KYUNGPOOK Math. J. 61(2021), 473-486
https://doi.org/10.5666/KMJ.2021.61.3.473
pISSN 1225-6951 eISSN 0454-8124

Lucas-Euler Relations Using Balancing and Lucas-Balancing Polynomials

Robert Frontczak
Landesbank Baden-Württemberg, Stuttgart 70173, Germany
e-mail : robert.frontczak@lbbw.de

Taras Goy*
Faculty of Mathematics and Computer Science, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk 76018, Ukraine
e-mail : taras.goy@pnu.edu.ua

Abstract. We establish some new combinatorial identities involving Euler polynomials and balancing (Lucas-balancing) polynomials. The derivations use elementary techniques and are based on functional equations for the respective generating functions. From these polynomial relations, we deduce interesting identities with Fibonacci and Lucas numbers, and Euler numbers. The results must be regarded as companion results to some Fibonacci-Bernoulli identities, which we derived in our previous paper.

1. Motivation and Preliminaries

In 1975, Byrd [1] derived the following identity relating Lucas numbers to Euler numbers:

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\left(\frac{5}{4}\right)^k L_{n-2k}E_{2k}=2^{1-n}. \tag{1.1}$$

In [18], Wang and Zhang obtained a more general result valid for $j\geq 1$ as follows

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\left(\frac{5}{4}\right)^k F_j^{2k}L_{j(n-2k)}E_{2k}=2^{1-n}L_j^n. \tag{1.2}$$

---

* Corresponding Author.
Received October 21, 2020; revised March 12, 2021; accepted March 23, 2021.
2020 Mathematics Subject Classification: 11B37, 11B65, 05A15.
Key words and phrases: Euler polynomials and numbers, Bernoulli numbers, balancing polynomials and numbers, Fibonacci numbers, generating function.
Statements and conclusions made in this article by R. Frontczak are entirely those of the author. They do not necessarily reflect the views of LBBW.

Castellanos [2] found

$$\sum_{k=0}^{n}\binom{2n}{2k}2^{-2k-1}L_{2(n-k)j}L_j^{2k}E_{2k}=\left(\frac{5}{4}\right)^n F_j^{2n}, \tag{1.3}$$

which expresses even powers of Fibonacci numbers in terms of Lucas and Euler numbers.

Here, as usual, Fibonacci and Lucas numbers satisfy the recurrence relation $u_n = u_{n-1} + u_{n-2}$, $n \geq 2$, with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively, whereas Euler numbers $(E_n)_{n\geq 0}$ are given by the power series

$$\sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \frac{1}{\cosh z}.$$

Fibonacci and Lucas numbers are entries A000045 and A000032 in the On-Line Encyclopedia of Integer Sequences [17], respectively.

The Lucas-Euler pair may be regarded as the twin of the Fibonacci-Bernoulli pair. In the last years, there has been a growing interest in deriving new relations for these two pairs of sequences. For example, Zhang and Ma [21] proved a relation between Fibonacci polynomials and Bernoulli numbers $(B_n)_{n\geq 0}$ defined by

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}.$$

The following identity is a special case of their result:

$$\sum_{k=0}^{n}\binom{n}{k}5^{\frac{n-k}{2}}F_k B_{n-k} = n\beta^{n-1},$$

where $\beta = (1-\sqrt{5})/2$, or, equivalently,

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^k F_{n-2k}B_{2k} = \frac{nL_{n-1}}{2}. \tag{1.4}$$

See also [14, 18, 19, 20] for other results in this direction. Recently, Frontczak [5], Frontczak and Goy [7], and Frontczak and Tomovski [8] proved some generalizations of existing results. For instance, from [7] we have

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^{n-k}F_{jk}B_{n-k} = nF_j\beta^{j(n-1)}, \tag{1.5}$$

which holds for all $j \geq 1$ and generalizes (1.4) to an arithmetic progression, and

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(20^k - 5^k)F_{2j}^{2k}L_{2j(n-2k)}B_{2k} = \frac{5n}{2}F_{2j}F_{2j(n-1)}. \tag{1.6}$$

Note, since $B_{2n+1} = 0$ for $n \geq 1$, from (1.5) we get Kelisky's formula [9]

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 5^k F_j^{2k} F_{j(n-2k)} B_{2k} = \frac{n}{2} F_j L_{j(n-1)}.$$

In this paper, we present new identities linking Lucas numbers to Euler numbers (polynomials). The results stated are polynomial generalizations of (1.1) and are complements of the recent discoveries from [5, 7].

