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On Axis-commutativity of Rings

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Abstract. We study a new ring property called axis-commutativity. Axis-commutative rings are seated between commutative rings and duo rings and are a generalization of division rings. We investigate the basic structure and several extensions of axis-commutative rings.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We denote the center and the group of units in R by $Z(R)$ and $U(R)$, respectively. We use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$, and $W(R)$ to denote the set of all nilpotents, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals), and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere. $\mathbb{Z}_n$ denotes the ring of integers modulo n.

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A ring is usually called *reduced* if it has no nonzero nilpotents. It is easily checked that a ring R is reduced if and only if $a^2 = 0$ for $a \in R$ implies $a = 0$. A ring is usually called *Abelian* if every idempotent is central. It is easily checked that reduced rings are Abelian.

Recall that for a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

The study of the trivial extension of generalized reduced rings plays a significant role in noncommutative ring theory to understand the ring structure. For example, the trivial extension of a reduced ring is not reduced but contained in some class of generalized reduced rings. In addition, a ring R is Abelian if and only if its trivial extension is by [6, Lemma 2], comparing with the fact that a ring R is commutative if and only if its trivial extension is. Moreover, we have the following.

Theorem 1.1. *Let R be a division ring. For the trivial extension $T(R, R)$ of R, we have $AT(R, R)B = BT(R, R)A$ for all $A, B \in T(R, R)$.*

Proof. Assume that R is a division ring and set $T = T(R, R)$. Let

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} \in T \backslash \{0\}$$

and compute the relation between ATB and BTA.

(Case 1) Suppose $a \neq 0$ and $a' \neq 0$. Then $A, B \in U(T)$ and hence $ATB = T = BTA$.

(Case 2) Suppose $a \neq 0$ and $a' = 0$. Then $A \in U(T)$ and hence

$$ATB = TB = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} = BT = BTA.$$

(Case 3) Suppose $a = 0$ and $a' \neq 0$. Then $B \in U(T)$ and hence

$$ATB = AT = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} = TA = BTA.$$

(Case 4) Suppose $a = 0$ and $a' = 0$. Then $ATB = 0 = BTA$.

Summarizing, we conclude that $AT(R, R)B = BT(R, R)A$ for any $A, B \in T(R, R)$. Note that $T(R, R)$ is clearly noncommutative. □

Example 1.2. (1) The condition 'R be a division ring' in Theorem 1.1 cannot be weakened by the condition 'R be a domain' as follows. Consider the free algebra $R = K\langle a, b\rangle$ generated by the noncommuting indeterminates a, b over a field K. Then R is a domain. Consider $T(R, R)$. For $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in T(R, R)$, we have $AB \notin BT(R, R)A$, since $ab \notin bRa$. This shows that $AT(R, R)B \neq BT(R, R)A$.

(2) The converse of Theorem 1.1 does not hold, in general. For example, consider a commutative ring R which is not a division ring. Then the trivial ring $T(R, R)$ is commutative and so it satisfies the conclusion of Theorem 1.1.

Based on the above, we define a new ring property as follows.

Definition 1.3. A ring R is called *axis-commutative* if $aRb = bRa$ for all $a, b \in R$.

Then we obtain the next results.

Proposition 1.4. (1) *If the trivial extension $T(R, R)$ of a ring R is axis-commutative, then so is R.*

(2) *$D_n(R)$ is not axis-commutative over any ring R when $n \geq 3$.*

(3) *Both $Mat_n(R)$ and $T_n(R)$ is not axis-commutative over any ring R when $n \geq 2$.*

Proof. (1) Suppose that $T = T(R, R)$ of a ring R is axis-commutative and consider aRb and bRa for $a, b \in R\backslash\{0\}$. Let

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in T.$$

Then $ATB = BTA$ by assumption. For any $C = \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \in T$, we have $ACB = \begin{pmatrix} arb & asb \\ 0 & arb \end{pmatrix} \in BTA$ and it implies $arb \in bRa$ for any $r \in R$. Thus $aRb \subseteq bRa$. Similarly, it can be obtained $bRa \subseteq aRb$.

Consequently, $aRb = bRa$, showing that R is axis-commutative.

(2) Let R be a ring and consider $D_n(R)$ for $n \geq 3$. Then $E_{12}D_n(R)E_{23} = RE_{13}$ but $E_{23}D_n(R)E_{12} = 0$. So $D_n(R)$ is not axis-commutative.

