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Convergence Theorem for Finding Common Fixed Points of N-generalized Bregman Nonspreading Mapping and Solutions of Equilibrium Problems in Banach Spaces

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Abstract. In this paper, we study some fixed point properties of n-generalized Bregman nonspreading mappings in reflexive Banach space. We introduce a hybrid iterative scheme for finding a common solution for a countable family of equilibrium problems and fixed point problems in reflexive Banach space. Further, we give some applications and numerical example to show the importance and demonstrate the performance of our algorithm. The results in this paper extend and generalize many related results in the literature.

1. Introduction

Let E be a real Banach space, and C be a nonempty, closed and convex subset of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction, the Equilibrium Problem with respect to g denoted by EP(g) is define as finding a point $z \in C$ such that

$$g(z, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The EP(g) was shown by Blum and Oettli [7] to cover several other optimization problems such as monotone inclusion problems, saddle point problems, minimization problems, variational inequality problems and Nash equilibria in non-cooperative games. In addition, there are many other important problems, for example, the complementarity problem and fixed point problems, which can be written in the

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form of EP(g) (1.1). Thus, the EP(g) is a unifying model for several problems arising in physics, engineering, science, optimization, economics etc.

In the last two decades, the existence of solutions of the EP(g) have been mentioned in many papers, see for instance [7, 11, 13, 26, 36, 39, 40], and several iterative methods have been proposed for solving EP(g) and related optimization problems, see for instance [1, 2, 4, 14, 15, 17, 18, 19, 28, 29, 32, 30, 31, 38, 41, 42] and reference therein. In solving the EP(g) (1.1) it is necessary to assume that the bifunction g satisfies the following assumptions:

(A1) $g(x,x) = 0$ for all $x \in C$;

(A2) g is monotone, that is $g(x,y) + g(y,x) \leq 0$ for all $x, y \in C$;

(A3) For all $x, y, z \in C$

$$\limsup_{t \downarrow 0^+} g(tz + (1-t)x, y) \leq g(x,y);$$

(A4) For all $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

Definition 1.1. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : dom\ f \times int(dom\ f) \to [0, +\infty)$ defined by

$$D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called a Bregman distance with respect to f.

From the definition, we know that the following properties are satisfied (see [6]):

(i) The three points identity, for any $x \in dom\ f$ and $y, z \in int(dom\ f)$

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle; \tag{1.2}$$

(ii) Four point identity, for any $x, w \in dom\ f$ and $y, z \in int(dom\ f)$

$$D_f(x,y) - D_f(x,z) - D_f(w,y) + D_f(w,z) = \langle \nabla f(z) - \nabla f(y), x - w \rangle. \tag{1.3}$$

Definition 1.2. Let C be a nonempty closed convex subset of $int(domf)$ and $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. We denote the set of all fixed points of T by $F(T)$. The mapping $T : C \to C$ is called

(a) Bregman nonexpansive [33] if

$$D_f(Tx, Ty) \leq D_f(x,y) \quad \forall x, y \in C;$$

(b) Bregman nonspreading [23] if

$$D_f(Tx,Ty) + D_f(Ty,Tx) \leq D_f(Tx,y) + D_f(Ty,x), \qquad \forall x, y \in C,$$

(c) $(\alpha, \beta, \gamma, \delta)$-generalized Bregman nonspreading [3, 16] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha D_f(Tx, Ty) + (1-\alpha)D_f(x, Ty) + \gamma\{D_f(Ty, Tx) - D_f(Ty, x)\} \\ \leq \beta D_f(Tx, y) + (1-\beta)D_f(x, y) + \delta\{D_f(y, Tx) - D_f(y, x)\}, \\ \forall x, y \in C.\end{aligned}$$

for all $x, y \in C$.

Next, we introduce a n-generalized Bregman nonspreading mapping in Banach spaces.

Definition 1.3. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function and C be a nonempty closed convex subset of $int(domf)$. A mapping $T : C \to C$ is called a n-generalized Bregman nonspreading mapping if there exist $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ such that

$$\begin{aligned}&\sum_{k=1}^{n}\alpha_k D_f(T^{n+1-k}x, Ty) + (1 - \sum_{k=1}^{n}\alpha_k)D_f(x, Ty) \\ &\quad + \sum_{k=1}^{n}\gamma_k\{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\} \\ &\quad \leq \sum_{k=1}^{n}\beta_k D_f(T^{n+1-k}x, y) + (1 - \sum_{k=1}^{n}\beta_k)D_f(x, y) \\ &\quad + \sum_{k=1}^{n}\delta_k\{D_f(y, T^{n+1-k}x) - D_f(y, x)\},\end{aligned} \tag{1.4}$$

for all $x, y \in C$.

Remark 1.4. From Definition 1.3,

(a) when $n = 2$, (1.4) becomes

$$\begin{aligned}&\alpha_1 D_f(T^2x, Ty) + \alpha_2 D_f(Tx, Ty) + (1 - \alpha_1 - \alpha_2)D_f(x, Ty) \\ &+ \gamma_1(D_f(Ty, T^2x) - D_f(Ty, x)) + \gamma_2(D_f(Ty, Tx) - D_f(Ty, x)) \\ &\leq \beta_1 D_f(T^2x, y) + \beta_2 D_f(Tx, y) + (1 - \beta_1 - \beta_2)D_f(x, y) \\ &+ \delta_1(D_f(y, T^2x) - D_f(y, x)) + \delta_2(D_f(y, Tx) - D_f(y, x)),\end{aligned}$$

which is called 2-generalized Bregman nonspreading in the sense of [44], where $f(x) = \frac{1}{2}||x||^2$.

(b) When $n = 1$, then (1.4) becomes

$$\begin{aligned}\alpha_1 D_f(Tx, Ty) + (1 - \alpha_1)D_f(x, Ty) + \gamma_1(D_f(Ty, Tx) - D_f(Ty, x)) \\ \leq \beta_1 D_f(Tx, y) + (1 - \beta_1)D_f(x, y) + \delta_1(D_f(y, Tx) - D_f(y, x)),\end{aligned}$$

which is the generalized Bregman nonspreading mapping in the sense of [3, 16]. Note that, the 2-generalized Bregman nonspreading mapping reduces to the generalized Bregman nonspreading mapping if $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$.

(c) The class of generalized Bregman nonspreading mapping reduces to Bregman nonspreading [23] if $\alpha_1 = \beta_1 = \gamma_1 = 1$ and $\delta_1 = 0$.

(d) The class of generalized Bregman nonspreading mapping reduces to Bregman nonexpansive [35] if $\alpha_1 = 1$ and $\beta_1 = \gamma_1 = \delta_1 = 0$.

We now present an example of Bregman nonspreading mapping which is not nonspreading in the usual Hilbert space setting.

Example 1.5. Let $E = \mathbb{R}$ with the usual metric. Let $f : E \to \mathbb{R}$ be defined by $f(x) = x^{10}$ for all $x \in \mathbb{R}$ and $T : [0, 0.85] \to [0, 0.85]$ be defined by $Tx = x^2$. We first show that T is not nonspreading, i.e.,

$$||Tx - Ty||^2 \leq ||x - y||^2 + 2\langle x - Tx, y - Ty\rangle \quad \forall x, y \in C,$$

does not hold. Taking $x = 0.5$ and $y = 0.85$, then

$$||Tx - Ty||^2 = (x^2 - y^2)^2 = [(0.5)^2 - (0.85)^2]^2 = 0.22325625,$$

while

$$\begin{aligned} ||x - y||^2 + 2\langle x - Tx, y - Ty\rangle &= (x - y)^2 + 2(x - x^2)(y - y^2) \\ &= (0.5 - 0.85)^2 + 2(0.5 - 0.5^2)(0.85 - 0.85^2) \\ &= 0.18625. \end{aligned}$$

Hence, T is not nonspreading. Put

$$h(x, y) = D_f(Tx, Ty) + D_f(Ty, Tx) - D_f(Tx, y) - D_f(Ty, x).$$

By simple calculations, we obtain

$$\begin{aligned} D_f(Tx, Ty) &= x^{20} + 9y^{20} - 10x^2y^{18}, \\ D_f(Ty, Tx) &= y^{20} + 9x^{20} - 10x^{18}y^2, \\ D_f(Tx, y) &= x^{20} + 9y^{10} - 10y^2x^9, \\ D_f(Ty, x) &= y^{20} + 9x^{10} - 10x^9y^2. \end{aligned}$$

Then

$$\begin{aligned} h(x, y) &= 9y^{10}(y^{10} - 1) + 9x^{10}(x^{10} - 1) - 10x^2y^9(y^9 - 1) - 10x^9y^2(x^9 - 1) \\ &\leq 0, \end{aligned}$$

for all $x, y \in [0, 0.85]$. Thus T is Bregman nonspreading.

We further give an example of 2-generalized Bregman nonspreading mapping which is not necessarily 1-generalized Bregman nonspreading.

Example 1.6. Let $E = \mathbb{R}$ and $f(x) = \frac{x^2}{2}$ then the associated Bregman distance is given by

$$\begin{aligned} D_f(x,y) &= f(x) - f(y) - \langle x - y, \nabla f(y)\rangle \\ &= \frac{1}{2}x^2 - \frac{1}{2}y^2 - (x-y)(y) \\ &= \frac{1}{2}(x-y)^2, \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

Define $T : [0,2] \to [0,2]$ by

$$Tx = \begin{cases} 0, & \text{if} \quad x \in [0,2), \\ 1, & \text{if} \quad x = 2. \end{cases} \tag{1.5}$$

It is easy to see that $F(T) = \{0\}$. Let

$$\begin{aligned} h(x,y) =& \alpha_1 D_f(T^2x, Ty) + \alpha_2 D_f(Tx, Ty) + (1 - \alpha_1 - \alpha_2)D_f(x, Ty) \\ &+ \gamma_1(D_f(Ty, T^2x) - D_f(Ty, x)) + \gamma_2(D_f(Ty, Tx) - D_f(Ty, x)) \\ &- \beta_1 D_f(T^2x, y) - \beta_2 D_f(Tx, y) - (1 - \beta_1 - \beta_2)D_f(x, y) \\ &- \delta_1(D_f(y, T^2x) - D_f(y, x)) - \delta_2(D_f(y, Tx) - D_f(y, x)), \end{aligned}$$

for all $x, y \in [0,2]$. We consider the following possible cases.