Throughout the paper, we will work with different kind of polynomials of a complex variable x: Euler polynomials $(E_n(x))_{n\geq 0}$, Bernoulli polynomials $(B_n(x))_{n\geq 0}$, balancing polynomials $(B_n^*(x))_{n\geq 0}$, and Lucas-balancing polynomials $(C_n(x))_{n\geq 0}$.

Euler and Bernoulli polynomials are famous mathematical objects and are fairly well understood. They are defined by [3, Chapter 24]

$$H(x,z) = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1} \qquad (|z| < 2\pi) \tag{1.7}$$

and

$$I(x,z) = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2e^{xz}}{e^z + 1} \qquad (|z| < \pi). \tag{1.8}$$

The numbers $B_n(0) = B_n$ are the famous Bernoulli numbers. Bernoulli numbers are rational numbers starting with $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, and so on. Also, as already mentioned, $B_{2n+1} = 0$ for $n \geq 1$. Euler numbers E_n are obtained from $I(1/2, 2z)$ that is

$$E_n = 2^n E_n(1/2). \tag{1.9}$$

In contrast to Bernoulli numbers, Euler numbers are integers where $E_0 = 1$, $E_2 = -1$, $E_4 = 5$ and $E_{2n+1} = 0$ for $n \geq 0$. Explicit formulas for the polynomials are

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \Big(x - \frac{1}{2}\Big)^{n-k}.$$

Euler polynomials can be expressed in terms of Bernoulli polynomials via

$$E_n(x) = \frac{2}{n+1} \Big(B_{n+1}(x) - 2^{n+1} B_{n+1}\Big(\frac{x}{2}\Big)\Big).$$

Particularly,

$$E_n(0) = \frac{2\,(1 - 2^{n+1})}{n+1} B_{n+1}. \tag{1.10}$$

Balancing polynomials are of younger age and are introduced in the next section.

2. Balancing and Lucas-Balancing Polynomials

Balancing polynomials $B_n^*(x)$ and Lucas-balancing polynomials $C_n(x)$ are generalizations of balancing and Lucas-balancing numbers [4]. These polynomials satisfy the recurrence $w_n(x) = 6xw_{n-1}(x) - w_{n-2}(x)$, $n \geq 2$, but with the respectively initial conditions $B_0^*(x) = 0$, $B_1^*(x) = 1$ and $C_0(x) = 1$, $C_1(x) = 3x$. The Binet formulas for these polynomials are

$$B_n^*(x) = \frac{\lambda^n(x) - \lambda^{-n}(x)}{2\sqrt{9x^2-1}} \quad \text{and} \quad C_n(x) = \frac{\lambda^n(x) + \lambda^{-n}(x)}{2},$$

where $\lambda(x) = 3x + \sqrt{9x^2-1}$. Also, the following explicit formulas hold [15, 16]

$$B_n^*(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} (6x)^{n-1-2k}, \qquad n \geq 0,$$

$$C_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (6x)^{n-2k}, \qquad n \geq 1.$$

Consult the papers [4, 6, 10, 11, 12, 13, 16] for more information about these polynomials. The numbers $B_n^*(1) = B_n^*$ and $C_n(1) = C_n$ are called balancing and Lucas-balancing numbers, respectively. These numbers are indexed in [17] under entries A001109 and A001541.

