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New Bounds for the Numerical Radius of a Matrix in Terms of Its Entries

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Abstract. In this work we give new upper and lower bounds for the numerical radius of a complex square matrix A using the entries and the trace of A.

1. Introduction

The numerical range of a complex $n \times n$ matrix A is the set defined as

$$W(A) = \{\langle Ax, x\rangle, \ \ x \in \mathbb{C}^n, \ \|x\| = 1\},$$

where $\langle x, y\rangle$ is the usual inner product of elements x and y in $\mathbb{C}^n$. The numerical range of the matrix A localizes its spectrum i.e $\Lambda(A) \subseteq W(A)$, where $\Lambda(A)$ denotes the spectrum of A. The numerical range has several properties.

The numerical radius $\omega(A)$ is defined by

$$\omega(A) = \sup_{\lambda \in W(A)} |\lambda| \qquad \text{or} \qquad \omega(A) = \max_{\|x\|=1} |\langle Ax, x\rangle|.$$

Numerous contributions related to numerical radius were made by various people including M. Goldberg, E. Tadmor and G. Zwas [1], also J. Merikoski and R. Kumar [4]. We cite here some properties of the numerical radius which are well known see [2]. Let A, B be two complex matrices and $\alpha \in \mathbb{C}$,

1. $\omega(A + B) \leq \omega(A) + \omega(B)$,
2. $\omega(\alpha A) = |\alpha|\omega(A)$,
3. $\omega(A) = \omega(A^*)$,

where A^* is the conjugate transpose of A.

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If M is any principle submatrix of A, then

$$\omega(M) \leq \omega(A).$$

In this paper, without knowing the numerical radius of the matrix A, we can estimate it by giving some upper and lower bounds using the entries and the trace of A.

Let A be a complex $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$, *the spectral radius* of A is defined by

$$\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|.$$

It is well known, see [1], that

$$\rho(A) \leq \omega(A) \leq \|A\| \leq 2\omega(A),$$

where $\|A\| = \max_{\|x\|=1} \|Ax\|$ is *the spectral norm.*
Let $tr(A) = \sum_{i=1}^{n} \lambda_i$ denote *the trace* of A and let $su(A) = \sum_{i,j=1}^{n} a_{ij}$ denote *the sum* of A.
Let e_i be the column vector whose i-th component is equal to 1 while all the remaining components are 0.
Let $R(A)$ and c denote the radius and center of the smallest disc $\mathcal{D}$ which contains all eigenvalues of A.

In [3] C. R. Johnson gave an upper bound for the numerical radius

$$\omega(A) \leq \max_{i} \left(\sum_{j=1}^{n} \frac{|a_{ij}| + |a_{ji}|}{2} \right).$$

J. K. Merikoski and R. Kumar [4] gave some lower bounds for the numerical radius $\omega(A)$ for example :

$$\max_{i} |a_{ii}| \leq \omega(A)$$

and

$$\left| \frac{su(A)}{n} \right| \leq \omega(A).$$

2. Bounds For the Numerical Radius

In this section, we give some upper and lowers bounds for the numerical radius of a given complex $n \times n$ matrix.

Proposition 2.1. *For any matrix A, we have*

$$R(A) \leq \omega(A).$$

Theorem 2.2. *Let $A = (a_{ij})$ be a normal $n \times n$ matrix, we have*

$$\max_{i \neq j} |a_{ij}| \leq \omega(A).$$

Proof. Let z be any complex number. For $i \neq j$,

$$\begin{aligned} |a_{ij}| = |e_i^*(A - zI)e_j| &\leq \|e_i\|.\|(A - zI)e_j\| = \|(A - zI)e_j\| \\ &\leq \sup_{\|u\|=1} \|(A - zI)u\| = \max_i |\lambda_i - z|. \end{aligned}$$

Since $\inf_z \max_i |\lambda_i - z| = R(A)$, then $\max_{i \neq j} |a_{ij}| \leq \omega(A)$. □

Corollary 2.3. *Let $A = (a_{ij})$ be a normal $n \times n$ matrix, we have*

$$\frac{1}{2} \max_{i \neq j}(|a_{ij}| + |a_{ji}|) \leq \omega(A).$$

Proof. Applying the result of the above theorem to the matrix $\dfrac{zA + \overline{z}A^*}{2}$, where $z \in \mathbb{C}$ with $|z| = 1$, it follows that $\dfrac{1}{2} \max_{i \neq j} |za_{ij} + \overline{za_{ji}}| \leq \omega(A)$. Since $\max_{|z|=1} |za_{ij} + \overline{za_{ji}}| = |a_{ij}| + |a_{ji}|$, then the required result is obtained. □