Case I: Suppose $x = y = 2$, then $Tx = Ty = 1$ and $T^2x = 0$. Thus

$$\begin{aligned} &D_f(Tx, Ty) = D_f(Ty, Tx) = D_f(x, y) = D_f(y, x) = 0, \\ &D_f(x, Ty) = D_f(Ty, x) = D_f(Tx, y) = D_f(y, Tx) = \frac{1}{2}, \\ &D_f(T^2x, Ty) = D_f(Ty, T^2x) = \frac{1}{2}, D_f(T^2x, y) = D_f(y, T^2x) = 2. \end{aligned}$$

Hence

$$h(x,y) = \frac{1}{2} - \frac{1}{2}(\alpha_2 + \gamma_2 + \beta_2 + \delta_2) - 2(\beta_1 + \delta_1).$$

Case II: Suppose $x = 2$ and $y \in [0,2)$, then $Tx = 1$ and $Ty = T^2x = 0$. Thus

$$\begin{aligned} &D_f(Tx, Ty) = D_f(Ty, Tx) = \frac{1}{2}, D_f(x, Ty) = D_f(Ty, x) = 2, \\ &D_f(Tx, y) = D_f(y, Tx) = \frac{1}{2}(y-1)^2, D_f(x, y) = D_f(y, x) = \frac{1}{2}(y-2)^2, \\ &D_f(T^2x, y) = D_f(y, T^2x) = \frac{y^2}{2}, D_f(T^2x, Ty) = D_f(Ty, T^2x) = 0. \end{aligned}$$

Hence

$$h(x,y) = -\tfrac{1}{2}(y^2-4y) - 2(\alpha_1+\gamma_1) - \tfrac{3}{2}(\alpha_2+\gamma_2) -2(y-2)(\beta_2+\delta_1) - \tfrac{1}{2}(2y-3)(\beta_2+\delta_2).$$

Case III: Suppose $x,y \in [0,2)$ then $Tx = Ty = T^2x = 0$. Thus

$$D_f(Tx,Ty) = D_f(Ty,Tx) = D_f(T^2x,Ty) = D_f(Ty,T^2x) = 0,$$
$$D_f(x,y) = D_f(y,x) = \frac{1}{2}(x-y)^2, D_f(x,Ty) = D_f(Ty,x) = \frac{x^2}{2},$$
$$D_f(Tx,y) = D_f(y,Tx) = D_f(T^2x,y) = D_f(y,T^2x) = \frac{y^2}{2}.$$

Hence

$$h(x,y) = (1-\alpha_1-\alpha_2)\frac{x^2}{2} - \frac{x^2}{2}(\gamma_1+\gamma_2) - \frac{y^2}{2}(\beta_1+\beta_2) - \frac{1}{2}(1-\beta_1-\beta_2)(x-y)^2 - \delta_1\left(xy-\frac{x^2}{2}\right) - \delta_2\left(xy-\frac{y^2}{2}\right).$$

Choosing suitable choices of $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma,\delta_1,\delta_2 \in \mathbb{R}$, for instance, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$ and $\delta_1 = \delta_2 = -1$, we see that $h(x,y) \leq 0$ for all the cases. Hence, T is 2-generalized Bregman nonspreading. However, in this case, T is not 1-generalized Bregman nonspreading (since $\alpha_1 \neq 0, \beta_1 \neq 0, \gamma_1 \neq 0, \delta_1 \neq 0$).

In 2010, by making use of the Bregman projection, Reich and Sabach [33] studied some approximation methods for finding common zeros of maximal monotone operators in reflexive Banach spaces. They also studied some approximation techniques for finding common solutions of finitely many Bregman nonexpansive operators, see [35]. In the same sense, Kassay et al. [20] studied the approximation of solutions of system of variational inequalities in reflexive Banach spaces. It is worth noting that extension of many theory from Hilbert space to general Banach space suffer some difficulties because many of the useful techniques employed in Hilbert space (for instance the inner product and the nonexpansiveness of resolvent operators) are no longer valid in Banach spaces setting.

Motivated by the works given in [21, 35, 46], we prove some properties of the n-generalized Bregman nonspreading mappings in reflexive Banach space. Further, we introduce a hybrid method for finding a common solution of countable family of equilibrium problem and finite family of fixed points of n-generalized Bregman nonspreading mapping in reflexive Banach space. We also discuss some applications and numerical example to demonstrate the applicability of our iterative algorithm and result. The method and results present in this paper generalized and unify many previously known related results, see for instance [21, 22, 35, 45, 46].

2. Preliminaries

In this section, we recall some definitions and preliminary results which will be used in the sequel. We denote the strong convergence (resp. weak convergence) of a sequence $\{x_n\} \subset E$ to a point $x \in E$ by $x_n \to x$ (resp. $x_n \rightharpoonup x$).

Let E be a real reflexive Banach space with the dual space E^* and C a nonempty closed convex subset of E. Throughout this paper, we shall assume that the mapping $f : E \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous and also denote the domain of f by dom$f$, where dom$f = \{x \in E : f(x) < \infty\}$. Let $x \in \text{int}(\text{dom}f)$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y),\ \forall y \in E\}$$

and the Frénchet conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(y^*) = \sup\{\langle y^*, x\rangle - f(x) : x \in E\}.$$

Let $x \in \text{int}(\text{dom}f)$, for any $y \in E$, the directional derivative of f at x is defined by

$$f^o(x, y) := \lim_{h\to 0} \frac{f(x + hy) - f(x)}{h}. \tag{2.1}$$

If the limit in (2.1) exists as $h \to 0$ for each y, then the function f is said to be Gâteaux differentiable at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y\rangle := f^o(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \text{int}(\text{dom}f)$. When the limit as $h \to 0$ in (2.1) is attained uniformly for any $y \in E$ with $||y|| = 1$, we say that f is Fréchet differentiable at x. It is well known that f is Gâteaux (resp. Fréchet) differentiable at $x \in \text{int}(\text{dom}f)$ if and only if the gradient ∇f is norm-to-weak* (resp. norm-to-norm) continuous at x (see [6]).
Let E be a reflexive Banach space. The function f is called Legendre if and only if it satisfies the following two conditions:

(L1) f is Gâteaux differentiable, $\text{int}(\text{dom } f) \ne \emptyset$ and $\text{dom } \nabla f = \text{int}(\text{dom } f)$,

(L2) f^* is Gâteaux differentiable, $\text{int}(\text{dom } f^*) \ne \emptyset$ and $\text{dom } \nabla f^* = \text{int}(\text{dom } f^*)$.

Since E is reflexive, we know that $(\nabla f)^{-1} = \nabla f^*$, this together with conditions (L1) and (L2) implies that

$$ran\nabla f = dom\nabla f^* = int(domf^*),$$

and

$$ran\nabla f^* = dom\nabla f = int(domf).$$

The notion of Legendre function in infinite dimensional spaces was first introduced by Bauschke, Borwein and Combettes in [6]. By their definition, the conditions (L1) and (L2) also yield that f and f^* are Gâteaux differentiable and strictly convex in

the interior of their respective domains. It follows that f is Legendre if and only if f^* is Legendre (see [6], Corollary 5.5, p. 634).
One important and interesting example of Legendre function is $\frac{1}{p}||\cdot||^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f coincide with the generalized duality mapping of E. More examples of Legendre functions can be found in [5, 6]. In the rest of this paper, we always assume that $f : E \to \mathbb{R} \cup \{+\infty\}$ is a Legendre function.

Definition 2.1. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in int(domf)$ onto the nonempty, closed and convex subset $C \subset domf$ is the necessarily unique vector $\text{Proj}_C^f(x) \in C$ satisfying

$$D_f(Proj_C^f(x), x) = \inf\Big\{D_f(y, x) : y \in C\Big\}.$$

Remark 2.2.

1. If E is a Hilbert space and $f(x) = \frac{1}{2}||x||^2$, then the Bregman projection $Proj_C^f(x)$ is reduced to the metric projection of x onto C.
2. If E is smooth and strictly convex and $f(x) = \frac{1}{p}||x||^p$ $(1 < p < \infty)$, then the Bregman projection $Proj_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$, which is defined by

$$D_p(\Pi_C(x), x) := \inf\{D_p(z, x) : z \in C\}.$$

It is known from [10] that $z = \text{Proj}_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0 \qquad for\ all\ y \in C.$$

We also have

$$D_f(y, Proj_C^f(x)) + D_f(Proj_C^f(x), x) \le D_f(y, x) \qquad for\ all\ x \in E,\ y \in C.$$

Similar to the metric projection in Hilbert space, the Bregman projection also has a variational characterization which is given below.

Lemma 2.3. ([33] (Characterization of Bregman Projection)). *Let f be totally convex on int(domf). Let C be a nonempty, closed and convex subset of int(domf) and $x \in$int(domf), if $\omega \in C$, then the following conditions are equivalent:*

(i) *the vector ω is the Bregman projection of x onto C, with respect to f,*

(ii) *the vector ω is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \ge 0 \qquad \forall y \in C, \tag{2.2}$$

(iii) *the vector* ω *is the unique solution of the inequality*

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x) \qquad \forall y \in C.$$

Definition 2.4. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function f is called:

(i) totally convex at x if its modulus of totally convexity at $x \in \text{int}(\text{dom} f)$, that is, the bifunction $v_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in dom f, ||y - x|| = t\} \tag{2.3}$$

is positive for any $t > 0$,

(ii) totally convex if it is totally convex at every point $x \in \text{int}(\text{dom } f)$,

(iii) totally convex on bounded subset B of E, if $v_f(B,t)$ is positive for any nonempty bounded subset B, where the function $v_f : \text{int}(\text{dom } f) \times [0, +\infty) \to [0, +\infty]$ is defined by

$$v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap int(dom f)\}, \quad t > 0. \tag{2.4}$$

(iv) cofinite if $\text{dom} f^* = E^*$,

(v) coercive if $\lim_{||x|| \to +\infty} \left(\frac{f(x)}{||x||}\right) = +\infty$,

(vi) sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{2.5}$$

For further details and examples on totally convex functions see [8, 9, 10].

