KYUNGPOOK Math. J. 61(2021), 495-512
https://doi.org/10.5666/KMJ.2021.61.3.495
pISSN 1225-6951 eISSN 0454-8124

Value Distribution of L-functions and a Question of Chung-Chun Yang

Xiao-Min Li* and Qian-Qian Yuan
Department of Mathematics, Ocean University of China, Qingdao, Shandong 266100, P. R. China
e-mail : lixiaomin@ouc.edu.cn *and* yuanqianqian92@163.com

Hong-Xun Yi
Department of Mathematics, Shandong University, Jinan, Shandong 250199, P. R. China
e-mail : hxyi@sdu.edu.cn

Abstract. We study the value distribution theory of L-functions and completely resolve a question from Yang [10]. This question is related to L-functions sharing three finite values with meromorphic functions. The main result in this paper extends corresponding results from Li [10].

1. Introduction and Main Results

Throughout this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We assume that the reader is familiar with the basic notions and results in the Nevanlinna theory, which can be found, for example, in [4, 9, 18, 19]. It will be convenient to let $E \subset (0, +\infty)$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r, h)$ the Nevanlinna characteristic function of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o(T(r, h))$, as $r \notin E$ and r runs to infinity. Let k be a positive integer, and let a be a complex value in the extended complex plane. Next we denote by $N_{(k}(r, 1/(h-a))$ the counting function of those a-points of the nonconstant meromorphic function h in $|z| < r$, where each point in $N_{(k}(r, 1/(h-a))$ is counted

* Corresponding Author.

Received February 22, 2017; revised May 2, 2021; accepted May 18, 2021.
2020 Mathematics Subject Classification: Primary 11M36; Secondary 30D35.
Key words and phrases: Nevanlinna theory, Meromorphic functions, L-functions, Shared Values, Uniqueness theorems.
This work was supported in part by the NSFC (No.11171184), the NSF of Shandong Province, China (No. ZR2019MA029) and the FRFCU (No. 3016000841964007).

according to its multiplicity, and each point in $N_{(k}(r, 1/(h-a))$ is of multiplicity $\geq k$. Here $N_{(k}(r, 1/(h-\infty))$ means $N_{(k}(r, h)$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points in the complex plane, and each common a-point of f and g has the same multiplicities related to f and g. We say that f and g share the value a IM, provided that f and g have the same a-points in the complex plane (cf.[18]). In terms of sharing values, two nonconstant meromorphic functions in the complex plane must be identically equal if they share five values IM, and one must be a Möbius transformation of the other one if they share four values CM; the numbers "five" and "four" are the best possible, as shown by Nevanlinna (cf. [15, 18]). L-functions, with the Riemann zeta function as a prototype, are important objects in number theory, and value distribution of L-functions has been studied extensively. See, for example, Ki[7], Li[10, 11, 12] Hu-Li[6] and Steuding [17].

This paper concerns the question of how an L-function is uniquely determined in terms of the pre-images of complex values in the extended complex plane, or sharing values. We refer the reader to the monograph [17] for a detailed discussion on the topic and related works. Throughout the paper, an L-function always means an L-function L in the Selberg class, which includes the Riemann zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$ and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined to be a Dirichlet series $L(s) = \sum\limits_{n=1}^{\infty} a(n)n^{-s}$ satisfying the following axioms (cf.[16, 17]):

(i) Ramanujan hypothesis. $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.

(ii) Analytic continuation. There is a nonnegative integer k such that $(s-1)^k L(s)$ is an entire function of finite order.

(iii) Functional equation. L satisfies a functional equation of type

$$\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product hypothesis. $L(s) = \prod_p \exp\left(\sum\limits_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, where the product is taken over all prime numbers p.

We first recall the following result due to Steuding [17], which actually holds without the Euler product hypothesis:

Theorem A.([17, p.152]) *If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $L_1 = L_2$.*

Remark 1.1. In 2016, Hu-Li [6] pointed out that Theorem A is false when $c = 1$. A counter example was given by Hu-Li [6, Remark 4] as follows: Let $L_1(s) = 1 + \frac{2}{4^s}$ and $L_2(s) = 1 + \frac{3}{9^s}$. Then L_1 and L_2 trivially satisfy axioms (i) and (ii). Also, one can check that L_1 satisfies the functional equation

$$2^s L(s) = 2^{1-s}\overline{L(1-\overline{s})},$$

and L_2 satisfies the functional equation

$$3^s L(s) = 3^{1-s}\overline{L(1-\overline{s})}.$$

Thus, L_1 and L_2 also satisfy axiom (iii). It is clear that $L_1 - 1$ and $L_2 - 1$ do not have any zeros and thus satisfy the assumptions of Theorem A with $c = 1$, but $L_1 \not\equiv L_2$.