Balancing and Lucas-balancing polynomials possess interesting properties. They are related to Chebyshev polynomials by simple scaling [4, Lemma 2.1]. The exponential generating functions for balancing and Lucas-balancing polynomials are derived in [4, 6]. Here, however, we will only need the results from [6]: Let $b_1(x,z)$ and $b_2(x,z)$ be the exponential generating functions of odd and even indexed balancing polynomials, respectively. Then

$$\begin{aligned} b_1(x,z) &= \sum_{n=0}^{\infty} B_{2n+1}^*(x) \frac{z^n}{n!} \\ &= \frac{e^{(18x^2-1)z}}{\sqrt{9x^2-1}} \left(3x \sinh(6x\sqrt{9x^2-1}z) + \sqrt{9x^2-1}\cosh(6x\sqrt{9x^2-1}z)\right) \end{aligned} \tag{2.1}$$

and

$$b_2(x,z) = \sum_{n=0}^{\infty} B_{2n}^*(x) \frac{z^n}{n!} = \frac{e^{(18x^2-1)z}}{\sqrt{9x^2-1}} \sinh(6x\sqrt{9x^2-1}z). \tag{2.2}$$

Similarly, the exponential generating functions for Lucas-balancing polynomials are found to be

$$
\begin{aligned}
c_1(x,z) &= \sum_{n=0}^{\infty} C_{2n+1}(x)\frac{z^n}{n!} \\
&= e^{(18x^2-1)z}\big(3x\cosh(6x\sqrt{9x^2-1}z)+\sqrt{9x^2-1}\sinh(6x\sqrt{9x^2-1}z)\big) \qquad (2.3)
\end{aligned}
$$

and

$$
c_2(x,z) = \sum_{n=0}^{\infty} C_{2n}(x)\frac{z^n}{n!} = e^{(18x^2-1)z}\cosh(6x\sqrt{9x^2-1}z). \qquad (2.4)
$$

Connections between Bernoulli polynomials $B_n(x)$ and balancing polynomials $B_n^*(x)$ have been established in the recent papers [5, 7]. They are interesting, as they instantly give relations between Bernoulli numbers and Fibonacci and Lucas numbers. The links are the following evaluations [4]

$$
B_n^*\Big(\omega_s\frac{L_s}{6}\Big) = \omega_s^{n-1}\frac{F_{sn}}{F_s}, \qquad C_n\Big(\omega_s\frac{L_s}{6}\Big) = \omega_s^{n}\frac{L_{sn}}{2}, \qquad (2.5)
$$

where $\omega_s = 1$, if s is even, and $\omega_s = i = \sqrt{-1}$, if s is odd. These links will be used to prove our results.

3. Relations Between Euler and Balancing (Lucas-Balancing) Polynomials

We start with the following result involving even indexed balancing and Lucas-balancing polynomials.

Theorem 3.1. *For each $n \geq 1$ and $x \in \mathbb{C}$, we have*

$$
\begin{aligned}
\sum_{k=1}^{\lfloor n/2\rfloor} \binom{n-1}{2k-1} C_{2(n-2k)}(x)\big(144x^2(9x^2-1)\big)^k E_{2k-1}(0)& \\
= 12x(1-9x^2)B_{2n-2}^*(x).& \qquad (3.1)
\end{aligned}
$$

Proof. Since $\tanh z = 1 - \dfrac{2}{e^{2z}+1}$, from (1.8) we get

$$
I(0,12x\sqrt{9x^2-1}z) = 1-\tanh(6x\sqrt{9x^2-1}z)
$$

and, by (2.2) and (2.4),

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}C_{2(n-k)}\big(12x\sqrt{9x^2-1}\big)^k E_k(0)\Big)\frac{z^n}{n!}\\
&\qquad = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n-1}\binom{n}{k}C_{2k}\big(12x\sqrt{9x^2-1}\big)^{n-k}E_{n-k}(0)+C_{2n}(x)\Big)\frac{z^n}{n!}\\
&\qquad = c_2(x,z)I(0,12x\sqrt{9x^2-1}z)\\
&\qquad = e^{(18x^2-1)z}\big(\cosh(6x\sqrt{9x^2-1}z)-\sinh(6x\sqrt{9x^2-1}z)\big)\\
&\qquad = c_2(x,z)-\sqrt{9x^2-1}b_2(x,z)\\
&\qquad = \sum_{n=0}^{\infty}\big(C_{2n}(x)-\sqrt{9x^2-1}B^*_{2n}(x)\big)\frac{z^n}{n!}.
\end{aligned}$$