Lemma 2.5. ([9]) *The function* $f : E \to \mathbb{R}$ *is totally convex on bounded subsets if and only if it is sequentially consistent.*

Lemma 2.6. ([34]) *Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in E$ *and the sequence* $\{D_f(x_0, x_n)\}$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

Lemma 2.7. ([10]) *Let* $f : E \to (-\infty, +\infty]$ *be a convex function whose domain contains at-least two points. Then the following statements holds:*

(i) f *is sequentially consistent if and only if it is totally convex on bounded subsets.*

(ii) *If* f *is lower semicontinuous, then* f *is sequential consistent if and only if it is uniformly convex on bounded subsets.*

(iii) *If f is uniformly strictly convex on bounded subsets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain, and the Fréchet derivative ∇f is uniformly continuous on bounded subsets.*

Lemma 2.8. ([33]) *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Let $f : E \to \mathbb{R}$ be a convex Legendre and Gâteaux differentiable function. The function $V_f : E \times E^* \to [0, \infty)$ associated with f defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \qquad \forall x \in E, x^* \in E^*.$$

Then, V_f is non-negative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. More so, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$ (see [24]). In addition, if $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function. Hence, V_f is convex in the second variable. Thus, for all $z \in E$

$$D_f\Big(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))\Big) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.6}$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^{N} t_i = 1$.
Let E be a Banach space and let $B_r := \{z \in E : ||z|| \leq r\}$ for all $r > 0$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E if $\rho_r(t) > 0$ for all $t \geq 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x,y \in B_r, ||x-y||=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)}. \tag{2.7}$$

The function ρ_r is called the gauge of uniform convexity of f. More so, the function $f : E \to (-\infty, +\infty]$ is called totally coercive if

$$\lim_{||x|| \to +\infty} \Big(\frac{f(x)}{||x||}\Big) = +\infty.$$

Lemma 2.9. ([27]) *Let $r > 0$ be a constant and let $f : E \to \mathbb{R}$ be a continuous uniformly convex function on bounded subsets of E. Then*

$$f\Big(\sum_{k=0}^{\infty} \alpha_k x_k\Big) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r^*(||x_i - x_j||), \tag{2.8}$$

for all $i, j \in \mathbb{N} \cup 0$, $x_k \in B_r$, $\alpha_k \in (0, 1)$ and $k \in \mathbb{N} \cup 0$ with $\sum_{k=0}^{\infty} \alpha_k = 1$, where ρ_r^* is the gauge of uniform convexity of f.

Let l^∞ be the Banach lattice of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional μ on l^∞ such that the following three conditions hold:

(i) if $\{t_n\}$ in l^∞ and $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) \geq 0$,

(ii) if $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) = 1$,

(iii) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\}$ in l^∞.

Here, $\{t_{n+1}\}$ denotes the sequence $(t_2, t_3, \dots, t_n, t_{n+1}, \dots)$ in l^∞. Such a functional μ is called a Banach limit and the value of μ at $\{t_n\}$ in l^∞ is denoted by $\mu_n t_n$. Therefore, condition (3) means $\mu_n t_n = \mu_n t_{n+1}$. If μ satisfies conditions (1) and (2), we call μ a mean on l^∞ (see, for example, [43] for more details).

Lemma 2.10. ([12]) *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $f : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable, locally bounded and local uniformly convex on E. Let $T : C \to C$ be a mapping and $\{x_n\}$ be a bounded sequence of C and μ be a mean on l^∞. Supposet that*

$$\mu_n D_f(x_n, Ty) \leq \mu_n D_f(x_n, y) \quad \forall y \in C.$$

Then, T has a fixed point in C.

Let T be a mapping from C into itself. A point $x \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Recall that a mapping $T : C \to C$ is said to be Bregman quasi-nonexpansive [27] if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x) \quad \forall x \in C, p \in F(T).$$

A mapping $T : C \to C$ is to be Bregman relatively nonexpansive [27] if the following conditions are satisfied:

(i) $F(T)$ is nonempty;

(ii) $D_f(p, Tv) \leq D_f(p, v)$, $\forall p \in F(T)$, $v \in C$;

(iii) $\hat{F}(T) = F(T)$.

Lemma 2.11. ([37]) *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and let $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). For all $\lambda > 0$ be any given number and $x \in E$, there exists $z \in C$ such that*

$$g(z, y) + \frac{1}{r}\langle \nabla(z) - \nabla(x), y - z\rangle \geq 0, \ \forall\ y \in C. \tag{2.9}$$

Define the resolvent mapping $T_r : E \to 2^C$ as follows

$$Res^f_{\lambda,g}(x) = \{z \in C : g(z,y) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0,\ \forall\ y \in C\}, \tag{2.10}$$

then, $Res^f_{\lambda,g}$ has the following properties:

(i) *$Res^f_{\lambda,g}$ is single-valued;*

(ii) *$Res^f_{\lambda,g}$ is a firmly nonexpansive mapping, that is;*

$$\begin{aligned}&\langle Res^f_{\lambda,g}z - Res^f_{\lambda,g}y, \nabla f(Res^f_{\lambda,g}z) - \nabla f(Res^f_{\lambda,g}y)\rangle\\ &\leq \langle Res^f_{\lambda,g}z - Res^f_{\lambda,g}y, \nabla f(z) - \nabla f(y)\rangle\end{aligned}$$

$\forall z, y \in E$;

(iii) *$F(Res^f_{\lambda,g}) = EP(g)$;*

(iv) *$EP(g)$ is closed and convex.*

It is easy to see that the resolvent operator satisfies the following inequality: for all $r > 0$, $u \in EP(g)$ and $x \in E$, then

$$D_f(x, Res^f_{\lambda,g}x) + D_f(Res^f_{\lambda,g}x, u) \leq D_f(x, u). \tag{2.11}$$

3. Main Results

In this section, we present the existence and some properties of fixed points of n-generalized Bregman nonspreading mapping in a reflexive Banach space. This result extend the corresponding results of [45] and [25] to reflexive Banach space.

Proposition 3.1. *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $C \subset int(domf)$ be a nonempty, closed and convex set and $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, the following are equivalent*

(i) *$F(T)$ is nonempty;*

(ii) *$\{T^m z\}$ is bounded for some $z \in C$ and $m \in \mathbb{N}$.*

Proof. First we show that (i) implies (ii). Suppose $F(T) \neq \emptyset$, then $\{T^m z\} = \{z\}$ for $z \in F(T)$. So $\{T^m z\}$ is bounded. Next, we show that (ii) implies (i). Let $\{T^m z\}$ be bounded for some $z \in C$. Since T is n-Bregman generalized nonspreading, then there exist $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$, such that

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k D_f(T^{n+1-k}x, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(x, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\} \\
&\leq \sum_{k=1}^{n} \beta_k D_f(T^{n+1-k}x, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(x, y) \\
&+ \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}x) - D_f(y, x)\}, \qquad (3.1)
\end{aligned}$$

for all $x, y \in C$. Replacing x by $T^{m-1}z$ in (3.1), we have that for any $y, z \in C$,

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k D_f(T^{n+1-k}T^{m-1}z, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(T^{m-1}z, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}T^{m-1}z) - D_f(Ty, T^{m-1}z)\} \\
&\leq \sum_{k=1}^{n} \beta_k D_f(T^{n+1-k}T^{m-1}z, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(T^{m-1}z, y) \\
&+ \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}T^{m-1}z) - D_f(y, T^{m-1}z)\}. \qquad (3.2)
\end{aligned}$$

Since $\{T^m z\}$ is bounded, we can apply Banach limit μ to both sides of (3.2), then we have

$$\begin{aligned}
&\mu_m \Big(\sum_{k=1}^{n} \alpha_k D_f(T^{m+n-k}z, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(T^{m-1}z, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{m+n-k}z) - D_f(Ty, T^{m-1}z)\} \Big) \\
&\leq \mu_m \Big(\sum_{k=1}^{n} \beta_k D_f(T^{m+n-k}z, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(T^{m-1}z, y) \\
&+ \sum_{k=1}^{n} \delta_k \{D_f(y, T^{m+n-k}z) - D_f(y, T^{m-1}z)\} \Big).
\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{n}\alpha_k\mu_m D_f(T^{m+n-k}z, Ty) + (1-\sum_{k=1}^{n}\alpha_k)\mu_m D_f(T^{m-1}z, Ty)\\
&+\sum_{k=1}^{n}\gamma_k\{\mu_m D_f(Ty, T^{m+n-k}z) - \mu_m D_f(Ty, T^{m-1}z)\}\\
&\leq \sum_{k=1}^{n}\beta_k\mu_m D_f(T^{m+n-k}z, y) + (1-\sum_{k=1}^{n}\beta_k)\mu_m D_f(T^{m-1}z, y)\\
&+\sum_{k=1}^{n}\delta_k\{\mu_m D_f(y, T^{m+n-k}z) - \mu_m D_f(y, T^{m-1}z)\}.
\end{aligned} \tag{3.3}
$$

Then

$$
\begin{aligned}
&\sum_{k=1}^{n}\alpha_k\mu_m D_f(T^{m}z, Ty) + (1-\sum_{k=1}^{n}\alpha_k)\mu_m D_f(T^{m}z, Ty)\\
&+\sum_{k=1}^{n}\gamma_k\{\mu_m D_f(Ty, T^{m}z) - \mu_m D_f(Ty, T^{m}z)\}\\
&\leq \sum_{k=1}^{n}\beta_k\mu_m D_f(T^{m}z, y) + (1-\sum_{k=1}^{n}\beta_k)\mu_m D_f(T^{m}z, y)\\
&+\sum_{k=1}^{n}\delta_k\{\mu_m D_f(y, T^{m}z) - \mu_m D_f(y, T^{m}z)\}.
\end{aligned}
$$

Hence

$$\mu_m D_f(T^m z, Ty) \leq \mu_m D_f(T^m z, y).$$

Therefore by Lemma 2.10, T has a fixed point in C. This completes the proof. □

The following results follow as direct consequences of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex Banach space E, let p be a real number such that $1 < p < +\infty$ and let f be a function defined by $f(x) = \frac{1}{p}||x||^p$ and $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, the following assertions are equivalent:*

- (i) *$F(T)$ is nonempty;*
- (ii) *$\{T^m z\}$ is bounded for some $z \in C$.*

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, T has a fixed point.*

Remark 3.4. Corollary 3.2 is a generalization of the corresponding result in Theorem 3.2 of [45], where the equivalence between the two assertions was shown for $p = 2$.

We now show another important property of the fixed points of n-generalized Bregman nonspreading mapping.