Theorem 2.4. *Let $A = (a_{ij})$ be a complex $n \times n$ matrix, we have*

$$\omega(A) \leq \max_i |a_{ii}| + (n - 1) \max_{i \neq j} |a_{ij}|.$$

Proof. Write $x = (x_1, x_2, \cdots, x_n)$ and let $\lambda \in W(A)$ then $\lambda = xAx^*$ with $\|x\| = 1$. Hence $\lambda = \sum_{i,j} a_{ij} x_j x_i^*$, thus $|\lambda| \leq \sum_{i,j} |a_{ij}| \xi_i \xi_j$ where $\xi_i = |x_i|$. It follows that

$$\begin{aligned} |\lambda| &\leq \sum_i |a_{ii}| \xi_i^2 + \sum_{i \neq j} |a_{ij}| \xi_j \xi_i \\ &\leq \max_i |a_{ii}| + \max_{i \neq j} |a_{ij}| \left(\sum_{i<j} 2\xi_i \xi_j \right) \\ &\leq \max_i |a_{ii}| + \max_{i \neq j} |a_{ij}| \left((n-1) \sum_i \xi_i^2 \right) \\ &= \max_i |a_{ii}| + (n - 1) \max_{i \neq j} |a_{ij}|. \end{aligned}$$

We have used the fact that $2\xi_i \xi_j \leq \xi_i^2 + \xi_j^2$. Since $\omega(A) = \max_{\lambda \in W(A)} |\lambda|$, then this completes the proof. □

Corollary 2.5. *Let $A = (a_{ij})$ be a complex $n \times n$ matrix, we have*

$$\omega(A) \leq n \max_{i,j} |a_{ij}|.$$

Theorem 2.6. *Let $A = (a_{ij})$ be a complex $n \times n$ matrix, we have*

$$\omega(A) \leq \max_i |a_{ii}| + \left(\sum_{i \neq j} |a_{ij}|^2 \right)^{1/2}.$$

Proof. Let $\lambda \in W(A)$ then $\lambda = \sum_{i,j} a_{ij} x_j x_i^*$. Hence $|\lambda| \leq \sum_{i,j} |a_{ij}| \xi_i \xi_j$ where $\xi_i = |x_i|$. It follows that $|\lambda| \leq \sum_i |a_{ii}| \xi_i^2 + \sum_{i \neq j} |a_{ij}| \xi_j \xi_i$. Rewriting $|a_{ij}| \xi_j \xi_i$ as $|a_{ij}| \times \xi_j \xi_i$ and applying the Cauchy Schwarz's inequality, we obtain

$$\begin{aligned} |\lambda| &\leq \max_i |a_{ii}| + \left(\sum_{i \neq j} |a_{ij}|^2 \right)^{1/2} \left(\sum_{i \neq j} \xi_i^2 \xi_j^2 \right)^{1/2} \\ &\leq \max_i |a_{ii}| + \left(\sum_{i \neq j} |a_{ij}|^2 \right)^{1/2} \left(\sum_i \xi_i^2 \cdot \sum_j \xi_j^2 \right)^{1/2} \\ &\leq \max_i |a_{ii}| + \left(\sum_{i \neq j} |a_{ij}|^2 \right)^{1/2}. \end{aligned}$$

Since $\omega(A) = \max_{\lambda \in W(A)} |\lambda|$, then the desired result is obtained. □

Let $A = (a_{ij})$ be a complex $n \times n$ matrix and let $L_i = \sum_j |a_{ij}| - |a_{ii}|$, $C_j = \sum_i |a_{ij}| - |a_{jj}|$.