(3) Let R be a ring and consider $T_n(R)$ for $n \geq 2$. Then $E_{22}T_n(R)E_{11} = 0$ but $E_{11}T_n(R)E_{22} = (RE_{11} + RE_{12} + \cdots + RE_{1n})E_{22} = RE_{12} \neq 0$. Thus $T_n(R)$ is not axis-commutative.

The computation for $Mat_n(R)$ when $n \geq 2$ is the same as the preceding one. □

Let $V_n(R)$ be the ring of all matrices $(a_{ij}) \in D_n(R)$ such that $a_{ij} = a_{(i+1)(j+1)}$ for $i = 1, \ldots, n-2$ and $j = 2, \ldots, n-1$.

Proposition 1.5. *Let R be a ring and $n \geq 2$.*

(1) *If R is a division ring, then $V_n(R)$ is axis-commutative.*

(2) *If the ring $V_n(R)$ is axis-commutative, then so does R.*

Proof. We apply the proof of Theorem 1.1 and Proposition 1.4. Let $V = V_n(R)$ and use $(a, a_2, \dots, a_n) \in V$ to denote

$$\begin{pmatrix} a & a_2 & a_3 & \cdots & a_n \\ 0 & a & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}.$$

(1) Suppose that R is a division ring. Let

$$A = (a, a_2, \dots, a_n), B = (b, b_2, \dots, b_n) \in V \backslash \{0\}.$$

If $a \neq 0$ and $b \neq 0$, then $A, B \in U(V)$ and so $AVB = V = BVA$. If $a \neq 0$ and $b = 0$, then $A \in U(V)$ and so $AVB = VB = (0, R, \dots, R) = BV = BVA$. If $a = 0$ and $b \neq 0$, then $B \in U(V)$ and so $AVB = AV = (0, R, \dots, R) = VA = BVA$.

Now we show that $AVB = 0 = BVA$ when $a = 0$ and $b = 0$. If $n = 2$, then V is axis-commutative by Theorem 1.1. We assume $n \geq 3$. Let $AB = 0$. Then we have the following equations:

$$a_2 b_2 = 0 \tag{1.1}$$

$$a_2 b_3 + a_3 b_2 = 0 \tag{1.2}$$

$$a_2 b_4 + a_3 b_3 + a_4 b_2 = 0 \tag{1.3}$$

$$\vdots$$

$$a_2 b_{n-1} + a_3 b_{n-2} + \cdots + a_{n-1} b_2 = 0 \tag{1.4}$$

We note that $ab = 0$ implies $aRb = 0$, and $ab^2 = 0$ implies $ab = 0$ and hence $aRb = 0$ for $a, b \in R$ because R is a division ring. We freely use these facts in the following computations.

With the help of $a_2 b_2 = 0$ and multiplying Equation(1.2) by b_2 on the right side, we get $a_3 b_2^2 = 0$ and so $a_3 b_2 = 0$ and $a_2 b_3 = 0$. Multiply Equation(1.3) on the right side by b_2 and b_3 in turn, we get $a_4 b_2 = 0$ and $a_3 b_3 = 0$, and so $a_2 b_4 = 0$.

Inductively we assume that $a_i b_j = 0$ for $i + j \leq n$ with $2 \leq i$. Multiply Equation(1.4) on the right side by $b_2, b_3, \dots, b_{n-2}$ in turn, we have $a_{n-1} b_2 = 0, a_{n-2} b_3 = 0, \dots, a_3 b_{n-2} = 0$, and hence $a_2 b_{n-1} = 0$.

Consequently, $a_ib_j = 0$ for all i and j with $i+j \le n+1$ and $2 \le i, j$, it implies that $a_iRb_j = 0$. Since R is a division ring, it is axis-commutative by Theorem 1.1 and Proposition 1.4(1), and thus $b_jRa_i = 0$ for all i and j with $i+j \le n+1$ and $2 \le i, j$. These yield $AVB = 0 = BVA$, proving that V is axis-commutative.