Theorem A implies that two L-functions with $a(1) = 1$ must be identically equal if they have the same zeros with counting multiplicities. Two L-functions with "enough" common zeros without counting multiplicities are expected to be dependent in a certain sense (cf.[1]). Since L-functions are analytically continued as meromorphic functions in the complex plane, in order to study how an L-function is uniquely determined by pre-images of complex values in the extended plane, one should examine the situation involving an arbitrary L-function and an arbitrary meromorphic function. The first observation on this uniqueness question is that the above theorem no longer holds for an L-function and a meromorphic function. For instance, the function ζ and ζe^g, where g is any entire function, share 0 CM, but they are not identically equal. It is natural to consider two sharing values, i.e., whether two sharing values with counting multiplicities would force an L-function and a meromorphic function to be identically equal. This turns out not to be the case either. For instance, consider the function $f = \frac{2\zeta}{\zeta+1}$. It is then clear that ζ and f share 0, 1 CM, but they are not identically equal. Observe, however, when considering L-functions, these functions have only one possible pole at $s = 1$, which is implicit in the conditions of the above theorem. Thus, this leads us to consider the natural objects of those meromorphic functions with finitely many poles (cf.[10]). The following uniqueness theorem was then established by Li [10] in 2009:

Theorem B.([10]) *Let f be a meromorphic function in the complex plane such that f has finitely many poles, and let a and b be two distinct finite values. If f and a nonconstant L-function L share a CM and b IM, then $f = L$.*

Remark 1.2. *The number "two" in Theorem B is the best possible, as shown by the above example with $L = \zeta$ and $f = \zeta e^g$.*

By Theorem B we can get the following result:

Corollary A.([10]) *Let f be a meromorphic function in the complex plane, and let a, b, c be three distinct values in the extended complex plane such that $a \in \mathbb{C}$ and $b = \infty$ or $c = \infty$. If f shares a CM and b, c IM with a nonconstant L-function L, then $f = L$.*

In a communication to Professor Li, Yang asked the following question:

Question A.([10]) *If f is a meromorphic function in $\mathbb{C}$ that shares three distinct values a, b CM and c IM with the Riemann zeta function ζ, where $c \notin \{a, b, 0, \infty\}$, is f equal to ζ ?*

Remark 1.3. *By taking $L = \zeta$ in Corollary A, we can find that the conclusion of Corollary A holds, which gives a positive answer to Question A provided that any one of a, b, c is ∞ in Question A.*

Next we consider the first, the second and the fourth Painlevé equations given respectively by

(PI) $\omega'' = z + 6\omega^2$,

(PII) $\omega'' = 2\omega^3 + z\omega + \alpha$ with $\alpha \in \mathbb{C}$,

(PIV) $2\omega\omega'' = (\omega')^2 + 3\omega^4 + 8z\omega^3 + 4(z^2 - \alpha)\omega^2 + \beta$ with $\alpha, \beta \in \mathbb{C}$.

In 2007, Lin-Tohge [13] obtained some results similar to Theorem B. Indeed, Lin-Tohge [13] studied some shared-value properties of the first, the second and the fourth Painlevé transcendents by applying their distinctive value distribution, and proved the following results:

Theorem C.([13, Theorem 1]) *Let ω be an arbitrary nonconstant solution of one of the equations (PI), (PII) and (PIV), and let f be a nonconstant meromorphic function that shares four distinct values a_1, a_2, a_3, a_4 IM with ω, where a_1, a_2, a_3, a_4 are four distinct values in the extended complex plane, then $f = \omega$.*

Theorem D.([13, Theorem 2]) *Let ω be an arbitrary solution of (PI) and f be a meromorphic function. Assume that f and ω share two distinct values a_1 and a_2 CM, where a_1 and a_2 are two distinct values in the extended complex plane, then we have*

$$\lim_{r\to\infty} \frac{N_0\left(r, \frac{1}{f-\omega}\right)}{T(r,\omega)} = 0 \quad or \quad \lim_{r\to\infty} \frac{N_0\left(r, \frac{1}{f-\omega}\right)}{T(r,\omega)} = \infty.$$

Theorem F.([13, Theorem 3]) *Let ω be an arbitrary solution of (PI). Then there does not exist a pair of two finite values a, b such that $E_\omega(\{a\}) \subseteq E_{\omega'}(\{b\})$. Here $E_\omega(\{a\})$ denotes the set of a-points of ω in the complex plane, where each a-point of ω with multiplicity m is counted m times in the set $E_\omega(\{a\})$. While $E_{\omega'}(\{b\})$*

denotes the set of b-points of ω' in the complex plane, where each b-point of ω' with multiplicity m is counted m times in the set $E_{\omega'}(\{b\})$.

