interval $[0, nT_0]$ for any natural number n, i.e., for the initial $u(nT_1)$ in the interval $[nT_1, (n+1)T_1]$, as analogous estimate (3.16) holds for the solution in $[0, (n+1)T_1]$. □

From the following result, we obtain that the solution mapping is continuous, which is useful for physical applications of the given equation.

Corollary 3.5. *Let the Assumptions (A), (F) and (G) be satisfied and $(x_0, k) \in H \times L^2(0,T;V^*)$ for each $T > 0$. Then the solution x of the equation (3.1) belongs to $x \in \mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$ and the mapping*

$$H \times L^2(0,T;V^*) \ni (x_0, k) \mapsto x \in \mathcal{W}_1(T)$$

is continuous.

Proof. From Theorem 3.4, it follows that if $(x_0, k) \in H \times L^2(0,T;V^*)$ then x belongs to $\mathcal{W}_1(T)$. Let $(x_{0i}, k_i) \in H \times L^2(0,T;V^*)$ and $x_i \in \mathcal{W}_1(T)$ be the solution of (3.1) with (x_{0i}, k_i) in place of (x_0, k) for $i = 1,\ 2$. Hence, we assume that x_i belongs to a ball $B_r(T) = \{y \in \mathcal{W}(T) : ||y||_{\mathcal{W}_1(T)} \le r\}$.

Let

$$\begin{aligned}(px_j)(t) = &- g(t, \int_0^t a(t,s,x_j(s))ds) + \int_0^t S(t-s)(Fx_j)(s)ds \\ &+ \int_0^t AS(t-s)\Big\{g\Big(s, \int_0^s a(s,\tau,x_j(\tau))d\tau\Big)ds.\end{aligned}$$

Then, by virtue of 2) of Proposition 2.1, we get

(3.17)
$$||x_1 - x_2||_{\mathcal{W}_1(T)} \quad \le C_1\{|x_{01} - x_{02}| + ||k_1 - k_2||_{L^2(0,T;V^*} + ||px_1 - px_2||_{L^2(0,T;V^*)}\}.$$

Set $||\cdot||_{L^2(0,T_0;V)} = ||\cdot||_{L^2}$ for brevity, where $T_0 > 0$ satisfies $\max\{M_1, M_2\} < 1$. Then, we have

(3.18)
$$\begin{aligned}&||px_1 - px_2||_{L^2(0,T;V^*)} \le ||px_1 - px_2||_{L^2} \\ &= || - g(t, \int_0^t a(t,s,x_j(s))ds) + g(t, \int_0^t a(t,s,x_{j-1}(s))ds)||_{L^2} \\ &+ \Big\|\int_0^t S(t-s)\{(Fx_1)(s)) - (Fx_2)(s))\}ds\Big\|_{L^2} \\ &+ \Big\|\int_0^t AS(t-s)\Big\{g\Big(s, \int_0^s a(s,\tau,x_1(\tau))d\tau\Big)ds - g\Big(s, \int_0^s a(s,\tau,x_2(\tau))\Big)\Big\}d\tau)ds\Big\|_{L^2} \\ &\le M_2||x_1 - x_2||_{L^2}.\end{aligned}$$

Hence, by (3.17), (3.18) and (iii) of Assumption (G), we see that

$$x_n \mapsto x \in \mathcal{W}_1(T_0) \equiv L^2(0,T_0;V)) \cap W^{1,2}(0,T_0;V^*).$$

This implies that $(x_n(T_0), (x_n)_{T_0}) \mapsto (x(T_0), x_{T_0})$ in $H \times L^2(0,T;V^*)$. Hence the same argument shows that $x_n \mapsto x$ in

$$L^2(T_0, \min\{2T_0, T\}; V)) \cap W^{1,2}(T_2, \min\{2T_0, T\}; V^*).$$

Repeating this process we conclude that $x_n \mapsto x$ in $\mathcal{W}_1(T)$. □

4. Controllability

Let U be a Banach space of control variables, and let B be an operator from

U to H, called controller. In this paper, we are concerned with the approximate controllability for the following the semilinear neutral control system with a control part Bu in place of k of (3.1):

(4.1)
$$\begin{cases} \dfrac{d}{dt}[x(t)+g(t,\int_0^t a(t,s,x(s))ds)]+Ax(t)=f(t,x(t))+Bu(t), \quad t\in(0,T], \\ x(0)=x_0. \end{cases}$$

Let $x(T;f,u)$ be a state value of the system (4.1) at time T corresponding to the nonlinear term f and the control u.