(2) Suppose that V is axis-commutative and let $a, b \in R$. For $A = (a, 0, \ldots, 0)$ and $B = (b, 0, \ldots, 0) \in V$, we have $AVB = BVA$ by assumption. This implies $aRb = bRa$. □

Recall that $V_n(R) \cong \frac{R[x]}{(x^n)}$, where (x^n) is an ideal of $R[x]$ generated by x^n. Hence, if the ring $R[x]/(x^n)$ is axis-commutative for any integer $n \ge 1$, then so is R.

The next example shows that the converse of Proposition 1.4(1) (also Proposition 1.5(2)) need not hold.

Example 1.6. Let $\mathbb{H}$ be the Hamilton quaternions over the real number field $\mathbb{R}$ and $R = T(\mathbb{H}, \mathbb{H})$. Then R is axis-commutative by Theorem 1.1, since $\mathbb{H}$ is a division ring. Consider $T = T(R, R)$ and

$$A = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix}, B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in T.$$

Then

$$ATB = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i\alpha k + j\alpha \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mid \alpha \in \mathbb{H} \right\}$$

and

$$BTA = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \beta j + k\beta i \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mid \beta \in \mathbb{H} \right\}.$$

This implies that $ATB \ne BTA$ since $i\alpha k + j\alpha \ne \beta j + k\beta i$ for any $\alpha, \beta \in \mathbb{H}\backslash\{0\}$. In fact, $i\alpha k + j\alpha = -2c - 2bk$ and $\beta j + k\beta i = -2d'i + 2a'j$ for any nonzero $\alpha = a + bi + cj + dk, \beta = a' + b'i + c'j + d'k$. Hence $T = T(R, R)$ is not axis-commutative.

2. Property of Axis-commutative Rings

Following Feller [4], a ring (possibly without identity) is called *right duo* if every right ideal is two-sided. Left duo rings are defined similarly. A ring is called *duo* if it is both left and right duo. Right or left duo rings are easily shown to be Abelian.

An axis-commutative ring is a generalization of division rings as noted in Section 1. In this section, we show that the class of axis-commutative rings is seated between commutative rings and duo rings, and investigate the basic structure and several extensions of axis-commutative rings.

In the next lemma we observe basic properties of an axis-commutative ring which do important roles throughout this article.

Lemma 2.1. (1) *A ring R is axis-commutative if and only if $ARB = BRA$ for any nonempty subsets A, B of R.*

(2) *Axis-commutative rings are duo (hence, Abelian).*

(3) *If R is an axis-commutative ring, then so is eRe for each $e^2 = e \in R$.*

(4) *For a ring R without identity, if either $R^2 = 0$ or $R^3 = 0$ then R is axis-commutative.*

(5) *Let $\{R_\gamma \mid \gamma \in \Gamma\}$ be a family of rings. Then R_γ is axis-commutative for every $\gamma \in \Gamma$ if and only if the direct product $R = \prod_{\gamma\in\Gamma} R_\gamma$ of R_γ's is axis-commutative.*

(6) *The class of axis-commutative rings is closed under homomorphic images.*

Proof. (1) Assume that R is axis-commutative. Let A and B be two nonempty subsets of R. By assumption, $ARB = \sum_{a\in A, b\in B} aRb = \sum_{a\in A, b\in B} bRa = BRA$. The converse is routine.

(2) Let R be a axis-commutative ring and $a \in R$. Then $rRa = aRr$ for all $r \in R$, hence we get both $ra \in aR$ and $ar \in Ra$. This implies both $Ra \subseteq aR$ and $aR \subseteq Ra$. Thus R is duo.

(3) Suppose that R is an axis-commutative ring and $e^2 = e \in R$. Let $a, b \in eRe$. Then $aeReb = ebRea$, since R is axis-commutative. By the fact that $er = r = re$ for any $r \in eRe$, we have $ebRea = beRea$ and so $aeReb = beRea$.

(4) The proof is clear.

(5) Suppose that $R = \prod_{\gamma\in\Gamma} R_\gamma$ is axis-commutative, and let $a, b \in R_\gamma$. Consider two sequences $\alpha = (a_\gamma)_{\gamma\in\Gamma}, \beta = (b_\gamma)_{\gamma\in\Gamma} \in R$ such that $a_\gamma = a$, $b_\gamma = b$, and $a_\delta = 0$, $b_\delta = 0$ for all $\delta \neq \gamma$. Since R is axis-commutative, Then $\alpha R\beta = \beta R\alpha$, and so this implies $aR_\gamma b = bR_\gamma a$.