Regarding Theorem B, one may ask, what can be said about the conclusion of Theorem B if we remove the assumption "f has finitely many poles in the complex plane" in Theorem B. In this direction, we will prove the following result that is an extension to Theorem B:

Theorem 1.1. *Let f be a meromorphic function in the complex plane, and let a, b, and c be three distinct finite values. If f and a nonconstant L-function L share a, b CM and c IM, then $f = L$.*

By Theorem 1.1 we get the following result:

Corollary 1.2. *If f is a meromorphic function in $\mathbb{C}$ that shares three distinct values a, b CM and c IM with the Riemann zeta function ζ, where $c \notin \{a, b, \infty\}$, then $f = \zeta$.*

As a special case of Corollary 1.2, we give the following result which completely resolves Question A:

Corollary 1.3. *If f is a meromorphic function in $\mathbb{C}$ that shares three distinct values a, b CM and c IM with the Riemann zeta function ζ, where $c \notin \{a, b, 0, \infty\}$, then $f = \zeta$.*

In the same manner as in the proof of Theorem 1.1, we can get the following result by Lemma 2.10 in Section 2 of the present paper:

Theorem 1.4. *Let f be a meromorphic function in the complex plane, let L be a nonconstant L-function, and let a, b, and c be three distinct finite values in the complex plane. Suppose that $f^{(k)}$ and $L^{(k)}$ share a, b CM and c IM, where $k \geq 1$ is a positive integer. Then $f^{(k)} = L^{(k)}$.*

Throughout this paper, we will apply Nevanlinna theory to prove the main result in this paper.

2. Preliminaries

In this section, we will give some important lemmas to prove the main result of the present paper. For convenience in stating the following first result from Gundersen [3], we shall use the following notation: we shall let (f, H) denote a pair that consists of a transcendental meromorphic function f and a finite set

$$H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$$

of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $1 \leq i \leq q$.

Lemma 2.1.([3, Corollary 2]) *Let (f, H) be a given pair where f has finite order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1, \infty)$ that has* *finite logarithmic measure, such that for all s satisfying $|s| \notin E \cup [0,1]$ and for all $(k,j) \in H$, we have*

$$\left|\frac{f^{(k)}(s)}{f^{(j)}(s)}\right| \leq |s|^{(k-j)(\rho-1+\varepsilon)}.$$

The following result is due to Mokhon-ko [14]:

Lemma 2.2.(Valiron-Mokhon-ko lemma, [14]) *Let f be a nonconstant meromorphic function, and let $F = \sum_{k=0}^{p} a_k f^k \Big/ \sum_{j=0}^{q} b_j f^j$ be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then $T(r,F) = d\,T(r,f) + O(1)$, where $d = \max\{p,q\}$.*

We also need the following result due to Lahiri-Sarkar [8]:

Lemma 2.3.([8, Lemma 6]) *Let F and G be two distinct nonconstant meromorphic functions that share $0, 1, \infty$ IM. If F is a Möbius transformation of G, then F and G satisfy one of the following six relations: (i) $FG = 1$; (ii) $(F-1)(G-1) = 1$; (iii) $F+G = 1$; (iv) $F = cG$; (v) $F-1 = c(G-1)$; (vi) $((c-1)F+1)((c-1)G-c) = -c$. Here $c \neq 0, 1$ is a complex number.*

The following result is from Gundersen [2]:

Lemma 2.4.([2, Theorem 3]) *Suppose that f and g are two nonconstant meromorphic functions that share 0, 1, ∞ IM. Then*

$$\left(\frac{1}{3} + o(1)\right) T(r,g) < T(r,f) < (3 + o(1))T(r,g),$$

as $r \to \infty$ and $r \notin E$, where $E \subset (0,+\infty)$ is a subset of finite linear measure.

Lemma 2.5.([20, proof of Lemma 4]) *Let f and g be two distinct nonconstant meromorphic functions that share 0, 1 CM and ∞ IM. If $\overline{N}(r,f) = S(r,f)$, then $N_{(2}\left(r, \frac{1}{f}\right) + N_{(2}\left(r, \frac{1}{f-1}\right) = S(r,f)$.*

Lemma 2.6.([21, Lemma 6]) *Let f_1 and f_2 be two nonconstant meromorphic functions such that*

$$\overline{N}(r,f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r)$$

for $1 \leq j \leq 2$. Then either $\overline{N}_0(r,1;f_1,f_2) = S(r)$ or there exist two integers p and q $(|p|+|q| > 0)$ such that $f_1^p \cdot f_2^q = 1$, where $\overline{N}_0(r,1;f_1,f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2, $T(r) = T(r,f_1) + T(r,f_2)$ and $S(r) = o(T(r))$, as $r \to \infty$ and $r \notin E$, $E \subset (0,+\infty)$ is a subset of r of finite linear measure.