Thus,

$$\sum_{k=0}^{n}\binom{n}{k}C_{2(n-k)}\big(12x\sqrt{9x^2-1}\big)^k E_k(0)=C_{2n}(x)-\sqrt{9x^2-1}B^*_{2n}(x).$$

Since $E_{2n-1}=0$ for $n\geq 1$, after some algebra we have (3.1). □

Corollary 3.2. *For each $n\geq 1$ and $j\geq 1$,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-1}{2k-1}5^{k-1}F_{2j}^{2k-1}L_{2j(n-2k)}E_{2k-1}(0)=-F_{2j(n-1)}. \tag{3.2}$$

Proof. Evaluate (3.1) at the $x=\omega_j L_j/6$ and use the links from (2.5). To simplify recall that $L_n^2-5F_n^2=(-1)^n4$ and $F_{2n}=F_nL_n$. □

Using (1.10), we can write (3.2) as

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-1}{2k-1}\frac{20^k-5^k}{k}F_{2j}^{2k-1}L_{2j(n-2k)}B_{2k}=5F_{2j(n-1)},$$

which is easily reduced to (1.6).

We also have the following interesting identity.

Theorem 3.3. *For each $n\geq 0$ and $x\in\mathbb{C}$, we have the relation*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_{2(n-2k)}(x)\big(36x^2(9x^2-1)\big)^k E_{2k}=\big(18x^2-1\big)^n. \tag{3.3}$$

Proof. The result is a consequence of the fact that

$$c_2(x,z)I(1/2,12x\sqrt{9x^2-1}z)=e^{(18x^2-1)z}. \qquad \square$$

Corollary 3.4. *For each $n\geq 0$ and $j\geq 1$,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\Big(\frac{5}{4}\Big)^k F_{2j}^{2k}L_{2j(n-2k)}E_{2k}=2^{1-n}L_{2j}^n. \tag{3.4}$$

Proof. Evaluate (3.3) at the point $x=\omega_j L_j/6$ and use the links from (2.5). When simplifying you will also need the formula $L_n^2-L_{2n}=(-1)^n 2$. $\square$

Interestingly, if $j=1/2$ from (3.4) we obtain Byrd's result (1.1). Also, when $j=1$ and $j=2$, from (3.4) we obtain the following Lucas-Euler relations:

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\Big(\frac{5}{4}\Big)^k L_{2(n-2k)}E_{2k}=2\Big(\frac{3}{2}\Big)^n,$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\Big(\frac{45}{4}\Big)^k L_{4(n-2k)}E_{2k}=2\Big(\frac{7}{2}\Big)^n, \tag{3.5}$$

respectively. The first example appears as equation (31) in [5].

A different expression for the sum on the left of (3.3) is stated next.

Theorem 3.5. *For each $n\geq 0$ and $x\in\mathbb{C}$, we have*

$$\begin{aligned}&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_{2(n-2k)}(x)\big(36x^2(9x^2-1)\big)^k E_{2k}\\ &\qquad=\sum_{k=0}^{n}\binom{n}{k}\big(C_{2k}(x)-\sqrt{9x^2-1}B_{2k}^*(x)\big)(6x\sqrt{9x^2-1})^{n-k}.\end{aligned} \tag{3.6}$$

Proof. We use the identity

$$I(1/2,2z)=e^z(1-\tanh z),$$

from which the functional equation follows

$$c_2(x,z)I(1/2,12x\sqrt{9x^2-1}z)=e^{(6x\sqrt{9x^2-1})z}\big(c_2(x,z)-\sqrt{9x^2-1}b_2(x,z)\big).$$