Definition 4.1 The system (4.1) is said to be approximately controllable in the time interval $[0,T]$ if for every desired final state $x_1\in H$ and $\epsilon>0$ there exists a control function $u\in L^2(0,T;U)$ such that the solution $x(T;f,u)$ of (4.1) satisfies $|x(T;f,u)-x_1|<\epsilon$.

In order to obtain results of controllability, we need the stronger hypotheses than those of Section 3:

Assumption (A1). Let $a:\mathbb{R}^+\times\mathbb{R}^+\times H\to H$ be a continuous function. Then there exists a constant L_a such that

$$|a(t,s,0)|\le L_a, \quad |a(t,s,x(s))-a(t,s,y(s))|\le L_a|x(s)-y(s)|.$$

Assumption (F1). Let $f:[0,T]\times H$ be a nonlinear mapping such that

(i) $t\to f(t,x)$ is measurable;

(ii) f is locally Lipschitz continuous respect to x, that is, for each $r>0$, there exists a constant $L_f=L(r)>0$ such that

$$|f(t,x)-f(t,y)|\le L_f|x-y|$$

hold for any $t\in[0,T]$, $|x|\le r$ and $|y|\le r$.

Assumption (G1). Let $g:[0,T]\times H\to D(A)$ be a nonlinear mapping such that there exists L_g satisfying the following conditions hold:

(i) (i) and (iii) of Assumption (G) in Section 3 are satisfied.

(ii) There exists positive constants L_g such that

$$g(0,0)=0, \quad |Ag(t,0)|\le L_g, \quad |Ag(t,x)-Ag(t,\hat{x})|\le L_g|x-\hat{x}|,$$

for all $t\in[0,T]$, and x, $\hat{x}\in H$.

We define the linear operator $\hat{S}$ from $L^2(0,T;H)$ to H by

$$\hat{S}p=\int_0^T S(T-s)p(s)ds \quad \text{for } \mathrm{p}\in \mathrm{L}^2(0,\mathrm{T};\mathrm{H}).$$

Assumption (S). For any $\varepsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} |\hat{S}p - \hat{S}\Phi u| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \le q||p||_{L^2(0,t;H)}, \quad 0 \le t \le T. \end{cases}$$

where q is a constant independent of p.

Here, we remark that Assumptions (A1), (F1) and (G1) are actually sufficient conditions for Assumptions (A), (F) and (G), respectively. So, if $(x_0, k) \in H \times L^2(0,T;V^*)$ then from Theorem 3.2 and Corollary 3.1, it follows that the solution x of the equation (4.1) belongs to $x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$ and the mapping

$$H \times L^2(0,T;U) \ni (x_0, u) \mapsto x \in \mathcal{W}_1(T)$$

is continuous.

Lemma 4.2. *Let u_1 and u_2 be in $L^2(0,T;U)$. Then under the Assumptions (A1), (F1), (G1), and $||x(\cdot;f,u_1)||_{C([0,T;H])} < r$, we have*

$$||x(\cdot\,;f,u_1) - x(\cdot\,;f,u_2)||_{C([0,T];H)} \le Me^{M_3}\sqrt{t}||Bu_1 - Bu_2||_{L^2(0,T;H)} \tag{4.2}$$

for $0 \le t \le T$, where

$$M_3 := (||A^{-1}||L_gL_a + ML_f + ML_aL_gT)T. \tag{4.3}$$

Proof. For brevity, we set $x_i(t) = x(t;f,u_i)(i = 1,2)$. Let

$$\begin{aligned}(px_i)(t) = &- g(t, \int_0^t a(t,s,x_i(s))ds) + \int_0^t S(t-s)f(t,x_j(s))ds \\ &+ \int_0^t AS(t-s)\Big\{g\Big(s, \int_0^s a(s,\tau,x_j(\tau))d\tau\Big)ds.\end{aligned}$$