For introducing the following result, we first give the following notation (cf.[20]): Let F and G be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞

IM. Next we use $N_0(r)$ to denote the counting function of those zeros of $f-g$ that are not zeros of F, $F-1$ and $1/F$, where each point in $N_0(r)$ is counted according to its multiplicity. We denote by $\overline{N}_0(r)$ the reduced form of $N_0(r)$.

The following lemma is essentially due to Zhang [21]:

Lemma 2.7.([21, proof of Theorem 1 and Theorem 2]) *Let F and G be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, and let $N_0(r) \neq S(r,f)$. If F is a Möbius transformation of G, then*

$$N_0(r) = T(r,F) + S(r,F).$$

If F is not any Möbius transformation of G, then

$$N_0(r) \leq \frac{1}{2}T(r,F) + S(r,F),$$

and F and G assume one of the following relations:

(i) $F = \frac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1}, G = \frac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1}$;

(ii) $F = \frac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1}, G = \frac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1}$;

(iii) $F = \frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}, G = \frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}$. *Here γ is a nonconstant entire function, s and $k \geq 2$ are positive integers such that s and $k+1$ are relatively prime and $1 \leq s \leq k$.*

Lemma 2.8.([22]) *Let $s\,(>0)$ and t are relatively prime integers, and let c be a finite complex number such that $c^s = 1$, then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.*

Finally we prove the following result which plays an important role in proving the main results of this paper:

Lemma 2.9. *Let F and G be two distinct nonconstant meromorphic functions sharing 0, 1 CM and ∞ IM. Suppose that F is not a Möbius transformation of G. If $\overline{N}(r,F) = S(r,F)$, then*

(i) $N_0\left(r, \frac{1}{F'}\right) = \overline{N}_0\left(r, \frac{1}{F'}\right) + S(r,F), \overline{N}\left(r, \frac{1}{F'}\right) = \overline{N}_0\left(r, \frac{1}{F'}\right) + S(r,F)$, *the same identities hold for G.*

(ii) $T(r,F) = \overline{N}\left(r, \frac{1}{G'}\right) + N_0(r) + S(r,F)$, $T(r,G) = \overline{N}\left(r, \frac{1}{F'}\right) + N_0(r) + S(r,F)$,
$N_0(r) = \overline{N}_0(r) + S(r,F)$.

Here $N_0(r, \frac{1}{F'})(\overline{N}_0(r, \frac{1}{F'}))$ denotes the counting function corresponding to the zeros of F' that are not zeros of F and $F-1$ (ignoring multiplicities) and $N_0(r)$ ($\overline{N}_0(r)$) is the counting function of the zeros of $F-G$ that are not zeros of G, $G-1$ and $\frac{1}{G}$ (ignoring multiplicities).

Proof. First of all, we set

$$\frac{F-1}{G-1}=h_1, \tag{2.1}$$

$$\frac{F}{G}=h_2 \tag{2.2}$$

and

$$h_0=\frac{h_1}{h_2}. \tag{2.3}$$

By Lemma 2.4 we have

$$S(r,F)=S(r,G). \tag{2.4}$$

By (2.4), Lemma 2.5 and the assumption of Lemma 2.9 we have

$$N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}(r,F)=S(r,F) \tag{2.5}$$

and

$$N_{(2}\left(r,\frac{1}{G}\right)+N_{(2}\left(r,\frac{1}{G-1}\right)+\overline{N}(r,G)=S(r,F). \tag{2.6}$$

By (2.1)-(2.3), (2.5), (2.6) and the assumption that F and G share 0, 1 CM and ∞ IM we get

$$\overline{N}\left(r,h_j\right)+\overline{N}\left(r,\frac{1}{h_j}\right)=S(r,F)\quad\text{with}\quad 0\le j\le 2. \tag{2.7}$$

By (2.1)-(2.3) and the assumption that F is not a Möbius transformation .of G we can see that none of h_1, h_2 and h_0 are constants. Therefore $h_1\not\equiv 1$, $h_2\not\equiv 1$ and $h_0\not\equiv 1$. This together with (2.1)-(2.3) gives

$$F=\frac{h_1-1}{h_0-1} \tag{2.8}$$

and

$$G=\frac{h_1^{-1}-1}{h_0^{-1}-1}. \tag{2.9}$$

Set

$$h=\frac{\frac{h_1'}{h_1}}{\frac{h_0'}{h_0}}=\frac{\frac{h_1'}{h_1}}{\frac{h_1'}{h_1}-\frac{h_2'}{h_2}}. \tag{2.10}$$