Proposition 3.5. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to C$ be a n-generalized Bregman nonspreading mapping such that $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Proof. Let $u \in F(T)$, then putting $u = x \in F(T)$ in (1.4), we have

$$\sum_{k=1}^{n} \alpha_k D_f(u, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(u, Ty) + \sum_{k=1}^{n} \gamma_k \{D_f(Ty, u) - D_f(Ty, u)\}$$
$$\leq \sum_{k=1}^{n} \beta_k D_f(u, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(u, y) + \sum_{k=1}^{n} \delta_k \{D_f(y, u) - D_f(y, u)\},$$

which implies that

$$D_f(u, Ty) \leq D_f(u, y), \qquad \forall u \in F(T), y \in C. \tag{3.4}$$

This means that T is quasi-Bregman nonexpansive. Now let $\{x_n\} \subset F(T)$ such that $x_n \to p$. Then

$$D_f(p, Tp) = \lim_{n\to\infty} D_f(x_n, Tp) \leq D_f(x_n, p) = D_f(p, p) = 0.$$

Hence, $p \in F(T)$. Therefore $F(T)$ is closed.
Next, we show that $F(T)$ is convex. For any $x, y \in F(T)$ and $\lambda \in (0, 1)$, let $z = \lambda x + (1 - \lambda) y$. Then

$$\begin{aligned}
D_f(z, Tz) &= f(z) - f(Tz) - \langle \nabla f(Tz), z - Tz \rangle \\
&= f(z) - f(Tz) - \langle \nabla f(Tz), \lambda x + (1 - \lambda) y - Tz \rangle \\
&= f(z) + \lambda D_f(x, Tz) + (1 - \lambda) D_f(y, Tz) - \lambda f(x) - (1 - \lambda) f(y) \\
&\leq f(z) + \lambda D_f(x, z) + (1 - \lambda) D_f(y, z) - \lambda f(x) - (1 - \lambda) f(y) \\
&= f(z) - f(z) - \langle \nabla f(z), \lambda x + (1 - \lambda) y - z \rangle \\
&= f(z) - f(z) - \langle \nabla f(z), z - z \rangle \\
&= 0. \qquad (3.5)
\end{aligned}$$

Hence, $z = Tz$. Therefore, $F(T)$ is convex. □

Using Corollary 3.3 and Proposition 3.5, we prove the following common fixed point theorem for a commutative family of n-generalized Bregman nonspreading mapping in a reflexive Banach space.

Theorem 3.6. *Let $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function, C be a nonempty bounded closed convex subset of a real reflexive Banach space E and let $\{T_\alpha\}_{\alpha\in I}$ be a commutative family of n-generalized Bregman nonspreading mappings from C into itself. Then $\{T_\alpha\}_{\alpha\in I}$ has a common fixed point.*

Proof. By Theorem 3.5, we know that $F(T_\alpha)$ is a closed convex subset of C. Since E is reflexive and C is a bounded closed and convex subset, C is weakly compact. To show that $\cap_{\alpha\in I}F(T_\alpha)$ is nonempty, it is sufficient to show that $\{F(T_\alpha)\}_{\alpha\in I}$ has a nonempty finite intersection property.

Now, let $\{T_1, T_2, \dots, T_N\}$ be a commutative finite family of n-generalized Bregman nonspreading mapping from C into itself. We prove by induction that $\{T_1, T_2, \dots, T_N\}$ has a common fixed point. To do this, we start by showing the case for $N = 2$. By Corollary 3.3 and Theorem 3.5, $F(T_1)$ is nonempty, bounded, closed and convex. Let $u \in F(T_1)$, since $T_1T_2 = T_2T_1$, then we have $T_1T_2u = T_2T_1u = T_2u$. This implies that $T_2u \in F(T_1)$. Hence, $F(T_1)$ is T_2-invariant. Thus, the restriction of T_2 to $F(T_1)$ is a n-generalized Bregman nonspreading self mapping. By Corollary 3.3, T_2 has a fixed point in $F(T_1)$, that is, we have $z \in F(T_1)$ such that $T_2z = z$. Hence, $z \in F(T_1) \cap F(T_2)$.

Suppose that for some $N \geq 2$, $\Gamma = \cap_{k=1}^{N} F(T_k)$ is nonempty. Then Γ is a nonempty, bounded, closed and convex subset of C and the restriction of T_{N+1} to Γ is a n-generalized Bregman nonspreading self mapping. By Corollary 3.3, T_{N+1} has a fixed point in Γ. This implies that $\Gamma \cap F(T_{N+1})$ is nonempty. Hence, $\cap_{k=1}^{N+1} F(T_k)$ is nonempty. This completes the proof. □

The following result will be used in the sequel.

Proposition 3.7. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, for any $x, y \in C$, $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$, for $i = 1, 2, \dots, n$, we have*

$$
\begin{aligned}
0 \quad \leq \quad & \sum_{k=1}^{n}(\beta_k - \alpha_k)\Big(D_f(T^{n+1-k}x, Ty) - D_f(x, Ty)\Big) + D_f(Ty, y) \\
& + \langle \nabla f(Ty) - \nabla f(y), \sum_{k=1}^{n} \beta_k (T^{n+1-k}x - x) + x - Ty \rangle \\
& + \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}x) - D_f(y, x)\} - \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\}. \qquad (3.6)
\end{aligned}
$$

Proof. From the definition of n-generalized Bregman nonspreading mapping, we have

$$\begin{aligned}&\sum_{k=1}^{n}\alpha_k D_f(T^{n+1-k}x,Ty)+(1-\sum_{k=1}^{n}\alpha_k)D_f(x,Ty)\\&\quad+\sum_{k=1}^{n}\gamma_k\{D_f(Ty,T^{n+1-k}x)-D_f(Ty,x)\}\\&\quad\leq\sum_{k=1}^{n}\beta_k D_f(T^{n+1-k}x,y)+(1-\sum_{k=1}^{n}\beta_k)D_f(x,y)\\&\quad+\sum_{k=1}^{n}\delta_k\{D_f(y,T^{n+1-k}x)-D_f(y,x)\},\end{aligned}\tag{3.7}$$

for all $x,y\in C$. This implies that

$$\begin{aligned}0\quad\leq\quad&\sum_{k=1}^{n}\beta_k D_f(T^{n+1-k}x,y)+(1-\sum_{k=1}^{n}\beta_k)D_f(x,y)\\&+\sum_{k=1}^{n}\delta_k\{D_f(y,T^{n+1-k}x)-D_f(y,x)\}\\&-\sum_{k=1}^{n}\alpha_k D_f(T^{n+1-k}x,Ty)-(1-\sum_{k=1}^{n}\alpha_k)D_f(x,Ty)\\&-\sum_{k=1}^{n}\gamma_k\{D_f(Ty,T^{n+1-k}x)-D_f(Ty,x)\}.\end{aligned}$$

Hence, from the three points identity (1.2), we have

$$\begin{aligned}0\quad\leq\quad&\sum_{k=1}^{n}\beta_k\Big(D_f(T^{n+1-k}x,Ty)+D_f(Ty,y)+\langle\nabla f(Ty)-\nabla f(y),T^{n+1-k}x-Ty\rangle\Big)\\&+(1-\sum_{k=1}^{n}\beta_k)\Big(D_f(x,Ty)+D_f(Ty,y)+\langle\nabla f(Ty)-\nabla f(y),x-Ty\rangle\Big)\\&-\sum_{k=1}^{n}\alpha_k D_f(T^{n+1-k}x,Ty)-(1-\sum_{k=1}^{n}\alpha_k)D_f(x,Ty)\\&-\sum_{k=1}^{n}\gamma_k\{D_f(Ty,T^{n+1-k}x)-D_f(Ty,x)\}\\&+\sum_{k=1}^{n}\delta_k\{D_f(y,T^{n+1-k}x)-D_f(y,x)\}.\end{aligned}$$

Therefore

$$
\begin{aligned}
0 \quad \leq \quad & \sum_{k=1}^{n}(\beta_k-\alpha_k)\Big(D_f(T^{n+1-k}x,Ty)-D_f(x,Ty)\Big)+D_f(Ty,y)\\
&+\langle\nabla f(Ty)-\nabla f(y),\sum_{k=1}^{n}\beta_k(T^{n+1-k}x-x)+x-Ty\rangle\\
&+\sum_{k=1}^{n}\delta_k\{D_f(y,T^{n+1-k}x)-D_f(y,x)\}\\
&-\sum_{k=1}^{n}\gamma_k\{D_f(Ty,T^{n+1-k}x)-D_f(Ty,x)\}.
\end{aligned}
$$

□

The following result is another important property which characterized the n-generalized Bregman nonspreading mapping.

Proposition 3.8. *Let $T:C\to C$ be a n-generalized Bregman nonspreading mapping. Suppose $F(T)\neq\emptyset$, then T is Bregman relatively nonexpansive.*

Proof. It is clear that

$$D_f(p,Tx)\leq D_f(p,x)\quad \forall p\in F(T), x\in C.$$

We show that $\hat{F}(T)=F(T)$. It is easy to see that $F(T)\subset\hat{F}(T)$. Now let $p\in\hat{F}(T)$, that is, there exist a sequence $\{x_n\}\subset C$ such that $x_n\rightharpoonup p$ and $||x_n-Tx_n||\to 0$. Since f is uniformly Fréchet differentiable on bounded subsets of E, then ∇f is uniformly continuous and thus

$$\lim_{n\to\infty}||f(x_n)-f(Tx_n)||=\lim_{n\to\infty}||\nabla f(x_n)-\nabla f(Tx_n)||=0. \tag{3.8}$$

Putting $x=x_n$ and $y=q$ in Proposition 3.7, we have

$$
\begin{aligned}
0 \quad \leq \quad & \sum_{k=1}^{n}(\beta_k-\alpha_k)\Big(D_f(T^{n+1-k}x_n,Tq)-D_f(x_n,Tq)\Big)+D_f(Tq,q)\\
&+\langle\nabla f(Tq)-\nabla f(q),\sum_{k=1}^{n}\beta_k(T^{n+1-k}x_n-x_n)+x_n-Tq\rangle\\
&+\sum_{k=1}^{n}\delta_k\{D_f(q,T^{n+1-k}x_n)-D_f(q,x_n)\}\\
&-\sum_{k=1}^{n}\gamma_k\{D_f(Tq,T^{n+1-k}x_n)-D_f(Tq,x_n)\}. \qquad (3.9)
\end{aligned}
$$