Then, we see

$$|x_1(t) - x_2(t)| \le |px_1 - px_2| + |\int_0^t S(t-s)B(u_1(s) - u_2(s))ds|. \tag{4.4}$$

Here, by Assumptions (F1) and(G1), he following inequalities hold:

$$|px_1 - px_2| \le (||A^{-1}||L_gL_a + ML_f + ML_aL_gt)\int_0^t |x_1(s) - x_2(s)|ds, \tag{4.5}$$

and

$$(4.6)\qquad |\int_0^t S(t-s)B(u_1(s)-u_2(s))ds| \le M\sqrt{t}||Bu_1 - Bu_2||_{L^2(0,T;H)}.$$

Thus from (4.4)-(4.6) and using Gronwall's inequality, it follows that

$$|x_1(t) - x_2(t)| \le M\sqrt{t}e^{(||A^{-1}||L_g L_a + ML_f + ML_a L_g T)T}||Bu_1 - Bu_2||_{L^2(0,t;H)}.$$

Therefore, (4.2) holds. □

Theorem 4.3. *Under the Assumptions (A1), (F1), (G1) and (S), the system (4.1) is approximately controllable on* $[0, T]$.

Proof. Let us define a reachable set for the system (4.1):

$$R_T = \{x(T; f, u) : u \in L^2(0, T; U)\}.$$

Then we will show that $D(A) \subset \overline{R_T}$, i.e., for given $\varepsilon > 0$ and $\xi_T \in D(A)$ there exists $u \in L^2(0, T; U)$ such that

$$|\xi_T - x(T; f, u)| < \varepsilon.$$

Noting that

$$\begin{aligned} g(t, \int_0^t a(t,s,x(s))ds) = \int_0^t S(t-s)\{g(t, &\int_0^t a(t,s,x(s))ds) \\ &+ sAg(t, \int_0^t a(t,s,x(s))ds)/t\}ds, \end{aligned}$$

the solution of (4.1) is represented as

$$x(t; f, u) = S(t)x_0 + \int_0^t S(t-s)G(s, x(s; f, u))ds + \int_0^t S(t-s)Bu(s)\}ds, \quad t \le T,$$

where

$$\begin{aligned} (4.7)\quad G(s, x(s)) =& g(t, \int_0^t a(t,s,x(s))ds) + sAg(t, \int_0^t a(t,s,x(s))ds)/t + f(s, x(s)) \\ &+ Ag\Big(s, \int_0^s a(s,\tau,x(\tau))d\tau\Big). \end{aligned}$$

As $\xi_T \in D(A)$ there exists a $h \in L^2(0, T; H)$ such that

$$\hat{S}h = \xi_T - S(T)x_0,$$

for instance, take $h(s) = (\xi_T + sA\xi_T) - S(s)x_0/T$. Let $u_1 \in L^2(0,T;U)$ be arbitrary fixed. Since by Assumption (S) there exists $u_2 \in L^2(0,T;U)$ such that

$$|\hat{S}(h - G(\,\cdot\,, x(\,\cdot\,, f, u_1))) - \hat{S}Bu_2| < \frac{\varepsilon}{4},$$

it follows

$$|\xi_T - S(T)x_0 - \hat{S}G(\,\cdot\,, x(\,\cdot\, f, u_1)) - \hat{S}Bu_2| < \frac{\varepsilon}{4}.$$

We can also choose $w_2 \in L^2(0,T;U)$ by Assumption (S) such that

$$|\hat{S}(G(\,\cdot\, x(\,\cdot\,; g, f, u_2)) - G(\,\cdot\, x(\,\cdot\,; g, f, u_1)) - \hat{S}Bw_2| < \frac{\varepsilon}{8} \tag{4.8}$$

and by Assumption (S)

$$\|Bw_2||_{L^2(0,t;H} \le q||G(\,\cdot\,, x(\,\cdot\,; f, u_1)) - G(\,\cdot\,, x(\,\cdot\,; f, u_2))||_{L^2(0,t;H)}$$

for $0 \le t \le T$. Choose a constant r_1 satisfying

$$||x(\,\cdot\,; f, u_1)||_{C([0,t];H)} \le r_1, \quad ||x(\,\cdot\,; f, u_2)||_{C([0,t];H)} \le r_1.$$