Then from (2.1), (2.2), (2.3) and (2.10) we can deduce

$$T(r,h)=S(r,F). \tag{2.11}$$

If

$$\frac{h_1'}{h_1}\cdot(h-1)-h'=0,$$

then

$$h_1=c_1(h-1), \tag{2.12}$$

where $c_1\neq 0$ is a finite complex number. By (2.11) and (2.12) we deduce

$$T(r,h_1)=S(r,F). \tag{2.13}$$

Again from (2.10) and (2.12) we have

$$\frac{h_0'}{h_0}=\frac{\frac{c_1h_1'}{h_1}}{h_1+c_1}=-\frac{(c_1h_1^{-1}+1)'}{c_1h_1^{-1}+1}. \tag{2.14}$$

By integrating two sides of (2.14) we can get

$$h_0=\frac{c_2}{c_1h_1^{-1}+1}, \tag{2.15}$$

where $c_2\neq 0$ is a finite complex number. By (2.13) and (2.15) we have

$$T(r,h_0)=T(r,h_1)+O(1)=S(r,F). \tag{2.16}$$

By (2.8), (2.13) and (2.16) we can get $T(r,F)=S(r,F)$, this is impossible. Thus

$$\frac{h_1'}{h_1}\cdot(h-1)-h'\not\equiv 0,$$

which together with (2.8) gives

$$F-h=\frac{h_1-h_0h+h-1}{h_0-1}. \tag{2.17}$$

Set

$$H=(F-h)(h_0-1)=h_1-h_0h+h-1. \tag{2.18}$$

By (2.10) and (2.18) we get

$$\frac{H'}{H}-\frac{h_1'}{h_1}=\frac{(h_1-h_0h+h-1)'-\frac{h_1'}{h_1}\cdot(h_1-h_0h+h-1)}{(F-h)(h_0-1)}=\frac{\frac{h_1'}{h_1}\cdot(h-1)-h'}{F-h},$$

and so we have

$$\frac{1}{F-h}=\frac{\frac{H'}{H}-\frac{h_1'}{h_1}}{\frac{h_1'}{h_1}\cdot(h-1)-h'}. \tag{2.19}$$

By (2.3), (2.11) and (2.19) we deduce

$$m\left(r,\frac{1}{F-h}\right)=S(r,F) \tag{2.20}$$

and

$$N_{(2}(r,\frac{1}{F-h})=S(r,F). \tag{2.21}$$

By (2.1), (2.3) and (2.9) we have

$$\frac{F-G}{G-1}=h_1-1 \quad \text{and} \quad G=\frac{h_1-1}{h_1-h_2}. \tag{2.22}$$

Thus

$$\frac{G'(F-G)}{G(G-1)}=\frac{(\frac{h_2'}{h_2}-\frac{h_1'}{h_1})\cdot h_1+\frac{h_1'}{h_1}\cdot h_0-\frac{h_2'}{h_2}}{h_0-1}. \tag{2.23}$$

On the other hand, by (2.10) and (2.19) we have

$$(F-h)\cdot(\frac{h_2'}{h_2}-\frac{h_1'}{h_1})=\frac{(\frac{h_2'}{h_2}-\frac{h_1'}{h_1})\cdot h_1+\frac{h_1'}{h_1}\cdot h_0-\frac{h_2'}{h_2}}{h_0-1}. \tag{2.24}$$

By (2.23) and (2.24) we have

$$-\frac{h_0'}{h_0}\cdot(F-h)=\frac{G'(F-G)}{G(G-1)}. \tag{2.25}$$

By (2.3), (2.5), (2.21), (2.22) and (2.25) we easily deduce

$$N\left(r,\frac{1}{F-h}\right)=N_0(r)+N_0\left(r,\frac{1}{G'}\right)+S(r,F), \tag{2.26}$$

$$N_0(r)=\overline{N}_0(r)+S(r,F) \tag{2.27}$$

and

$$N_0\left(r,\frac{1}{G'}\right)=\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,F). \tag{2.28}$$

By (2.28) and Lemma 2.5 we deduce

$$N_0\left(r, \frac{1}{G'}\right) = \overline{N}\left(r, \frac{1}{G'}\right) + S(r, F). \tag{2.29}$$