Theorem 2.7. *Let $A = (a_{ij})$, L_i and C_j be as described above and let $L = \max(L_i)$, $C = \max(C_j)$. Then*

$$\omega(A) \leq \max_i |a_{ii}| + (LC)^{1/2}.$$

Proof. Let $\lambda \in W(A)$ then $\lambda = \sum_{i,j} a_{ij} x_j x_i^*$. Hence $|\lambda| \leq \sum_{i,j} |a_{ij}| \xi_i \xi_j$ where $\xi_i = |x_i|$. Thus $|\lambda| \leq \sum_i |a_{ii}| \xi_i^2 + \sum_{i \neq j} |a_{ij}| \xi_j \xi_i$.
Rewriting $|a_{ij}| \xi_j \xi_i$ as $|a_{ij}|^{1/2} \xi_i \times |a_{ij}|^{1/2} \xi_j$ and applying the Cauchy Schwarz's in-

equality, it follows that

$$\begin{aligned}
|\lambda| &\leq \max_i |a_{ii}| + \left(\sum_{i\neq j} |a_{ij}|\xi_i^2\right)^{1/2} \left(\sum_{i\neq j} |a_{ij}|\xi_j^2\right)^{1/2} \\
&= \max_i |a_{ii}| + \left(\sum_i L_i\xi_i^2\right)^{1/2} \left(\sum_j C_j\xi_j^2\right)^{1/2} \\
&\leq \max_i |a_{ii}| + \left(L\sum_i \xi_i^2\right)^{1/2} \left(C\sum_j \xi_j^2\right)^{1/2} \\
&= \max_i |a_{ii}| + (LC)^{1/2}.
\end{aligned}$$

Since $\omega(A) = \max_{\lambda\in W(A)} |\lambda|$, then the assertion follows immediately. □

Theorem 2.8. *Let $A = (a_{ij})$, L_i and C_i be as described above and let $S_i = \dfrac{L_i + C_i}{2}$, $S = \max_i S_i$. Then*

$$\omega(A) \leq \max_i |a_{ii}| + S.$$

Proof. Let $\lambda \in W(A)$ then $\lambda = \sum_{i,j} a_{ij}x_jx_i^*$. Hence $|\lambda| \leq \sum_{i,j} |a_{ij}|\xi_i\xi_j$ where $\xi_i = |x_i|$. It follows that

$$\begin{aligned}
|\lambda| &\leq \sum_i |a_{ii}|\xi_i^2 + \sum_{i\neq j} |a_{ij}|\xi_j\xi_i \\
&\leq \max_i |a_{ii}| + \frac{1}{2}\sum_{i\neq j} |a_{ij}|(\xi_i^2 + \xi_j^2) \\
&= \max_i |a_{ii}| + \frac{1}{2}\sum_i L_i\xi_i^2 + \frac{1}{2}\sum_j C_j\xi_j^2 \\
&= \max_i |a_{ii}| + \sum_i S_i\xi_i^2 \\
&\leq \max_i |a_{ii}| + S.
\end{aligned}$$

Since $\omega(A) = \max_{\lambda\in W(A)} |\lambda|$, then the result follows directly. □

Lemma 2.9. *If $z_1, \cdots, z_n$ are complex numbers, then*

$$\left|\frac{z_1 + \cdots + z_n}{n}\right| \leq \max_i |z_i|.$$

Corollary 2.10. *Let A be a complex $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then*

$$\left|\frac{tr(A)}{n}\right| \leq \omega(A).$$

Proof. Using the previous lemma, $z_i = \lambda_i$, it follows that $\left|\frac{tr(A)}{n}\right| \leq \rho(A) \leq \omega(A)$. □

Theorem 2.11. *Let $A = (a_{ij})$ be a complex $n \times n$ matrix. Then*

$$\max_{i \neq j}\left|\frac{a_{ii} + a_{jj} - a_{ij} - a_{ji}}{2}\right| \leq \omega(A).$$

Proof. For $i \neq j$, we have $\left|\frac{(e_i - e_j)^* A (e_i - e_j)}{2}\right| = \left|\frac{a_{ii} + a_{jj} - a_{ij} - a_{ji}}{2}\right| \leq \omega(A)$.
□

Theorem 2.12. *Let $A = (a_{ij})$ be a complex $n \times n$ matrix. Then*

$$\frac{n}{n-1}\left|\frac{tr(A)}{n} - \frac{su(A)}{n^2}\right| \leq \omega(A).$$

Proof. Using Lemma 2.9. where the z's are the $n(n-1)$ numbers $z_{ij} = \dfrac{a_{ii} + a_{jj} - a_{ij} - a_{ji}}{2}, i \neq j$, thus

$$\begin{aligned} \max_{i \neq j}\left|\frac{a_{ii} + a_{jj} - a_{ij} - a_{ji}}{2}\right| &\geq \frac{1}{n(n-1)}\left|\sum_i \sum_{j \neq i} \frac{a_{ii} + a_{jj} - a_{ij} - a_{ji}}{2}\right| \\ &= \frac{1}{n(n-1)}\left|n\sum_{i=1}^n a_{ii} - \sum_{i,j=1}^n a_{ij}\right| \\ &= \frac{n}{n-1}\left|\frac{tr(A)}{n} - \frac{su(A)}{n^2}\right|. \end{aligned}$$