According to a simple calculation of (4.7), from Lemma 4.1 we have

(4.9)

$$\begin{aligned}
&|G(s, x(s; f, u_1)) - G(s, x(s; f, u_2))| \\
&\le \{(||A^{-1}|| + 2)L_a L_g T + L_f\}||x(\,\cdot\,; f, u_1) - x(\,\cdot\,; f, u_2)||_{C([0,t];H)}, \quad s \in (0,t], \\
&\le \{(||A^{-1}|| + 2)L_a L_g T + L_f\}Me^{M_3(r_1)}\sqrt{t}||Bu_1 - Bu_2||_{L^2(0,T;H)}
\end{aligned}$$

where M_3 is the constant of (4.3). For the sake of simplicity, set

$$\hat{L} := \{(||A^{-1}|| + 2)L_a L_g T + L_f\}Me^{M_3}, \quad r > 0.$$

Thus, in view of (4.9) and Assumption (S), we see

$$\begin{aligned}
||Bw_2||_{L^2(0,t;H)} &\le q\{\int_0^t |G(\tau, x(\tau; f, u_2)) - G(\tau, x(\tau; f, u_1))|^2 d\tau\}^{\frac{1}{2}} \\
&\le q\hat{L}\{\int_0^t \tau||Bu_2 - Bu_1||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}} \\
&\le q\hat{L}(\int_0^t \tau d\tau)^{\frac{1}{2}}||Bu_2 - Bu_1||_{L^2(0,t;H)} \\
&= q\hat{L}(\frac{t^2}{2})^{\frac{1}{2}}||Bu_2 - Bu_1||_{L^2(0,t;H)}.
\end{aligned}$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$\begin{aligned}
&|\hat{S}(G(\,\cdot\,, x(\,\cdot\,; f, u_3)) - G(\,\cdot\,, x(\,\cdot\,; f, u_2))) - \hat{S}Bw_3| < \frac{\varepsilon}{8}, \\
&||Bw_3||_{L^2(0,t;H)} \le q||G(\,\cdot\,, x(\,\cdot\,; f, u_3)) - G(\,\cdot\,, x(\,\cdot\,; f, u_2))||_{L^2(0,t;H)}
\end{aligned}$$

for $0 \leq t \leq T$. Let r_2 be a constant satisfying $r_2 \geq r_1$ and

$$||x(\,\cdot\,; f, u_3)||_{C([0,t];H)} \leq r_2.$$

Then, we have

$$\begin{aligned}
&||Bw_3||_{L^2(0,t;H)} \\
&\leq q\{\int_0^t ||G(\tau, x(\tau; f, u_3)) - G(\tau, x(\tau; f, u_2))||^2 d\tau\}^{\frac{1}{2}} \\
&\leq q\hat{L}\{\int_0^t \tau ||Bu_3 - Bu_2||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}} \\
&\leq q\hat{L}\{\int_0^t \tau ||Bw_2||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}} \\
&\leq q\hat{L}\{\int_0^t \tau (q\hat{L})^2 \frac{\tau^2}{2} ||Bu_2 - Bu_1||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}} \\
&\leq q^2\hat{L}^2 (\int_0^t \frac{\tau^3}{2} d\tau)^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)} \\
&= q^2\hat{L}^2 (\frac{t^4}{2\cdot 4})^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)}.
\end{aligned}$$