By (2.11), (2.20) and (2.26) we deduce

$$T(r, F) = N_0(r) + N_0\left(r, \frac{1}{G'}\right) + S(r, F). \tag{2.30}$$

In the same manner as above we get

$$N_0\left(r, \frac{1}{F'}\right) = \overline{N}_0\left(r, \frac{1}{F'}\right) + S(r, F), \quad N_0\left(r, \frac{1}{F'}\right) = \overline{N}\left(r, \frac{1}{F'}\right) + S(r, F) \tag{2.31}$$

and

$$T(r, G) = N_0(r) + N_0\left(r, \frac{1}{F'}\right) + S(r, F). \tag{2.32}$$

By (2.28), (2.29) and (2.31) we get the conclusion (i) of Lemma 2.9. By (2.27), (2.30) and (2.32) we get (ii) of Lemma 2.9. This completely proves Lemma 2.9. □

Lemma 2.10.([5]) *Let f be a transcendental meromorphic function in C. Then, for each $K > 1$, there exists a set $M(K)$ of the lower logarithmic density at most $d(K) = 1 - (2e^{K-1} - 1)^{-1} > 0$, that is*

$$\underline{\log dens} M(K) = \liminf_{r\to\infty} \frac{1}{\log r} \int_{M(K)\cap[1,r]} \frac{dt}{t} \le d(K),$$

such that, for every positive integer k,

$$\limsup_{\substack{r\to\infty \\ r\notin M(K)}} \frac{T(r, f)}{T(r, f^{(k)})} \le 3eK.$$

3. Proof of Theorem 1.1.

First of all, we denote by d the degree of L. Then $d = 2\sum_{j=1}^{K} \lambda_j > 0$ (cf.[17, p.113]), where K and λ_j are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-function. Therefore, by Steuding [17, p.150] we have

$$T(r, L) = \frac{d}{\pi} r \log r + O(r), \tag{3.1}$$

which together with the definition of the order of a meromorphic function implies that

$$\rho(L) = 1. \tag{3.2}$$

By noting that L has only one possible pole at $s = 1$, we have

$$N(r, L) \leq \log r + O(1), \quad \text{as} \quad r \to \infty. \tag{3.3}$$

On the other hand, by the assumption that f and L share a, b CM and c IM, we have by the second fundamental theorem that

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + \overline{N}\left(r, \frac{1}{f-c}\right) + O(\log r + \log T(r,f)) \\ &= \overline{N}\left(r, \frac{1}{L-a}\right) + \overline{N}\left(r, \frac{1}{L-b}\right) + \overline{N}\left(r, \frac{1}{L-c}\right) + O(\log r + \log T(r,f)) \\ &\leq 3T(r,L) + O(\log r + \log T(r,f)), \end{aligned}$$

i.e.,

$$T(r,f) \leq 3T(r,L) + O(\log r + \log T(r,f)), \tag{3.4}$$

as $r \to \infty$ possibly outside of an exceptional set of finite linear measure. Similarly

$$T(r,L) \leq 3T(r,f) + O(\log r + \log T(r,L)), \tag{3.5}$$

as $r \to \infty$ possibly outside of an exceptional set of finite linear measure.

By (3.4), (3.5), the definition of the order of a meromorphic function and the standard reasoning of removing an exceptional set we deduce

$$\rho(f) = \rho(L) = 1. \tag{3.6}$$

Now we set

$$\psi = \frac{f'}{(f-a)(f-b)} - \frac{L'}{(L-a)(L-b)} = \frac{1}{a-b}\left(\frac{f'}{f-a} - \frac{f'}{f-b} - \frac{L'}{L-a} + \frac{L'}{L-b}\right). \tag{3.7}$$

By the assumption that f and L share a, b CM, we can deduce from (3.7) that ψ is an entire function. Therefore, by (3.6) and Lemma 2.1 we have

$$|\psi(z)| \leq O(|z|^{\varepsilon}) \tag{3.8}$$

for all z satisfying $|z| \notin E \cup [0,1]$. Here $E \subset (1, \infty)$ is some subset that has finite logarithmic measure.

By (3.8) we can see that ψ is reduced to a constant, say $\psi = c_1$. This together with (3.7) gives

$$\frac{(f(z)-a)(L(z)-b)}{(f(z)-b)(L(z)-a)} = A_1 e^{c_1(a-b)z}, \tag{3.9}$$

where $A_1 \neq 0$ is a constant. By noting that

$$T\left(r, A_1 e^{c_1(a-b)z}\right) = \frac{|c_1(a-b)|r}{\pi}(1+o(1)) + O(1), \quad \text{as} \quad |z| = r \to \infty, \tag{3.10}$$

we deduce by (3.1), (3.5) and (3.10) that

$$T\left(r, A_1 e^{c_1(a-b)z}\right) = o(T(r,L)) \quad \text{and} \quad T\left(r, A_1 e^{c_1(a-b)z}\right) = o(T(r,f)), \tag{3.11}$$

as $r \to \infty$.