Thus,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}C_{2k}(x)(6x\sqrt{9x^2-1})^{n-k}E_{n-k}\\ &\qquad=\sum_{k=0}^{n}\binom{n}{k}\big(C_{2k}(x)-\sqrt{9x^2-1}B_{2k}^*(x)\big)(6x\sqrt{9x^2-1})^{n-k},\end{aligned}$$

that is equivalent to (3.6). $\square$

Theorem 3.6. *For each $n \geq 0$ and $x \in \mathbb{C}$, it is true that*

$$\sum_{k=0}^{n} \binom{n}{k} C_{2(n-k)}(x) \left(12x\sqrt{9x^2-1}\right)^k E_k(x) = \left(18x^2 - 1 + 6x(2x-1)\sqrt{9x^2-1}\right)^n.$$

Proof. The functional relation $\cosh(z/2)I(x,z) = e^{(x-1/2)z}$ produces immediately

$$c_2(x,z)I(x, 12x\sqrt{9x^2-1}z) = e^{\left(18x^2-1+6x(2x-1)\sqrt{9x^2-1}\right)z}.$$

Comparing the coefficients of z in the power series expansions on both sides gives the identity. $\square$

When $x = 1/2$, then we recover (3.5), by (1.9).

4. Other Special Polynomial Identities

The following result appears as Theorem 13 in [7]: For each $n \geq 0$, $j \geq 1$, and $x \in \mathbb{C}$, we have

$$\sum_{k=0}^{n} \binom{n}{k} F_{jk}(\sqrt{5}F_j)^{n-k} B_{n-k}(x) = nF_j\big((\sqrt{5}x + \beta)F_j + F_{j-1}\big)^{n-1},$$

$$\sum_{k=0}^{n} \binom{n}{k} F_{jk}(-\sqrt{5}F_j)^{n-k} B_{n-k}(x) = nF_j\big((\alpha - \sqrt{5}x)F_j + F_{j-1}\big)^{n-1},$$

where $\alpha = (1+\sqrt{5})/2$ is the golden ratio and $\beta = (1-\sqrt{5})/2 = -1/\alpha$.

Now, we present the analogue result for the Lucas-Euler pair:

Theorem 4.1. *The following polynomial identity is valid for all $n \geq 0$, $j \geq 1$, and $x \in \mathbb{C}$:*

$$\sum_{k=0}^{n} \binom{n}{k} L_{jk}(\sqrt{5}F_j)^{n-k} E_{n-k}(x) = 2\big((\sqrt{5}x + \beta)F_j + F_{j-1}\big)^n, \tag{4.1}$$

$$\sum_{k=0}^{n} \binom{n}{k} L_{jk}(-\sqrt{5}F_j)^{n-k} E_{n-k}(x) = 2\big((\alpha - \sqrt{5}x)F_j + F_{j-1}\big)^n. \tag{4.2}$$

Proof. Let $L(z)$ be the exponential generating function for $(L_{jn})_{n\geq 0}$, $j \geq 1$. Then, using the Binet formula for L_n, we get

$$L(z) = 2e^{(1/2F_j+F_{j-1})z}\cosh\Big(\frac{\sqrt{5}F_j}{2}z\Big).$$

Thus, it follows that

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}L_{jk}(\sqrt{5}F_j)^{n-k}E_{n-k}(x)\Big)\frac{z^n}{n!} = L(z)I(x,\sqrt{5}F_j z)$$
$$= 2\,e^{\big((x-1/2)\sqrt{5}F_j+1/2F_j+F_{j-1}\big)z}$$
$$= 2\,e^{\big((\sqrt{5}x+\beta)F_j+F_{j-1}\big)z}.$$

This proves the first equation. The second follows upon replacing x by $1-x$ and using $E_n(1-x)=(-1)^nE_n(x)$ and $\alpha-\beta=\sqrt{5}$. □

Note that the relations (4.1) and (4.2) provide a generalization of (3.4). To see this, notice that they can be written more compactly as

$$\sum_{k=0}^{n}\binom{n}{k}L_{jk}(\pm\sqrt{5}F_j)^{n-k}E_{n-k}(x) = 2^{1-n}\big(L_j\pm\sqrt{5}F_j(2x-1)\big)^n. \tag{4.3}$$