By proceeding this process, the following holds

$$\begin{aligned}
&||B(u_n - u_{n+1})||_{L^2(0,t;H)} = ||Bw_n||_{L^2(0,t;H)} \\
&\leq q^{n-1}\hat{L}^{n-1} (\frac{t^{2n-2}}{2\cdot 4\cdots(2n-2)})^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)} \\
&= (\frac{tq}{\sqrt{2}})^{n-1} \hat{L}^{n-1} \frac{1}{\sqrt{(n-1)!}} ||Bu_2 - Bu_1||_{L^2(0,t;H)},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} ||Bu_{n+1} - Bu_n||_{L^2(0,T;H)} \\
&\leq \sum_{n=0}^{\infty} (\frac{tq}{\sqrt{2}})^n \hat{L}^n) \frac{1}{\sqrt{n!}} ||Bu_2 - Bu_1||_{L^2(0,T;H)} < \infty.
\end{aligned}$$

Therefore, by virtue of Assumption (F1), there exists $u^* \in L^2(0,T;U)$ such that

$$\lim_{n\to\infty} Bu_n = u^* \quad \text{in} \quad L^2(0,T;H). \tag{4.10}$$

From (4.8), (4.9) it follows that

$$\begin{aligned}
&|\xi_T - S(T)g - \hat{S}G(\cdot, x(\cdot; f, u_2)) - \hat{S}Bu_3| \\
&= |\xi_T - S(T)g - \hat{S}G(\cdot, x(\cdot; f, u_1)) - \hat{S}Bu_2 + \hat{S}Bw_2 \\
&\ - \hat{S}[G(\cdot, x(\cdot; f, u_2)) - G(\cdot, x(\cdot; f, u_1))]| \\
&< (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.
\end{aligned}$$

By choosing choose $w_n \in L^2(0, T; U)$ by the Assumption (S) such that

$$|\hat{S}(G(\,\cdot\, x(\,\cdot\,; f, u_n)) - G(\,\cdot\, x(\,\cdot\,; f, u_{n-1})) - \hat{S}Bw_n| < \frac{\varepsilon}{2^{n+1}},$$

putting $u_{n+1} = u_n - w_n$, we have

$$\begin{aligned}
&|\xi_T - S(T)g - \hat{S}G(\cdot, x(\cdot; f, u_n)) - \hat{S}Bu_{n+1}| \\
&< (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1\ 2,\ \cdots.
\end{aligned}$$

Therefore, for $\varepsilon > 0$ there exists integer N such that

$$||\hat{S}Bu_{N+1} - \hat{S}Bu_N|| < \frac{\varepsilon}{2}$$

and

$$\begin{aligned}
&|\xi_T - S(T)g - \hat{S}G(\cdot, x(\cdot; f, u_N)) - \hat{S}Bu_N| \\
&\le |\xi_T - S(T)g - \hat{S}G(\cdot, x(\cdot; f, u_N)) - \hat{S}Bu_{N+1}| + |\hat{S}Bu_{N+1} - \hat{S}Bu_N| \\
&< (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \le \varepsilon.
\end{aligned}$$

Thus the system (4.1) is approximately controllable on $[0, T]$ as N tends to infinity. □

5. Example

Let

$$H = L^2(0, \pi),\ V = H_0^1(0, \pi),\ V^* = H^{-1}(0, \pi).$$

Consider the following semilinear neutral differential control system in Hilbert space H:

(5.1)

$$\begin{cases}
\frac{d^\alpha}{dt^\alpha}[x(t, y) + \sum_{n=1}^\infty \int_0^T e^{n^2(t-s)}(\int_0^t a(t+s)x(s, y)ds)dt = Ax(t, y) \\
\qquad\qquad + \frac{\sigma(x(t,y)-x(0,y))}{1+|x(t,y)-x(0,y)|} + (Bu(t))(y), \quad (t, y) \in [0, T] \times [0, \pi],\ \ \sigma > 0, \\
x(0, y) = x_0(y), \quad y \in [0, \pi],
\end{cases}$$

where $h > 0$, $a_1(\cdot)$ is Hölder continuous, and $A_1 \in \mathcal{L}(H)$. Let

$$a(u, v) = \int_0^\pi \frac{du(y)}{dy} \overline{\frac{dv(y)}{dy}} dy.$$

Then

$$A = \partial^2/\partial y^2 \quad \text{with} \quad D(A) = \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $z_n(y) = (2/\pi)^{1/2} \sin ny$, respectively. Moreover,

(a1) $\{z_n : n \in N\}$ is an orthogonal basis of H and

$$S(t)x = \sum_{n=1}^{\infty} e^{n^2 t}(x, z_n)z_n, \quad \forall x \in H, \ t > 0.$$

Moreover, there exists a constant M such that $||S(t)||_{\mathcal{L}(H)} \leq M$.