Conversely, suppose that every R_γ is axis-commutative and let $\alpha = (a_\gamma)_{\gamma\in\Gamma}, \beta = (b_\gamma)_{\gamma\in\Gamma} \in R$. Consider $\alpha R\beta$. Then $a_\gamma R_\gamma b_\gamma = b_\gamma R_\gamma a_\gamma$ for all $\gamma \in \Gamma$, since R_γ is axis-commutative. Thus we get $\alpha R\beta = \beta R\alpha$, concluding that R is axis-commutative.

(6) Suppose that R is an axis-commutative ring, and let $\bar{R} = R/I$ and $\bar{r} = r + I$ for $r \in R$ for an ideal I of R. For $\bar{a}, \bar{b} \in \bar{R}$, we have $\bar{a}\bar{R}\bar{b} = aRb + I = bRa + I = \bar{b}\bar{R}\bar{a}$ by assumption. Thus $\bar{R} = R/I$ is also axis-commutative. □

Observe that $Mat_n(R)$ and $T_n(R)$ over any ring R for $n \geq 2$ is not Abelian, and so they are not duo. Thus $Mat_n(R)$ and $T_n(R)$ are not axis-commutative by Lemma 2.1(2). This provides another proof of Proposition 1.4(3).

As corollaries of Lemma 2.1(2, 5), we have the following.

Corollary 2.2. (1)*A ring R is axis-commutative if and only if R is right (resp., left) duo with $RaRb = RbRa$ (resp., $aRbR = bRaR$) for $a, b \in R$.*

(2) *Let R be a ring and $e^2 = e \in Z(R)$. Then R is axis-commutative if and only if both eR and $(1-e)R$ are axis-commutative.*

Proof. (1) Let R be a right duo ring and $RaRb = RbRa$ for $a, b \in R$. Then $aRb \subseteq RaRb = RbRa \subseteq bRRa = bRa$ and $bRa \subseteq RbRa = RaRb \subseteq aRRb = aRb$, and so $aRb = bRa$. The proof for R being a left duo ring is similar.

The converse is clear by Lemma 2.1(2) and definition.

(2) It comes from the facts $R = eR \oplus (1-e)R$ for $e^2 = e \in R$ and R is Abelian, by Lemma 2.1(3, 5). □

The following example shows that the converse of Lemma 2.1(2) does not hold as well as the condition '$RaRb = RbRa$ for $a, b \in R$' in Corollary 2.2(1) cannot be dropped.

Example 2.3. We follow the construction and argument in [15, Example 2]. Let $A = \mathbb{Z}_2\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over $\mathbb{Z}_2$. Let I be the ideal of A generated by

$$x^3, y^3, yx, x^2 - xy, y^2 - xy.$$

Set $R = A/I$, and identify x, y with the images in A/I for simplicity. Then R is duo by the argument in [15, Example 2].

We will show that R is not axis-commutative. First note that every element in R is of the form

$$k_1 + k_2x + k_3y + k_4x^2 \text{ with } k_i \in \mathbb{Z}_2$$

because $x^2 = y^2 = xy$. So we get

$$\begin{aligned} x(k_1 + k_2x + k_3y + k_4x^2)y &= k_1xy + k_2x^2y + k_3xy^2 + k_4x^3y \\ &= k_1xy + k_2y^3 + k_3x^3 + k_4x^3y \\ &= xy \neq 0 \end{aligned}$$

when $k_1 = 1$ and

$$y(k_1 + k_2x + k_3y + k_4x^2)x = k_1yx + k_2yx^2 + k_3y^2x + k_4yx^3 = 0.$$

These entail $xRy \neq 0$ and $yRx = 0$, hence R is not axis-commutative.

The following example shows that the converse of Lemma 2.1(6) does not hold. That is there exists a ring R which is not axis-commutative such that for a nonzero proper ideal I of R, R/I is axis-commutative and I is axis-commutative as a ring without identity.

Example 2.4. Consider $R = T_3(F)$ over a division ring F. Then R is not axis-commutative by Proposition 1.4(3). For a nonzero proper ideal $I = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$, we have I is axis-commutative since $I^3 = 0$ by Lemma 2.1(4), and $R/I \cong F$ is also axis-commutative by Proposition 1.5.

A ring R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are Abelian and semilocal.