Observe that

$$\begin{aligned} D_f(T^{n+1-k}x_n, Tq) - D_f(x_n, Tq) &= f(T^{n+1-k}x_n) - f(Tq) \\ &\quad -\langle \nabla f(Tq), T^{n+1-k}x_n - Tq\rangle \\ &\quad -f(x_n) + f(Tq) + \langle \nabla f(Tq), x_n - Tq\rangle \\ &= f(T^{n+1-k}x_n) - f(x_n) \\ &\quad +\langle \nabla f(Tq), x_n - Tq\rangle \\ &\quad -\langle \nabla f(Tq), T^{n+1-k}x_n - Tq\rangle \\ &= f(T^{n+1-k}x_n) - f(x_n) \\ &\quad +\langle \nabla f(Tq), x_n - T^{n+1-k}x_n\rangle. \end{aligned} \tag{3.10}$$

Similarly

$$\begin{aligned} D_f(q, T^{n+1-k}x_n) - D_f(q, x_n) &= f(x_n) - f(T^{n+1-k}x_n) + \langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ &\quad +\langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), q - x_n\rangle, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} D_f(Tq, T^{n+1-k}x_n) - D_f(Tq, x_n) &= f(x_n) - f(T^{n+1-k}x_n) \\ &\quad +\langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ &\quad +\langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), Tq - x_n\rangle. \end{aligned} \tag{3.12}$$

Substituting (3.10), (3.11) and (3.12) into (3.9), we have

$$\begin{aligned} 0 \leq\ & \sum_{k=1}^{n}(\beta_k - \alpha_k)\Big(f(T^{n+1-k}x_n) - f(x_n) + \langle \nabla f(Tq), x_n - T^{n+1-k}x_n\rangle\Big) \\ & +D_f(Tq, q) \\ & +\langle \nabla f(Tq) - \nabla f(q), \sum_{k=1}^{n}\beta_k(T^{n+1-k}x_n - x_n) + x_n - Tq\rangle \\ & +\sum_{k=1}^{n}\delta_k\{f(x_n) - f(T^{n+1-k}x_n) + \langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ & +\langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), q - x_n\rangle\} \\ & -\sum_{k=1}^{n}\gamma_k\{f(x_n) - f(T^{n+1-k}x_n) + \langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ & +\langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), Tq - x_n\rangle\}. \end{aligned} \tag{3.13}$$

Taking limit as $n \to \infty$ in (3.13) and using (3.8), we have

$$0 \leq D_f(Tq, q) + \langle \nabla f(Tq) - \nabla f(q), q - Tq\rangle.$$

Using the four points identity (1.3), we have

$$\begin{aligned}0 &\leq D_f(Tq,q)+D_f(Tq,Tq)-D_f(Tq,q)-D_f(q,Tq)+D_f(q,q)\\ &= -D_f(q,Tq).\end{aligned}$$

Thus $D_f(q,Tq) \leq 0$ and then $D_f(q,Tq) = 0$. Since f is strictly convex, we have $q = Tq$. Hence, $q \in F(T)$. Therefore $\hat{F}(T) \subset F(T)$. This thus implies that $\hat{F}(T) = F(T)$. □

4. Convergence Analysis

In this section, we introduce a hybrid algorithm for finding common solutions of countable family of equilibrium problem and finite fixed points of n-generalized Bregman nonspreading mapping in reflexive Banach space.

Let $\{\alpha_{n,i} : n,i \in \mathbb{N}, 1 \leq i \leq \mathbb{N}\}$ be sequences of real numbers such that $\{\alpha_{n,i}\} \subset (0,1)$. We define the following $W_n : C \to C$ mapping generated by T^i, $i = 1,2,\ldots,N$ and $\{\alpha_{n,i}\}$, where $T^i : C \to C$ is a finite family of n-generalized Bregman nonspreading mappings.

$$\begin{aligned}S_{n,0}x &= x,\\ S_{n,1}x &= \nabla f^*[\alpha_{n,1}\nabla f(T^1x)+(1-\alpha_{n,1})\nabla f(x)]\\ S_{n,2}x &= \nabla f^*[\alpha_{n,2}\nabla f(T^2S_{n,1}x)+(1-\alpha_{n,2})\nabla f(S_{n,1}x)]\\ S_{n,3}x &= \nabla f^*[\alpha_{n,3}\nabla f(T^3S_{n,2}x)+(1-\alpha_{n,3})\nabla f(S_{n,2}x)]\\ &\vdots\\ S_{n,N-1}x &= \nabla f^*[\alpha_{n,N-1}\nabla f(T^{N-1}S_{n,N-2}x)+(1-\alpha_{n,N-1})\nabla f(S_{n,N-2}x)]\\ W_n = S_{n,N} &= \nabla f^*[\alpha_{n,N}\nabla f(T^NS_{n,N-1}x)+(1-\alpha_{n,N})\nabla f(S_{n,N-1}x)].\end{aligned} \tag{4.1}$$

Using the above definition, we have the following lemma.

Proposition 4.1. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{T^i\}_{i=1}^N$ be a finite famiy of n-generalized Bregman nonspreading mapping of C into itself such that $\bigcap_{i=1}^N F(T^i) \neq \emptyset$. Let $\{\alpha_{n,i}\}$ be real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\forall i \in \{1,2,\ldots,N\}$. Let W_n be a Bregman W-mapping generated by $T^1,T^2,\ldots,T^N$ in (4.1). Then*

(i) $\cap_{i=1}^N F(T^i) = F(W_n)$,

(ii) *W_n is Bregman quasi-nonexpansive,*

(iii) *If in addition, T^i is Bregman relatively nonexpansive mapping, for each i, then W_n is Bregman relatively nonexpansive.*

Proof. Let $x \in \cap_{i=1}^{N} F(T^i)$. Then $T^i x = x$, $i = 1, 2, \ldots, N$. From (4.1), we have that $S_{n,1}x = x$, $S_{n,2}x = x, \ldots, S_{n,N}x = x$. Thus $\cap_{i=1}^{N} F(T^i) \subset F(W_n)$. Conversely, let $y \in F(W_n)$ and $x \in \cap_{i=1}^{N} F(T^i)$. Then

$$\begin{aligned}
D_f(x,y) &= D_f(x, W_n y)\\
&= D_f(x, \nabla f^*(\alpha_{n,N}\nabla f(T^N S_{n,N-1}y) + (1-\alpha_{n,N})\nabla f(S_{n,N-1}y)))\\
&= f(x) - \langle x, \alpha_{n,N}\nabla f(T^N S_{n,N-1}y)\rangle + (1-\alpha_{n,N})\nabla f(S_{n,N-1}y)\rangle\\
&\quad + f^*(\alpha_{n,N}\nabla f(T^N S_{n,N-1}y) + (1-\alpha_{n,N})\nabla f(S_{n,N-1}y))\\
&\leq \alpha_{n,N}(f(x) - \langle x, \nabla f(T^N S_{n,N-1}y) + f^*(\nabla f(T^N S_{n,N-1}y)))\\
&\quad + (1-\alpha_{n,N})(f(x) - \langle x, \nabla f(S_{n,N-1}y)\rangle + f^*(\nabla f(T^N S_{n,N-1}y)))\\
&\quad - \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||)\\
&= \alpha_{n,N} D_f(x, T^N S_{n,N-1}y) + (1-\alpha_{n,N}) D_f(x, S_{n,N-1}y)\\
&\quad - \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y))\\
&\leq D_f(x, S_{n,N-1}y)\\
&\quad - \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||)\\
&\;\;\vdots\\
&\leq D_f(x,y) - \alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 y) - \nabla f(y)||)\\
&\quad - \alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}y) - \nabla f(S_{n,1}y)||)\\
&\quad - \cdots - \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||). \qquad (4.2)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 y) - \nabla f(y)||)\\
&\qquad = \alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}y) - \nabla f(S_{n,1}y)||)\\
&\qquad = \cdots = \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||) = 0.
\end{aligned}$$

Then by the property of ρ_r^* from Lemma 2.9 and the norm-to-norm continuity of ∇f^*, we have

$$\begin{aligned}
T^1 y &= y,\\
T^2 S_{n,1} y &= S_{n,1} y,\\
&\;\;\vdots\\
T^N S_{n,N-1} &= S_{n,N-1} y.
\end{aligned}$$

It follows that

$$\begin{aligned}
D_f(y, S_{n,1}y) &= D_f(y, \nabla f^*(\alpha_{n,1}\nabla f(T^1 y) + (1-\alpha_{n,1})\nabla f(y)))\\
&\leq \alpha_{n,1} D_f(y, T^1 y) + (1-\alpha_{n,1}) D_f(y,y) = 0.
\end{aligned}$$

Therefore $y \in F(S_{n,1})$ and consequently, $y \in F(T^1)$. Following similar argument, we have that $y \in F(T^i)$ for $i = 1, 2, \ldots, N$ and hence $y \in \bigcap_{i=1}^{N} F(T^i)$.

(ii) Let $y \in F(W_n)$. Then

$$\begin{aligned} D_f(y, W_n x) &= D_f(y, \nabla f^*(\alpha_{n,N}\nabla f(T^N S_{n,N-1}x) + (1-\alpha_{n,N})\nabla f(S_{n,N-1}x))) \\ &\leq \alpha_{n,N} D_f(y, T^N S_{n,N-1}x) + (1-\alpha_{n,N}) D_f(y, S_{n,N-1}x) \\ &\leq \alpha_{n,N} D_f(y, S_{n,N-1}x) + (1-\alpha_{n,N}) D_f(y, S_{n,N-1}x) \\ &= D_f(y, S_{n,N-1}x) \\ &= D_f(y, \nabla f^*(\alpha_{n,N-1}\nabla f(T^{N-1}S_{n,N-2}x) \\ &\quad +(1-\alpha_{n,N-1})\nabla f(S_{n,N-2}x))) \\ &\leq \alpha_{n,N-1} D_f(y, T^{N-1}S_{n,N-2}x) + (1-\alpha_{n,N-1}) D_f(y, S_{n,N-2}x) \\ &\leq D_f(y, S_{n,N-2}x) \\ &\ \vdots \\ &\leq D_f(y, x). \end{aligned}$$

(iii) Let $\{x_n\} \subset C$ such that $x_n \rightharpoonup \bar{x}$ and $||W_n x_n - x_n|| \to 0$ as $n \to \infty$. From (4.2), we have

$$\begin{aligned} D_f(\bar{x}, W_n x_n) &\leq D_f(\bar{x}, x_n) - \alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 x_n) - \nabla f(x_n)||) \\ &- \alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}x_n) - \nabla f(S_{n,1}x_n)||) \\ &- \cdots - \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}x_n) - \nabla f(S_{n,N-1}x_n)||). \end{aligned} \tag{4.3}$$