Now, if $x=1/2$, we get

$$\sum_{k=0}^{n}\binom{n}{k}\big(\pm\sqrt{5}F_j\big)^{n-k}2^kL_{jk}E_{n-k} = 2L_j^n,$$

which is equivalent to (3.4). We also mention the nice and curious identities

$$\sum_{k=0}^{n}\binom{n}{k}\big(\pm\sqrt{5}F_j\big)^{n-k}L_{jk}E_{n-k}(\alpha) = 2(\pm1)^nL_{j\pm1}^n,$$
$$\sum_{k=0}^{n}\binom{n}{k}\big(\pm\sqrt{5}F_j\big)^{n-k}L_{jk}E_{n-k}(\beta) = 2(\mp1)^nL_{j\mp1}^n,$$

which can be deduced from (4.3) and $5F_n=L_{n+1}+L_{n-1}$.

We conclude this presentation with the following interesting corollary.

Corollary 4.2. *Let n, j and q be integers with $n,\, j\geq 1$ and q odd. Then it holds that*

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^{n-k}\big(q^{-(n-k)}-1\big)L_{jk}E_{n-k}(0) = 2q^{-n}\sum_{r=1}^{q-1}(-1)^r\big(r\alpha^j+(q-r)\beta^j\big)^n.$$

Proof. The known multiplication formula for Euler polynomials for odd q [3, Chapter 24]

$$q^n\sum_{r=0}^{q-1}(-1)^rE_n\Big(x+\frac{r}{q}\Big) = E_n(qx)$$

yields

$$\sum_{r=1}^{q-1}(-1)^r E_n\Big(\frac{r}{q}\Big) = \big(q^{-n}-1\big)E_n(0).$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}L_{jk}(\sqrt{5}F_j)^{n-k}\big(q^{-(n-k)}-1\big)E_{n-k}(0)&\\ &= 2\sum_{r=1}^{q-1}(-1)^r\Big(\Big(\sqrt{5}\,\frac{r}{q}+\beta\Big)F_j+F_{j-1}\Big)^n\\ &= 2q^{-n}\sum_{r=1}^{q-1}(-1)^r\big(\sqrt{5}rF_j+q(\beta F_j+F_{j-1})\big)^n\\ &= 2q^{-n}\sum_{r=1}^{q-1}(-1)^r\big(r\alpha^j+(q-r)\beta^j\big)^n. \qquad \square\end{aligned}$$

The special instances for $j=1$, and $q=3$ and $q=5$, respectively, take the form

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^{n-k}\big(3^{-(n-k)}-1\big)E_{n-k}(0) = 2\sqrt{5}\cdot 3^{-n}F_{2n}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^{n-k}\big(5^{-(n-k)}-1\big)L_kE_{n-k}(0) = \begin{cases}2\cdot 5^{(1-n)/2}(F_{2n}-F_n), & \text{if } n \text{ is even;}\\ 2\cdot 5^{-n/2}(L_{2n}-L_n), & \text{if } n \text{ is odd.}\end{cases}$$

5. Mixed Polynomial Identities

In this section, we derive some mixed identities involving Bernoulli (Euler) polynomials and Bernoulli, Fibonacci and Lucas numbers.

Theorem 5.1. *For each $n, j \geq 0$ and $x \in \mathbb{C}$,*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(\pm\sqrt{5}F_j)^k L_{j(n-k)}B_k(x)&\\ &= 2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}20^k F_j^{2k}\big(\pm\sqrt{5}F_j(2x-1)+L_j\big)^{n-2k}B_{2k}. \qquad (5.1)\end{aligned}$$

Proof. The result follows from the functional relation

$$H(x,\pm\sqrt{5}F_j z)L(z)=\sqrt{5}F_j e^{(\pm(x-1/2)\sqrt{5}F_j+1/2L_j)z}z\coth\left(\frac{\sqrt{5}F_j}{2}z\right)$$

and the well-known power series $\coth z=\sum_{n=0}^{\infty}\frac{4^n B_{2n}}{(2n)!}z^{2n-1}$. □

If $x=1/2$, from (5.1) using $B_n(1/2)=(2^{1-n}-1)B_n$, we obtain the following result.