Proposition 2.5. *A ring R is axis-commutative and semiperfect if and only if R is a finite direct sum of local axis-commutative rings.*

Proof. Suppose that R is axis-commutative and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \dots, e_n\}$ of local idempotents whose sum is 1 by [9, Proposition 3.7.2], say $R = \sum_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. By Lemma 2.1(2), R is Abelian and so $e_i R = e_i R e_i$ for each i. But each $e_i R$ is also axis-commutative by Lemma 2.1(3).

Conversely assume that R is a finite direct sum of local axis-commutative rings. Then R is semiperfect since local rings are semiperfect by [9, Corollary 3.7.1], and moreover R is axis-commutative by Lemma 2.1(5). □

As a generalization of a reduced ring, Cohn [2] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Narbonne [14] called a ring R *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is well-known that reduced rings are reversible; and reversible rings are semicommutative; and semicommutative rings are Abelian; and one-sided duo rings are semicommutative, but not conversely in general.

Remark 2.6. Let R be an axis-commutative ring.

(1) Then $W(R) = N_*(R) = N^*(R) = N(R)$ by Lemma 2.1(2) because it is easily shown that $N(R) = N_*(R)$ and RaR is nilpotent for all $a \in N(R)$ when R is a semicommutative ring.

(2) If $J(R)$ is nil, then $W(R) = N_*(R) = N^*(R) = N(R) = J(R)$.

(3) If $J(R) = 0$, then R is reduced.

Following Goodearl [5], a ring R (possibly without identity) is called (*von Neumann*) *regular* if for every $a \in R$ there exists $b \in R$ such that $a = aba$. It is easily shown that $J(R) = 0$ if R is regular. Thus R is a axis-commutative and regular ring, then it is reduced by Remark 2.6(3).

Due to [13], a right ideal I of a ring R is called *reflexive* if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$, and a ring R is called *reflexive* if 0 is a reflexive ideal (i.e.,

$aRb = 0$ implies $bRa = 0$ for $a, b \in R$). It can be easily checked that reversible rings are reflexive.

Proposition 2.7 (1) *Every axis-commutative ring is reversible, and hence it is reflexive.*

(2) *Let R be an axis-commutative ring. Then R is semiprime if and only if it is reduced.*

Proof. (1) Let R be an axis-commutative ring. Then R is semicommutative since R is duo by Lemma 2.1(2). If $ab = 0$ for $a, b \in R$, then $aRb = 0$ and so $0 = aRb = bRa$ by assumption. This implies $ba = 0$, showing that R is reversible.

(2) Suppose that R is semiprime and let $a^2 = 0$ for $a \in R$. Since R is semicommutative as noted in (1) above, $aRa = 0$ and so $a = 0$. Thus R is reduced. The converse is evident. □

Remark 2.8. (1) Notice that there exists a domain which is not axis-commutative. Recall the domain $R = K\langle a, b\rangle$ with $aRb \neq bRa$, as noted in Example 1.2(1).

(2) Related to Proposition 2.7(1), note that (i) there exists an axis-commutative ring which is not reduced (and hence not a domain) by help of Theorem 1.1, i.e., the condition '$J(R) = 0$' in Remark 2.6(3) is not superfluous; (ii) it is obvious that the semicommutativity coincides with the reversibility whenever given a ring is axis-commutative as seen in the proof of Proposition 2.7(1).

The next example illuminates that the converse of Proposition 2.7(1) does not hold.

Example 2.9. (1) For a reflexive ring R and $n \geq 2$, $Mat_n(R)$ is also reflexive by [8, Theorem 2.6(2)], but it is not axis-commutative by Proposition 1.4(3) to follow.

(2) Consider the ring R in [7, Example 2.1]. Let

$$A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra with noncommuting indeterminates

$$a_0, a_1, a_2, b_0, b_1, b_2, c$$

over $\mathbb{Z}_2$. Next let L be the ideal of A generated by

$$\begin{aligned} &a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\\ &b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,\\ &(a_0+a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \text{ and } r_1r_2r_3r_4, \end{aligned}$$

where the constant terms of $r, r_1, r_2, r_3, r_4 \in A$ are zero. Now set $R = A/L$. Then R is a reversible ring by the argument in [7, Example 2.1]. Let a_1 and b_1 coincide with their images in R for simplicity. Then $a_1b_1 \notin b_1Ra_1$, entailing that R is not axis-commutative.