Using three points identity (1.2), we obtain

$$\begin{aligned} D_f(\bar{x}, x_n) - D_f(\bar{x}, W_n x_n) &= \langle \bar{x} - x_n, \nabla f(W_n x_n) - \nabla f(x_n)\rangle \\ &\quad -D_f(x_n, W_n x_n). \end{aligned} \tag{4.4}$$

Since $x_n \rightharpoonup \bar{x}$ and $\lim_{n\to\infty} ||x_n - W_n x_n|| = 0$, we obtain

$$\begin{aligned} |D_f(\bar{x}, x_n) - D_f(\bar{z}, W_n x_n)| &\leq ||\bar{x} - x_n|| ||\nabla f(W_n x_n) - \nabla f(x_n)|| \\ &\quad -D_f(x_n, W_n x_n) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.5}$$

Therefore from (4.3), we have

$$\begin{aligned} &\alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 x_n) - \nabla f(x_n)||) + \alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}x_n) \\ &\quad - \nabla f(S_{n,1}x_n)||) + \dots \\ &\quad + \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}x_n) - \nabla f(S_{n,N-1}x_n)||) \\ &\quad \leq D_f(\bar{x}, x_n) - D_f(\bar{x}, x_n). \end{aligned}$$

Taking limit as $n \to \infty$, using (4.5) and property of ρ_r^*, yields

$$\begin{aligned} \lim_{n\to\infty} ||\nabla f(T^1 x_n) - \nabla f(x_n)|| &= \lim_{n\to\infty} ||\nabla f(T^2 S_{n,1}x_n) - \nabla f(S_{n,1}x_n)|| = \\ &\cdots = \lim_{n\to\infty} ||\nabla f(T^N S_{n,N-1}x_n) - \nabla f(S_{n,N-1}x_n)|| = 0. \end{aligned}$$

By the norm-to-norm uniform continuity of ∇f on bounded subset of E^*, it follows that

$$\begin{aligned}\lim_{n\to\infty}||T^1x_n - x_n|| &= \lim_{n\to\infty}||T^2S_{n,1}x_n - S_{n,1}x_n|| = \cdots \\ &= \lim_{n\to\infty}||T^NS_{n,N-1}x_n - S_{n,N-1}x_n|| = 0. \end{aligned} \tag{4.6}$$

We next prove that $S_{n,i}x_n - x_n \to 0$ for each $i = 1, 2, \ldots, N-1$. From (4.1), we get

$$\begin{aligned} D_p(x_n, S_{n,1}x_n) &= D_f(x_n, \nabla f^*[\alpha_{n,1}\nabla f(T^1x_n) + (1-\alpha_{n,1})\nabla f(x_n)]) \\ &\leq \alpha_{n,1}D_f(x_n, T^1x_n) + (1-\alpha_{n,1})D_f(x_n, x_n). \end{aligned}$$

Taking limit as $n \to \infty$ and using (4.6), we have

$$\lim_{n\to\infty} D_f(x_n, S_{n,1}x_n) = 0,$$

hence

$$\lim_{n\to\infty}||S_{n,1}x_n - x_n|| = 0.$$

Thus

$$||T^2S_{n,1}x_n - x_n|| \leq ||T^2S_{n,1}x_n - S_{n,1}x_n|| + ||S_{n,1}x_n - x_n|| \to 0 \quad n \to \infty.$$

Similarly, we have

$$\begin{aligned} D_f(x_n, S_{n,2}x_n) &= D_f(x_n, \nabla f^*[\alpha_{n,2}\nabla f(T^2S_{n,1}x_n) + (1-\alpha_{n,2})\nabla f(S_{n,1}x_n)]) \\ &\leq \alpha_{n,2}D_f(x_n, T^2S_{n,1}x_n) + (1-\alpha_{n,2})D_f(x_n, S_{n,1}x_n). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} D_f(x_n, S_{n,2}x_n) = 0,$$

and hence

$$\lim_{n\to\infty}||S_{n,2}x_n - x_n|| = 0.$$

Following similar approach as above, we have

$$\lim_{n\to\infty}||S_{n,3}x_n - x_n|| = \lim_{n\to\infty}||S_{n,4}x_n - x_n|| = \cdots = \lim_{n\to\infty}||S_{n,N-1}x_n - x_n|| = 0.$$

Therefore

$$\lim_{n\to\infty}||S_{n,i}x_n - x_n|| = 0 \quad \text{for each} \;\; i = 1, 2, \ldots, N-1.$$

This together with the Bregman relative nonexpansiveness of each T^i for $i = 1, 2, \ldots, N$, implies that $\bar{x} \in F(S_{n,i})$ for $i = 1, 2, \ldots, N$. Hence $\bar{x} \in F(W_n)$. This therefore implies that W_n is Bregman relatively nonexpansive. □

We are now in position to introduce our iterative algorithm.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. For $i = 1, 2, \dots, N$, let $\{\alpha_{n,i}\} \subset (0,1)$, $T^i : C \to C$ be finite family of n-generalized Bregman nonspreading mappings and $W_n : C \to C$ be a Bregman W-mapping generated by $\{\alpha_{n,i}\}$ and $T^1, T^2, \dots, T^N$ in (4.1). Let $g_j : C \times C \to \mathbb{R}$ be bifunctions satisfying assumptions (A1)-(A4) and suppose $\Gamma := \bigcap_{i=1}^N F(T^i) \cap \bigcap_{j=1}^\infty EP(g_i) \neq \emptyset$. Define the sequence $\{x_n\}$ by the following process*

$$\begin{cases} x_0 = x \in C, C_0 = Q_0 = C, \\ z_n = \nabla f^*[\beta_{n,0}\nabla f(x_n) + \sum_{j=1}^\infty \beta_{n,j}\nabla f(Res^f_{\lambda_n,g_j}x_n)], \\ y_n = \nabla f^*[\delta_n \nabla f(x_n) + (1-\delta_n)\nabla f(W_n z_n)], \\ C_n = \Big\{z \in C : D_f(z, y_n) \leq D_f(z, x_n)\Big\}, \\ Q_n = \Big\{z \in C : \langle \nabla f(x) - \nabla f(x_n), x_n - z\rangle \geq 0\Big\}, \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x, \end{cases} \tag{4.7}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_{n,j}\}$ and $\{\delta_n\}$ are sequences in $[0,1)$ satisfying the following control conditions:

(i) $\sum_{j=0}^\infty \beta_{n,j} = 1$, $\forall\ n \in \mathbb{N} \cup \{0\}$;

(ii) *There exists* $k \in \mathbb{N}$ *such that* $\liminf_{n\to\infty} \beta_{n,j}\beta_{n,k} > 0$, $\forall j \in \mathbb{N} \cup \{0\}$;

(iii) $0 \leq \delta_n < 1$, $\forall n \in \mathbb{N}$ *and* $\liminf_{n\to\infty} \delta_n < 1$;

(iv) $\liminf_{n\to\infty} \lambda_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $Proj^f_\Gamma x$ as $n \to \infty$.

Proof. We divide the proof into several steps.

Step 1: We show that $\Gamma \subset C_n \cap Q_n$ and x_{n+1} is well defined.
It is clear that C_n and Q_n are closed and convex. Then $C_n \cap Q_n$ is closed and convex for $n \geq 0$. Obviously, $\Gamma \subset C_0 \cap Q_0$. Suppose $\Gamma \subset C_m \cap Q_m$ for some $m \in \mathbb{N}$.

Let $p \in \Gamma$, then

$$\begin{aligned}
D_f(p, y_m) &= D_f(p, \nabla f^*[\delta_m \nabla f(x_m) + (1-\delta_m)\nabla f(W_m z_m)]) \\
&= V_f(p, \delta_m \nabla f(x_m) + (1-\delta_m)\nabla f(W_m z_m)) \\
&= f(p) - \langle p, \delta_m \nabla f(x_m) + (1-\delta_m)\nabla f(W_m z_m)\rangle \\
&\quad + f^*(\delta_m \nabla f(x_m) + (1-\delta_m)\nabla f(W_m z_m)) \\
&\leq \delta_m[f(p) - \langle p, \nabla f(x_m)\rangle + f^*(x_m)] \\
&\quad + (1-\delta_m)[f(p) - \langle p, \nabla f(W_m z_m)\rangle + f^*(W_m z_m)] \\
&\quad - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_m z_m||) \\
&\leq \delta_m D_f(p, x_m) + (1-\delta_m) D_f(p, z_m) - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_n z_m||) \\
&= \delta_n D_f(p, x_m) + (1-\delta_m) D_f(p, \nabla f^*[\beta_{m,0}\nabla f(x_m) \\
&\quad + \sum_{j=1}^{\infty} \beta_{m,j} \nabla f(Res^f_{EP(g)} x_m)]) \\
&\quad - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_m z_m||).
\end{aligned}$$

Hence

$$\begin{aligned}
D_f(p, y_m) &\leq \delta_m D_f(p, x_m) + (1-\delta_m)[\beta_{m,0} D_f(p, x_m) \\
&\quad + \sum_{j=1}^{\infty} \beta_{m,j} D_f(p, Res^f_{EP(g)} x_m) \\
&\quad - \beta_{m,0} \sum_{j=1}^{\infty} \beta_{m,j} \rho_r^*(||x_m - Res^f_{EP(g)} x_m||)] \\
&\quad - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_m z_m||) \\
&\leq \delta_m D_f(p, x_m) + (1-\delta_m)[\beta_{m,0} D_f(p, x_m) + \sum_{j=1}^{\infty} \beta_{m,j} D_f(p, x_m)] \\
&\quad - (1-\delta_m)\beta_{m,0} \sum_{j=1}^{\infty} \beta_{m,j} \rho_r^*(||x_m - Res^f_{EP(g)} x_m||) \\
&\quad - \delta_n(1-\delta_m)\rho_r^*(||x_m - W_m z_m||) \\
&= D_f(p, x_m) - (1-\delta_m)\beta_{m,0} \sum_{j=1}^{\infty} \beta_{m,j} \rho_r^*(||x_m - Res^f_{EP(g)} x_m||) \\
&\quad - \delta_n(1-\delta_n)\rho_r^*(||x_m - W_m z_m||) \qquad (4.8) \\
&\leq D_f(p, x_m).
\end{aligned}$$

Hence $p \in C_m$, which implies that $\Gamma \in C_m$. Since $x_{m+1} = Proj^f_{C_m \cap Q_m} x$, then $\langle \nabla f(x) - \nabla f(x_{m+1}), z - x_{m+1}\rangle \leq 0 \ \forall \ z \in C_m \cap Q_m$. In particular, $\langle \nabla f(x) - \nabla f(x_{m+1}), p - x_{m+1}\rangle \leq 0 \ \forall p \in \Gamma$. Thus $p \in Q_{m+1}$. This proves that $\Gamma \subset C_{m+1} \cap$

Q_{m+1}. Therefore $\Gamma \subset C_n \cap Q_n\ \forall\ n \geq 0$. Consequently, since $C_n \cap Q_n$ is closed and convex, then $x_{n+1} = Prof^f_{C_n \cap Q_n} x$ is well-defined.