By (3.9) we consider the following two cases:

Case 1. Suppose that there exists a subset $I \subset (0, +\infty)$ with infinite linear measure such that

$$\lim_{\substack{r\to\infty \\ r\in I}} \frac{\overline{N}\left(r, \frac{1}{f-c}\right)}{r} = +\infty. \tag{3.12}$$

Next we prove

$$A_1 e^{c_1(a-b)z} \equiv 1. \tag{3.13}$$

Indeed, if

$$A_1 e^{c_1(a-b)z} \not\equiv 1, \tag{3.14}$$

by (3.9), (3.11), (3.14) and the assumption that f and L share c IM we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f-c}\right) &\leq \overline{N}\left(r, \frac{1}{A_1 e^{c_1(a-b)z} - 1}\right) = T\left(r, A_1 e^{c_1(a-b)z}\right) + O(1) \\ &= \frac{|c_1(a-b)|r}{\pi}(1+o(1)) + O(1),\end{aligned}$$

which contradicts (3.12), and so (3.13) is valid. By (3.9) and (3.13) we get the conclusion of Theorem 1.1.

Case 2. Suppose that at most there exists a subset $E \subset (0, +\infty)$ with finite linear measure such that

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}\left(r, \frac{1}{f-c}\right)}{r} < +\infty. \tag{3.15}$$

Then, by (3.15) we have

$$\overline{N}\left(r, \frac{1}{f-c}\right) \leq A_2 r, \tag{3.16}$$

as $r \to \infty$ and $r \notin E$, where $A_2 > 0$ is a constant. Now we set

$$F = \frac{f-a}{f-c} \cdot \frac{b-c}{b-a}, \quad G = \frac{L-a}{L-c} \cdot \frac{b-c}{b-a}. \tag{3.17}$$

Noting the assumption that f and L share a, b CM, and c IM, we have by (3.17) that F and G share 0, 1 CM, and ∞ IM. Moreover, by (2.4), (3.1), (3.4), (3.5), (3.6), (3.16), (3.17) and Lemma 2.2 we deduce

$$\overline{N}(r, F) = S(r, F), \quad \overline{N}(r, G) = S(r, F). \tag{3.18}$$

We discuss the following two subcases:

Subcase 2.1. Suppose that F is a Möbius transformation of G. Then, by Lemma 2.3 we can see that F and G satisfy one of the six relations (i)-(vi) of Lemma 2.3. We consider the following two subcases:

Subcase 2.1.1. Suppose that F and G satisfy one of (i), (ii) and (vi) of Lemma 2.3. We discuss this as follows:

Suppose that F and G satisfy (i) of Lemma 2.3. Then, 0 and ∞ are exceptional values of F and G. Therefore,

$$F = e^{\alpha}, \quad G = e^{-\alpha}, \tag{3.19}$$

where α is a nonconstant entire function. By the right formulae of (3.17) and (3.19) we have

$$\frac{L-a}{L-c} \cdot \frac{b-c}{b-a} = e^{-\alpha}. \tag{3.20}$$

By (3.20) and Lemma 2.2 we have

$$T(r, L) = T(r, e^{\alpha}) + O(1). \tag{3.21}$$

By (3.6), (3.21) and the definition of the order of a meromorphic function we have $\rho(e^{\alpha}) = 1$, which implies that α is a polynomial of degree $\deg \alpha = 1$, say $\alpha(z) = A_3 z + B_1$, where $A_3 \neq 0$ and B_1 are complex constants. This implies that (3.21) can be rewritten as

$$T(r, L) = T(r, e^{\alpha}) + O(1) = \frac{|A_3| r}{\pi}(1 + o(1)) + O(1),$$

which contradicts (3.1).

Suppose that F and G satisfy one of (ii) and (vi) of Lemma 2.3. Then, in the same manner as the above discussion we can get a contradiction.

Subcase 2.1.2. Suppose that F and G satisfy one of (iii), (iv) and (v) of Lemma 2.3, say F and G satisfy (iii) of Lemma 2.3. Then, 0 and 1 are Picard exceptional values of F and G. Therefore

$$\frac{G}{G-1} = e^{\beta}, \tag{3.22}$$

where β is an entire function. By (3.17), (3.22) and Lemma 2.2 we have

$$T(r, L) = T(r, e^{\beta}) + O(1). \tag{3.23}$$

By (3.6), (3.23) and the definition of the order of a meromorphic function we have $\rho(e^{\beta}) = 1$, which implies that β is a polynomial of degree $\deg \beta = 1$, say $\beta(z) = A_4 z + B_2$, where $A_4 \neq 0$ and B_2 are complex constants. This implies that (3.23) can be rewritten as

$$T(r, L) = T(r, e^{\beta}) + O(1) = \frac{|A_4| r}{\pi}(1 + o(1)) + O(1),$$

which contradicts (3.1).