(a2) Let $0 < \alpha < 1$. Then the fractional power $A^\alpha : D(A^\alpha) \subset H \to H$ of A is given by

$$A^\alpha x = \sum_{n=1}^{\infty} n^{2\alpha}(x, z_n)z_n, \ D(A^\alpha) := \{x : A^\alpha x \in H\}.$$

In particular,

$$A^{-1/2}x = \sum_{n=1}^{\infty} \frac{1}{n}(x, z_n)z_n, \ \text{ and } \ ||A^{-1/2}|| = 1.$$

The nonlinear mapping that appears on the control system for a diffusion and reaction process in an enzyme membrane is defined as

$$f(x(t,y)) = \frac{\sigma(x(t,y) - x(0,y))}{1 + |x(t,y) - x(0,y)|}.$$

Then since

$$|f(x_1(t,y)) - f(x_2(t,y))| \leq \frac{\sigma\big(1 + 2|x_2(t,y) - x(0,y)|\big) \cdot |x_1(t,y) - x_2(t,y)|}{(1 + |x_1(t,y) - x(0,y)|)(1 + |x_2(t,y) - x(0,y)|},$$

we can see that f satisfies Assumption (F1).

Define $g : [0,T] \times H \to H$ as

$$g(t, \int_0^t a(t,s,x(s))ds) = \sum_{n=1}^{\infty} \int_0^T e^{n^2(t-s)}(\int_0^t a(t+s)x(s,y)ds)dt, \quad , t > 0.$$

Then it can be checked that Assumption (G) is satisfied. Indeed, for $x \in \Pi$, we know

$$Ag(t, \int_0^t a(t,s,x(s))ds) = (I - S(t)) \int_0^t a(t+s)x(s,y)ds)dt,$$

where I is the identity operator form H to itself and, we assume that there is a constant $\rho > 0$ such that

$$|a(0)| \leq \rho, \quad |a(s) - a(\tau)| \leq \rho(s-\tau)^\kappa, \quad s,\ \tau \in [0,T]$$

for a constant $\kappa > 0$. Hence we have

$$\begin{aligned} |Ag(t, \int_0^t a(t,s,x(s))ds)| \leq &(M+1)\big\{\big|\int_0^t (a(t+s) - a_2(0))x(s)ds\big| \\ &+ \big|\int_0^t a(0)x(s)ds\big|\big\} \\ \leq &(M+1)\rho\big\{(2\kappa+1)^{-1}h^{2\kappa+1} + h\big\}||x||_{L^2(0,T;V)}. \end{aligned}$$

It is immediately seen that Assumption (G1) has been satisfied. A simple example of the controller operator B which satisfies Assumption (S) is introduced by Naito [16] as follows. Consider $H = U$ and define the intercept operator $B_{(\alpha,T)}$, $0 < \alpha < T$, on $L^2(0,T;H)$ by

$$(B_{(\alpha,T)}u)(t) = \begin{cases} 0, & 0 \leq t < \alpha \\ u(t), & \alpha \leq t \leq T, \quad u \in L^2(0,T;H). \end{cases}$$

Then as seen in [16], for a given $p \in L^2(0,T;H)$ there exists a control $u \in L^2(0,T;H)$such that $\hat{S}p = \hat{S}B_{(\alpha,T)}u$. Thus, all the conditions stated in Theorem 3.1 have been satisfied for the equation (5.1), and so there exists a solution of (5.1) belongs to $\mathcal{W}_1(T) = L^2(0,T;V)) \cap W^{1,2}(0,T;V^*) \hookrightarrow C([0,T];H)$, and by virtue of Theorem 4.1, the system (4.1) is approximately controllable on $[0,T]$.

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