For an algebra R over a commutative ring S, the *Dorroh extension* of R by S is the Abelian group $D = R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$ ([3]).

Proposition 2.10. *Let R be an algebra with identity over a commutative ring S. Then R is axis-commutative if and only if the Dorroh extension D of R by S is.*

Proof. Note that $s \in S$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$. Suppose that R is axis-commutative. Consider $\alpha \in (r_1, s_1)D(r_2, s_2)$ for $(r_1, s_1), (r_2, s_2) \in D$. Then for any $(r, s) \in D$,

$$\begin{aligned}\alpha &= (r_1, s_1)(r, s)(r_2, s_2)\\ &= (r_1rr_2 + s_1rr_2 + sr_1r_2 + s_2r_1r + s_1sr_2 + s_1s_2r + ss_2r_1, s_1ss_2)\\ &= ((r_1 + s_1)(r + s)(r_2 + s_2), s_1ss_2).\end{aligned}$$

Thus we have $(r_1, s_1)D(r_2, s_2) = ((r_1 + s_1)R(r_2 + s_2), s_1Ss_2)$. Since R is axis-commutative and S is commutative, $((r_1+s_1)R(r_2+s_2), s_1Ss_2) = ((r_2+s_2)R(r_1+s_1), s_2Ss_1) = (r_2, s_2)D(r_1, s_1)$. Consequently, $(r_1, s_1)D(r_2, s_2) = (r_2, s_2)D(r_1, s_1)$.

Conversely, assume that D is axis-commutative. Let $c \in aRb$ for $a, b \in R$. Then $c = a(r+s)b$ for some $(r, s) \in D$. This implies $c = (a, 0)(r, s)(b, 0) \in (a, 0)D(b, 0) = (b, 0)D(a, 0)$, since D is axis-commutative. Thus $c = (b, 0)(r', s')(a, 0)$ for some $(r', s') \in D$ and it implies that $c = b(r' + s')a$ with $r' + s' \in R$. So $c \in bRa$, proving that $aRb \subseteq bRa$. Similarly, we obtain $bRa \subseteq aRb$. Therefore R is axis-commutative. $\square$

Recall that when K is a commutative ring and G is a finite group, the group ring KG is right duo if and only if KG is left duo by the argument in [11, Example 7].

Proposition 2.11. *For a field K of $ch(K) = 0$ and the group ring $R = KQ_8$, where Q_8 denotes the quaternion group. Then the following statements are equivalent:*

(1) *R is an axis-commutative ring;*
(2) *R is a reversible ring;*
(3) *R is a duo ring;*
(4) *R is a right(left) duo ring;*
(5) *R is a semicommutative ring;*
(6) *R is Abelian;*
(7) *The equation $1 + x^2 + y^2 = 0$ has no solutions in K.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3): By Proposition 2.7(1) and Lemma 2.1(2), respectively. The relations of (2) $\Rightarrow$ (6) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are well-known. Let R be Abelian. Then, by [12, Theorem 7.4.6 and Lemma 7.4.9], R is isomorphic to $K \times K \times K \times K \times D$ for some division ring D. So R is an axis-commutative ring by Lemma 2.1(5) because division rings are axis-commutative by Proposition 1.5, and so (6) $\Rightarrow$ (1) follows. The equivalence relation of (3) and (7) is shown by [1, Theorem 2.1]. □

Theorem 2.12. *For a ring R, the following statements are equivalent:*

(1) *R is commutative;*

(2) *$R[x]$ is commutative;*

(3) *$R[x]$ is axis-commutative;*

(4) *$R[x]$ is duo;*

(5) *$R[x]$ is right (or left) duo.*

Proof. (1) $\Leftrightarrow$ (2), (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5) are obvious. (3)$\Rightarrow$ (4) follows from Lemma 2.1(2), and (5) $\Rightarrow$ (2) comes from [10, Theorem 1]. □

Observe that if the polynomial ring $R[x]$ is axis-commutative then so is R by Proposition 2.12, but the axis-commutativity does not pass to polynomial rings. Indeed, the ring $R = D_2(A)$ over noncommutative division ring A is axis-commutative by Theorem 1.1 but not commutative. Thus it implies that $R[x]$ is not axis-commutative by Proposition 2.12.

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