Step 2: We prove that $\{x_n\}, \{y_n\}, \{z_n\}, \{Res^f_{\lambda_n, g_j} x_n\}$ and $\{W_n z_n\}$ are bounded. Since $\Gamma \subset C_n \cap Q_n$ for every $n \geq 0$ and $x_{n+1} = Proj^f_{C_n \cap Q_n} x$, then

$$D_f(p, x_{n+1}) \leq D_f(p, x) \quad \forall\ n \geq 0. \tag{4.9}$$

So $\{D_f(p, x_n)\}$ is bounded and hence there exists a constant $M > 0$ such that

$$D_f(p, x_n) \leq M \quad \forall\ n \in \mathbb{N} \cup \{0\}.$$

In view of Lemma 2.6, we conclude that the sequence $\{x_n\}$ is bounded. Similarly, the sequences $\{y_n\}, \{z_n\}, \{Res^f_{\lambda_n, g_j} x_n\}$ and $\{W_n z_n\}$ are bounded.

Step 3: Next, we show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, $\lim_{n\to\infty} ||Res^f_{\lambda_n, g_j} x_n - x_n|| = 0$ and $\lim_{n\to\infty} ||W_n z_n - z_n|| = 0$.

Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = Proj^f_{Q_n}(x)$, we have

$$D_f(x_{n+1}, Proj^f_{Q_n}(x)) + D_f(Proj^f_{Q_n}(x_1), x) \leq D_f(x_{n+1}, x).$$

Thus

$$D_f(x_{n+1}, x_n) + D_f(x_n, x) \leq D_f(x_{n+1}, x). \tag{4.10}$$

Therefore the sequence $\{D_f(x_n, x)\}$ is non-decreasing and thus $\lim_{n\to\infty} D_f(x_n, x)$ exists. Hence, it follows that $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$, and by Lemma 2.5, we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{4.11}$$

Also, since $x_{n+1} \in C_n$, we have

$$D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n).$$

This yields that $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$ and thus

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0.$$

Therefore from (4.11) and (4.12), we get

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{4.12}$$

By the uniform continuity of f and ∇f on bounded subsets of E and E^* respectively, we have

$$\lim_{n\to\infty} ||f(y_n) - f(x_n)|| = 0 \tag{4.13}$$

and

$$\lim_{n\to\infty} ||\nabla f(y_n) - \nabla f(x_n)||_* = 0. \tag{4.14}$$

Furthermore,

$$\begin{aligned} D_f(p, x_n) - D_f(p, y_n) &= f(p) - f(x_n) - \langle p - x_n, \nabla f(x_n)\rangle \\ &\quad - f(p) + f(y_n) + \langle p - y_n, \nabla f(y_n)\rangle \\ &= f(y_n) - f(x_n) + \langle p - y_n, \nabla f(y_n)\rangle - \langle p - x_n, \nabla f(x_n)\rangle \\ &= f(y_n) - f(x_n) + \langle x_n - y_n, \nabla f(y_n)\rangle \\ &\quad - \langle p - x_n, \nabla f(y_n) - \nabla f(x_n)\rangle. \end{aligned}$$

Therefore from (4.12) - (4.14), we get

$$\lim_{n\to\infty} [D_f(p, x_n) - D_f(p, y_n)] = 0. \tag{4.15}$$

Note that from (4.8), we have

$$\begin{aligned} D_f(p, y_n) &\leq D_f(p, x_n) - (1 - \delta_n)\beta_{n,0} \sum_{j=1}^{\infty} \beta_{n,j} \rho_r^*(||x_n - Res^f_{\lambda_n, g_j} x_n||) \\ &\quad - \delta_n(1 - \delta_n)\rho_r^*(||x_n - W_n z_n||). \end{aligned}$$

Using the property of ρ_r^* and conditions (ii) and (iii) together with (4.15), we have

$$\lim_{n\to\infty} ||x_n - Res^f_{\lambda_n, g_j} x_n|| = 0 \tag{4.16}$$

and

$$\lim_{n\to\infty} ||x_n - W_n z_n|| = 0. \tag{4.17}$$

By the uniform continuity of ∇f on bounded subsets of E^*, we have

$$\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(Res^f_{\lambda_n, g_j} x_n)|| = 0.$$

Hence from (4.7), we get

$$\lim_{n\to\infty} ||\nabla f(z_n) - \nabla f(x_n)||' = \lim_{n\to\infty} \sum_{j=1}^{\infty} \beta_{n,j} ||\nabla f(Res^f_{\lambda_n, g_j} x_n) - \nabla f(x_n)||' = 0.$$

Furthermore, since f is Fréchet differentiable on bounded subset of E, then ∇f^* is uniformly continuous on bounded subsets of E^*. Thus

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{4.18}$$

Therefore

$$\lim_{n\to\infty}||W_nz_n - z_n|| = \lim_{n\to\infty}[||W_nz_n - x_n|| + ||x_n - z_n||] = 0. \tag{4.19}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $q \in E$. Since $||W_nz_n - z_n|| \to 0$ and $||z_n - x_n|| \to 0$ as $n \to \infty$, then from Lemma 2.11 we have that $q \in F(W_n)$. Hence $q \in \bigcap_{i=1}^N F(T^i)$.
Also from Lemma 2.11, we have for each $j = 1, 2, \ldots$

$$g_j(Res^f_{\lambda_n,g_j}x_n, y) + \frac{1}{\lambda_n}\langle y - Res^f_{\lambda_n,g_j}x_n, \nabla f(Res^f_{\lambda_n,g_j}x_n) - \nabla f(x_n)\rangle \geq 0 \quad \forall y \in C.$$

Hence

$$g_j(Res^f_{\lambda_{n_k},g_j}x_{n_k}, y) + \frac{1}{\lambda_{n_k}}\langle y - Res^f_{\lambda_{n_k},g_j}x_{n_k}, \nabla f(Res^f_{\lambda_{n_k},g_j}x_{n_k}) - \nabla f(x_{n_k})\rangle \geq 0 \quad \forall y \in C.$$

From the assumption (A2), we have

$$\begin{aligned}\frac{1}{\lambda_{n_k}}||y - &Res^f_{\lambda_{n_k},g_j}x_{n_k}||||\nabla f(Res^f_{\lambda_{n_k}g_j}x_{n_k}) - \nabla f(x_{n_k})|| \\ &\geq \frac{1}{\lambda_{n_k}}\langle y - Res^f_{\lambda_{n_k},g_j}x_{n_k}, \nabla f(Res^f_{\lambda_{n_k},g_j}x_{n_k}) - \nabla f(x_{n_k})\rangle \\ &\geq -g_j(Res^f_{\lambda_{n_k}g_j}x_{n_k}, y) \geq g_j(y, Res^f_{\lambda_{n_k},g_j}x_{n_k}) \quad \forall y \in C.\end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequality, from (A4) and condition (iv), we have $x_{n_k} \to q$, $||\nabla f(Res^f_{\lambda_{n_k},g_j}x_{n_k}) - \nabla f(x_{n_k})|| \to 0$, we have that $g_j(y, q) \leq 0$ for all $y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)q$. Noting that $y_t \in C$, which yields $g_j(y_t, q) \leq 0$. It therefore follows from (A1) that

$$0 = g_j(y_t, y_t) \leq tg_j(y_t, y) + (1-t)g_j(y_t, q) \leq tg_j(y_t, y).$$

That is $g_j(y_t, y) \geq 0$.
Let $t \downarrow 0$, from (A3), we obtain $g_j(q, y) \geq 0$ for any $y \in C$, $j = 1, 2, \ldots$. This implies that $q \in \bigcap_{j=1}^\infty EP(g_j)$. Therefore $q \in \Gamma := \bigcap_{i=1}^N F(T^i) \cap \bigcap_{j=1}^\infty EP(g_j)$.
Now since $x_{n+1} = Proj^f_{C_n\cap Q_n}x$, we have

$$\langle \nabla f(x) - \nabla f(x_{n+1}), x_{n+1} - z\rangle \geq 0 \quad \forall z \in C_n \cap Q_n.$$

Since $\Gamma \subset C_n \cap Q_n$, we have

$$\langle \nabla f(x) - \nabla f(x_{n+1}), x_{n+1} - z\rangle \geq 0 \quad \forall z \in \Gamma.$$

Taking the limit of the above inequality, we have

$$\langle \nabla f(x) - \nabla f(q), q - z\rangle \geq 0 \quad \forall z \in \Gamma.$$

Therefore $q = Proj^f_\Gamma x$. This completes the proof. □

5. Application to Zeros of Maximal Monotone Operators

Sabach [37] showed that under some properties of the function f, the solution set of the equilibrium problem is equivalent to the set of zeros of a maximal monotone operator, that is the points $x^* \in dom\ A$ such that

$$0^* \in Ax^*, \tag{5.1}$$

where $A : E \to 2^{E^*}$ is a maximal monotone operator. We denotes the set of zeros of A by $A^{-1}(0^*)$. An operator $A : E \to 2^{E^*}$ is said to be monotone if for any $x, y \in dom\ A$, we have

$$\xi \in Ax \quad \text{and} \quad \mu \in Ay \Rightarrow \langle \xi - \mu, x - y \rangle \geq 0.$$

A monotone operator A is said to be maximal if the graph of A, $Gr(A) := \{(x, \xi) : \xi \in Ax\}$ is not contained in the graph of any other monotone operator. The problem of finding the zeros of monotone operators is very important due to its applications in differential equations, evolution equations, optimization and other related fields. Many algorithms have also been introduced to find its solutions in Hilbert and Banah spaces.
Let $g : C \times C \to \mathbb{R}$ be a bifunction and define the following operator $A_g : E \to 2^{E^*}$ in the following manner

$$A_g(x) = \begin{cases} \{\xi \in E^* : g(x, y) \geq \langle \xi, y - x \rangle \quad \forall y \in C\}, & x \in C, \\ \emptyset & x \notin C. \end{cases} \tag{5.2}$$

The following result was proved for the mapping A_g in [37].