Using the previous theorem then the required statement follows immediately. □

Theorem 2.13. *Let A be a complex $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then*

$$\sqrt{\frac{1}{n}}\left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|tr(A)|^2}{n}\right)^{\frac{1}{2}} \leq \omega(A).$$

Proof. We have

$$\sum_{i=1}^n |\lambda_i - c|^2 \leq nR^2(A),$$

where c and $R(A)$ are the center and the radius of the smallest disc $\mathcal{D}$, respectively. On the other hand,

$$\begin{aligned}\sum_{i=1}^{n}|\lambda_i - c|^2 &= \sum_{i=1}^{n}\left(|\lambda_i|^2 - c\overline{\lambda_i} - \overline{c}\lambda_i + |c|^2\right)\\ &= \sum_{i=1}^{n}|\lambda_i|^2 - \frac{|tr(A)|^2}{n} + n\left|c - \frac{tr(A)}{n}\right|^2.\end{aligned}$$

It is clear that the choice $c = tr(A)/n$ gives the smallest possible value for this last expression. Hence $\frac{1}{n}\left(\sum_{i=1}^{n}|\lambda_i|^2 - \frac{|tr(A)|^2}{n}\right) \leq R^2(A) \leq \omega^2(A)$. □

Corollary 2.14. *Let A be a normal $n \times n$ matrix. Then*

$$\sqrt{\frac{1}{n}}\left(\|A\|_{Fr}^2 - \frac{|tr(A)|^2}{n}\right)^{\frac{1}{2}} \leq \omega(A),$$

where $\|A\|_{Fr}^2 = \sum_{i,j=1}^{n}|a_{ij}|^2 = trAA^*$ is the Frobenius norm.

Proof. Since A is normal, then $\sum_{i=1}^{n}|\lambda_i|^2 = \|A\|_{Fr}^2$. Hence the desired result follows. □

Theorem 2.15. *Let $A = (a_{ij})$ be a Hermitian $n \times n$ matrix. Then*

$$\frac{1}{2}\max_{i\neq j}\left\{a_{ii} + a_{jj} + \sqrt{(a_{ii} - a_{jj})^2 + 4|a_{ij}|^2}\right\} \leq \omega(A).$$

Proof. Let M be any principal submatrix of A. Let $1 \leq i < j \leq n$ and

$$M = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj}\end{pmatrix},$$

then

$$\rho(M) = \frac{1}{2}\left\{a_{ii} + a_{jj} + \sqrt{(a_{ii} - a_{jj})^2 + 4|a_{ij}|^2}\right\} \leq \omega(M) \leq \omega(A).$$

□

3. The Areal Numerical Radius of Matrices

Let $\Gamma(A)$ denotes the area of the smallest disc $\mathcal{D}$ which contains all eigenvalues of the matrix A.

R. A. Smith and L. Mirsky in [5] called areal spread of the matrix A the ratio $\frac{\sigma(A)}{\|A\|^2}$ where $\sigma(A)$ is the minimal area in the complex plane and $\|.\|$ is the euclidean matrix norm. In analogy with this concept, let $\dfrac{\Gamma(A)}{\omega^2(A)}$ be the areal numerical radius of A. In the following theorem we give an estimate to the supremum of the areal numerical radius of A as A ranges over all nonzero $n \times n$ matrices.

Theorem 3.1. *Let A be a complex $n \times n$ matrix and let $\Gamma(A)$ be as described above. Then*

$$\sup\left(\frac{\Gamma(A)}{\omega^2(A)}\right) = \pi,$$

where the supremum is taken over all nonzero $n \times n$ matrices A.

Proof. Since $\Gamma(A) = \pi R^2(A)$, it is sufficient to prove that $\sup \dfrac{R(A)}{\omega(A)} = 1$. We have $R(A) \leq \rho(A) \leq \omega(A)$, on the other hand, taking $A = diag(-1, 0, \cdots, 0, 1)$, it follows that $R(A) = 1$ and $\omega(A) = 1$. Hence $\sup \dfrac{R(A)}{\omega(A)} = 1$, this completes the proof. $\square$

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