Corollary 5.2. *For each $n,j\ge 0$,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^k(2^{1-2k}-1)F_j^{2k}L_{j(n-2k)}B_{2k}=2^{1-n}L_j^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}20^k\left(\frac{F_j}{L_j}\right)^{2k}B_{2k}.$$

For example,

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(2^{1-2k}-1)5^kB_{2k}L_{n-2k}=2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}20^kB_{2k}$$

and

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(2^{1-2k}-1)20^kB_{2k}L_{3(n-2k)}=2^{1+n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^kB_{2k}.$$

If $x=\alpha$ and $x=\beta$, where $\alpha=(1+\sqrt{5})/2$ is the golden ratio and $\beta=-1/\alpha$, from (5.1) we have the following corollary.

Corollary 5.3. *For each $n,j\ge 0$,*

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^kL_{j(n-k)}B_k(\alpha)=2L_{j+1}^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^k\left(\frac{F_j}{L_{j+1}}\right)^{2k}B_{2k},$$

$$\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5}F_j)^kL_{j(n-k)}B_k(\alpha)=2(-1)^nL_{j-1}^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^k\left(\frac{F_j}{L_{j-1}}\right)^{2k}B_{2k}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^kL_{j(n-k)}B_k(\beta)=2(-1)^nL_{j-1}^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^k\left(\frac{F_j}{L_{j-1}}\right)^{2k}B_{2k},$$

$$\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5}F_j)^kL_{j(n-k)}B_k(\beta)=2L_{j+1}^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}5^k\left(\frac{F_j}{L_{j+1}}\right)^{2k}B_{2k}.$$

Finally, we present the theorem for Fibonacci-Euler pair.

Theorem 5.4. *For each $n \geq 0$ and $j \geq 1$,*

$$\begin{aligned}&(n+1)\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^k F_{j(n-k)}E_k(x)\\&= 4\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1}{2k}5^{k-1}(4^k-1)F_j^{2k-1}\Big(\frac{F_j}{2}+F_{j-1}+\sqrt{5}F_j\Big(x-\frac{1}{2}\Big)\Big)^{n+1-2k}B_{2k}\end{aligned}$$

and

$$\begin{aligned}&(-1)^n(n+1)\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5}F_j)^k F_{j(n-k)}E_k(x)\\&= 4\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1}{2k}5^{k-1}(4^k-1)F_j^{2k-1}\Big(\frac{F_j}{2}+F_{j-1}-\sqrt{5}F_j\Big(x-\frac{1}{2}\Big)\Big)^{n+1-2k}B_{2k}.\end{aligned}$$

Proof. Let $F(z)$ be the exponential generating function for $(F_{jn})_{n\geq 0}$ with $j \geq 1$. Then, using the Binet formula for F_n, we get

$$F(z) = \frac{2}{\sqrt{5}}e^{(1/2F_j+F_{j-1})z}\sinh\Big(\frac{\sqrt{5}F_j}{2}z\Big).$$

The relations follows from

$$I(x,\pm\sqrt{5}F_j z)F(z) = \frac{2}{\sqrt{5}}e^{\left(F_j/2+F_{j-1}+(x-1/2)\sqrt{5}F_j\right)z}\tanh\Big(\frac{\sqrt{5}F_j}{2}z\Big)$$

and power series

$$\tanh z = \sum_{n=0}^{\infty}\frac{4^n(4^n-1)B_{2n}}{(2n)!}z^{2n-1}. \qquad \square$$

In particularly, from Theorem 5.4 we have the following Euler-Bernoulli-Fibonacci-Lucas identity:

$$\begin{aligned}&(n+1)\sum_{k=1}^{n}\binom{n}{k}\Big(\pm\frac{\sqrt{5}F_j}{2}\Big)^k F_{j(n-k)}E_k\\&= \frac{L_j^n}{2^{n-2}}\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1}{2k}5^{k-1}(4^k-1)\left(\frac{2F_j}{L_j}\right)^{2k-1}B_{2k}.\end{aligned}$$

6. Conclusion

In this paper, we have documented identities relating Euler numbers (polynomials) to balancing and Lucas-balancing polynomials. We have also derived a general identity involving Euler polynomials and Lucas numbers in arithmetic progression. All results must be seen as companion results to the Fibonacci-Bernoulli pair from [7]. In the future, we will work on more identities connecting Bernoulli/Euler numbers (polynomials) with Fibonacci/Lucas numbers (polynomials).