Proposition 5.1. *(Sabach [37]) Let C be a nonempty, closed and convex subset of a reflexive Banach space E and let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then:*

(i) $EP(g) = A_g^{-1}(0^*)$;

(ii) A_g *is maximal monotone operator;*

(iii) $Res_g^f = Res_{A_g}^f$.

Based on the above result, we propose the following which can be obtain from Theorem 4.2 for finding common fixed point of finite family of n-generalized Bregman nonspreading mapping and zeros of maximal monotone operators in reflexive Banach space.

Theorem 5.2. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded,* *uniformly Fréchet differentiable and totally convex on bounded subsets of E. For* $i = 1, 2, \dots, N$, *let* $\{\alpha_{n,i}\} \subset (0,1)$, $T^i : C \to C$ *be finite family of* n*-generalized Bregman nonspreading mappings and* $W_n : C \to C$ *be a Bregman W-mapping generated by* $\{\alpha_{n,i}\}$ *and* $T^1, T^2, \dots, T^N$ *in* (4.1). *Let* $g_j : C \times C \to \mathbb{R}$ *be bifunctions satisfying assumptions (A1)-(A4),* $A_{g_j} : E \to 2^{E^*}$ *be as defined in* (5.3) *for* $j = 1, 2, \dots$ *and suppose* $\Gamma := \bigcap_{i=1}^{N} F(T^i) \cap \bigcap_{j=1}^{\infty} A_{g_j}^{-1}(0^*) \neq \emptyset$. *Define the sequence* $\{x_n\}$ *by the following process*

$$
\begin{cases}
x_0 = x \in C, C_0 = Q_0 = C,\\
z_n = \nabla f^*[\beta_{n,0}\nabla f(x_n) + \sum_{j=1}^{\infty}\beta_{n,j}\nabla f(Res^f_{A_{g_j}} x_n)],\\
y_n = \nabla f^*[\delta_n \nabla f(x_n) + (1-\delta_n)\nabla f(W_n z_n)],\\
C_n = \Big\{z \in C : D_f(z, y_n) \leq D_f(z, x_n)\Big\},\\
Q_n = \Big\{z \in C : \langle \nabla f(x) - \nabla f(x_n), x_n - z\rangle \geq 0\Big\},\\
x_{n+1} = Proj^f_{C_n \cap Q_n} x,
\end{cases}
\tag{5.3}
$$

for all $n \geq 0$, *where* $\{\beta_{n,j}\}$ *and* $\{\delta_n\}$ *are sequences in* $[0,1)$ *satisfying the following control conditions:*

(i) $\sum_{j=0}^{\infty}\beta_{n,j} = 1$, $\forall\ n \in \mathbb{N}\cup\{0\}$;

(ii) *There exists* $k \in \mathbb{N}$ *such that* $\liminf_{n\to\infty}\beta_{n,j}\beta_{n,k} > 0$, $\forall j \in \mathbb{N}\cup\{0\}$;

(iii) $0 \leq \delta_n < 1$, $\forall n \in \mathbb{N}$ *and* $\liminf_{n\to\infty}\delta_n < 1$.

Then, the sequence $\{x_n\}$ *converges strongly to* $Proj^f_{\Gamma}x$ *as* $n \to \infty$.

6. Numerical Example

We give a numerical example to demonstrate the performance of our algorithm 4.7.

Example 6.1. Let $E = \mathbb{R}$, $C = [-10, 10]$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{2}{3}x^2$. Let $g : C \times C \to \mathbb{R}$ be defined by $g(x,y) = x(y-x)$, $\forall x, y \in C$ and $T : C \to C$ be defined by $T^i x = \frac{1}{3i}x$, $i = 1, 2, \dots, N$. It is easy to observe that f is coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of $\mathbb{R}$ and $\nabla f(x) = \frac{4}{3}x$. Also since $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{R}\}$, then $f^*(z) = \frac{3}{8}z^2$ and $\nabla f^*(z) = \frac{3}{4}z$. Further, T^i is 1-generalized Bregman nonspreading mapping and $Res^f_{\lambda_n, g_j} z = \frac{z}{2 - 3\lambda_n j}$.
Choose $\{\alpha_{n,i}\} = \Big\{\frac{1}{(n+i)^2}\Big\}$, $\{\delta_n\} = \Big\{\frac{1}{(n+1)^2}\Big\}$, $\{\lambda_n\} = \big\{\frac{1}{2}\big\}$ and for each $n \in \mathbb{N}\cup\{0\}$, and $j \geq 0$, let $\{\beta_{n,j}\}$ be defined by

$$
\beta_{n,j} =
\begin{cases}
\frac{1}{3^{j+1}}\left(\frac{n}{n+1}\right), & n > j,\\
1 - \frac{n}{n+1}\sum_{k=1}^{n}\frac{1}{3^k} & n = j,\\
0 & n < j.
\end{cases}
$$

Observe that g satisfy Assumption (A1)-(A4) and $\Gamma = \{0\} \neq \emptyset$. After simplification, the hybrid iterative scheme (4.7) reduces to the following: Given x_0,

$$(6.1)\qquad \begin{cases} z_n = \frac{3}{4}\left[\beta_{n,0}\frac{4}{3}(x_n) + \sum_{j=1}^{\infty}\beta_{n,j}\frac{2x_n}{3(2-3j)}\right]; \\ y_n = \frac{3}{4}\left[\frac{4}{3(n+1)^2}(x_n) + \left(1 - \frac{1}{(n+1)^2}\right)\frac{4}{3}(W_n z_n)\right]; \\ C_n = \left[0, \frac{2(x_n^2+y_n^2)}{3}\right]; \\ Q_n = [0, x_n]; \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x_0, \end{cases}$$

where $W_n z_n$ is computed as follow:

$$(6.2)\qquad \begin{aligned} S_{n,0}z_n &= z_n, \\ S_{n,1}z_n &= \frac{z_n}{3(n+1)^2} + \left(1 - \frac{1}{(n+1)^2}\right)z_n; \\ S_{n,2}z_n &= \frac{z_n}{6(n+2)^2} + \left(1 - \frac{1}{(n+2)^2}\right)S_{n,1}z_n; \\ &\vdots \\ W_n z_n = S_{n,N} &= \frac{z_n}{3N(n+N)^2} + \left(1 - \frac{1}{(n+N)^2}\right)S_{n,N-1}z_n. \end{aligned}$$

Finally, we select the following values

$$\begin{aligned} &\text{Case(i): } N = 10 \text{ and } x_0 = -1, \\ &\text{Case(ii): } N = 50 \text{ and } x_0 = 0.5, \\ &\text{Case(iii): } N = 100 \text{ and } x_0 = 2. \end{aligned}$$

Using Matlab 2016(b) and $\epsilon = 10^{-6}$ as stopping criterion, we plot the graphs of error $||x_{n+1} - x_n||$ against number of iteration in each case. The computational results can be found in Figure 1.

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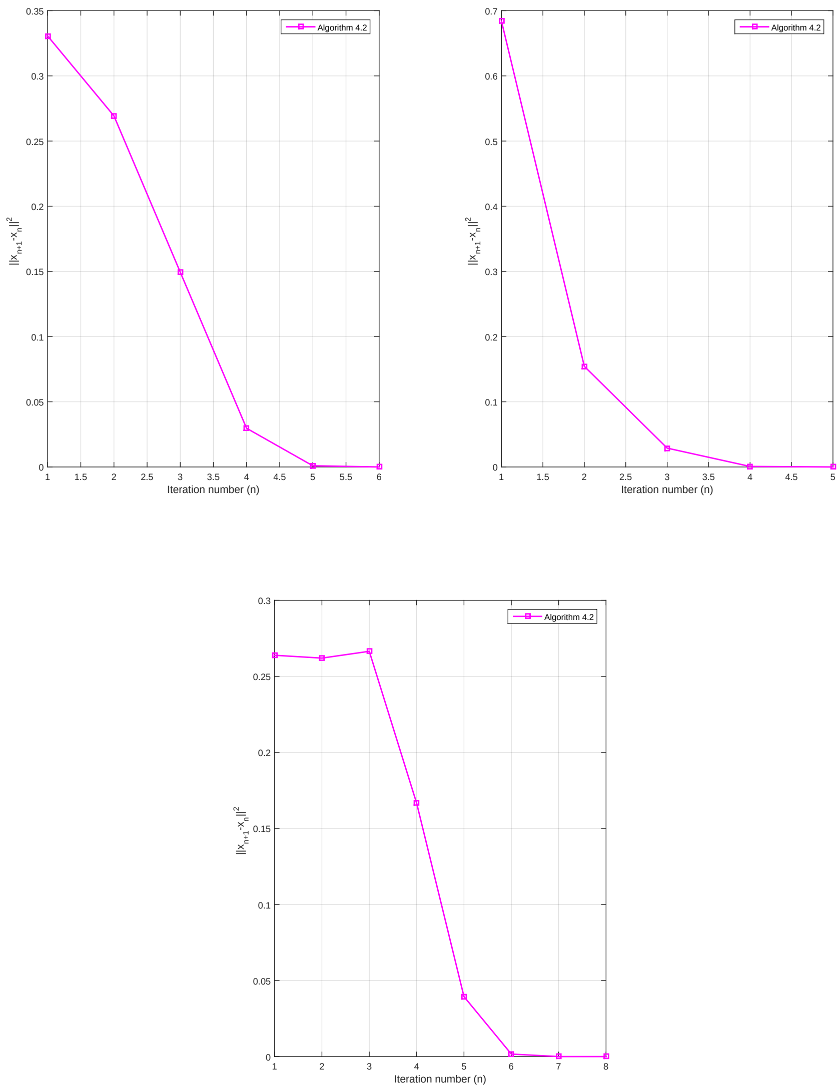

Figure 1: Example 6.1, Top-Left: Case(i); Top-Right: Case(ii); Bottom: Case(iii).

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