Acknowledgements. We would like to thank the referee for valuable suggestions.

References

[1] P. F. Byrd, *Relations between Euler and Lucas numbers*, Fibonacci Quart., **13**(1975), 111–114.

[2] D. Castellanos, *A generalization of Binet's formula and some of its consequences*, Fibonacci Quart., **27(5)**(1989), 424–438.

[3] K. Dilcher, *Bernoulli and Euler Polynomials*, in: F. W. J. Olver, D. M. Lozier, R. F. Boisvert, C. W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.

[4] R. Frontczak, *On balancing polynomials*, Appl. Math. Sci., **13**(2019), 57–66.

[5] R. Frontczak, *Relating Fibonacci numbers to Bernoulli numbers via balancing polynomials*, J. Integer Seq., **22**(2019), Article 19.5.3.

[6] R. Frontczak and T. Goy, *Additional close links between balancing and Lucas-balancing polynomials*, Adv. Stud. Contemp. Math., **31(3)**(2021), 287–300.

[7] R. Frontczak and T. Goy, *More Fibonacci-Bernoulli relations with and without balancing polynomials*, Math. Comm., **26**(2021), 215–226.

[8] R. Frontczak and Z. Tomovski, *Generalized Euler-Genocchi polynomials and Lucas numbers*, Integers, **20**(2020), #A52.

[9] R. P. Kelisky, *On formulas involving both the Bernoulli and Fibonacci numbers*, Scripta Math., **23**(1957), 27–32.

[10] D. S. Kim and T. Kim, *On sums of finite products of balancing polynomials*, J. Comput. Appl. Math., **377**(2020), 112913.

[11] T. Kim, C. S. Ryoo, D. S. Kim and J. Kwon, *A difference of sum of finite product of Lucas-balancing polynomials*, Adv. Stud. Contemp. Math., **30(1)**(2020), 121–134.

[12] T. Kim, D. S. Kim, D. V. Dolgy and J. Kwon, *A note on sums of finite products of Lucas-balancing polynomials*, Proc. Jangjeon Math. Soc., **23(1)**(2020), 1–22.

[13] Y. Meng, *A new identity involving balancing polynomials and balancing numbers*, Symmetry, **11**(2019), 1141.

[14] G. Ozdemir and Y. Simsek, *Identities and relations associated with Lucas and some special sequences*, AIP Conf. Proc., **1863**(2017), 300003.

[15] B. K. Patel, N. Irmak and P. K. Ray, *Incomplete balancing and Lucas-balancing numbers*, Math. Reports, **20(70)(1)**(2018), 59–72.

[16] P. K. Ray, *Balancing polynomials and their derivatives*, Ukrainian Math. J., **69(4)** (2017), 646–663.

[17] N. J. A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, available at `https://oeis.org`.

[18] T. Wang and Z. Zhang, *Recurrence sequences and Nörlund-Euler polynomials*, Fibonacci Quart., **34(4)**(1996), 314–319.

[19] P. T. Young, *Congruences for Bernoulli-Lucas sums*, Fibonacci Quart., **55(5)**(2017), 201–212.

[20] Z. Zhang and L. Guo, *Recurrence sequences and Bernoulli polynomials of higher order*, Fibonacci Quart., **33(3)**(1995), 359–362.

[21] T. Zhang and Y. Ma, *On generalized Fibonacci polynomials and Bernoulli numbers*, J. Integer Seq., **8**(2005), Article 05.5.3.