Suppose that F and G satisfy one of (iv) and (v) of Lemma 2.3. Then, in the same manner as the above discussion we can get a contradiction.

Subcase 2.2. Suppose that F is not a Möbius transformation of G. In the same manner as in the proof of Lemma 2.9 we have (2.1)-(2.4) and (2.7)-(2.9). By (2.8) and (2.9) we have

$$F - G = \frac{(h_1 - 1)\,(1 - h_2^{-1})}{h_0 - 1}. \tag{3.24}$$

By (2.1)-(2.4), (2.7)-(2.9) and (3.24) we deduce

$$N_0(r) = \overline{N}_0(r, 1; h_1, h_0) + S(r, f) = \overline{N}_0(r, 1; h_1, h_2) + S(r, F). \tag{3.25}$$

We consider the following two subcases:

Subcase 2.2.1. Suppose that

$$N_0(r) \neq S(r, F). \tag{3.26}$$

Then, by (3.25) and (3.26) we have

$$\overline{N}_0(r, 1; h_1, h_2) \neq S(r, F). \tag{3.27}$$

By (2.7), (3.27) and Lemma 2.6 we know that there exist two integers s and t ($|s| + |t| > 0$), such that

$$h_1^s h_2^t = 1. \tag{3.28}$$

By substituting (2.1) and (2.2) into (3.28) we get

$$F^t(F-1)^s = G^t(G-1)^s. \tag{3.29}$$

By noting that F is not a Möbius transformation of G, we deduce by (3.29) that s, t are nonzero integers such that $|s| \neq |t|$, this together with the assumption that F and G share ∞ IM implies that F and G share ∞ CM. Combining this with Lemma 2.7 and the assumption that F and G share 0, 1 CM, we can see that F and G satisfy one of the three relations (i)-(iii) of Lemma 2.7, say F and G satisfy (i). Then

$$F = \frac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1} \quad \text{and} \quad G = \frac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1}, \tag{3.30}$$

where γ is a nonconstant entire function, $k \geq 2$ and s are positive integers such that s and $k+1$ are relatively prime and $1 \leq s \leq k$. Moreover, $N_0(r)$ is such that $N_0(r) \leq \frac{1}{2}T(r,F) + S(r,F)$. By (3.17), (3.30), Lemma 2.2 and Lemma 2.8 we have

$$T(r,f) = T(r,F) + O(1) = T(r,G) + O(1) = T(r.L) + O(1) = kT(r,e^{\gamma}) + O(1). \tag{3.31}$$

By (3.6) and (3.31) we have $\rho(e^{\gamma}) = 1$, this implies that γ is a polynomial of degree $\deg\gamma = 1$, say $\gamma = A_5 z + B_3$, where $A_5 \neq 0$ and B_3 are complex constants. This implies that (3.23) can be rewritten as

$$T(r,L) = kT(r,e^{\gamma}) + O(1) = \frac{k|A_5|r}{\pi}(1+o(1)) + O(1),$$

which contradicts (3.1).

Subcase 2.2.2. Suppose that

$$N_0(r) = S(r,F). \tag{3.32}$$

By noting that L has a pole $z = 1$ in the complex plane at most, we have by (3.3) and (3.17) that

$$N\left(r, \frac{1}{G-a_1}\right) \leq \log r + O(1), \tag{3.33}$$

where $a_1 = \frac{b-c}{b-a} \notin \{0, 1, \infty\}$. Therefore, by (2.4), (3.18), (3.32), (3.33), the conclusion (ii) of Lemma 2.9 and the second fundamental theorem we have

$$\begin{aligned}
2T(r,G) &\le \overline{N}(r,G) + N\left(r,\frac{1}{G}\right) + N\left(r,\frac{1}{G-1}\right) + N\left(r,\frac{1}{G-a_1}\right) - N\left(r,\frac{1}{G'}\right) \\
&\quad + O(\log r) \\
&= \overline{N}(r,G) + N\left(r,\frac{1}{G}\right) + N\left(r,\frac{1}{G-1}\right) + N\left(r,\frac{1}{G-a_1}\right) + N_0(r) \\
&\quad - T(r,F) + O(\log r) \\
&\le N\left(r,\frac{1}{G}\right) + N\left(r,\frac{1}{G-1}\right) - T(r,F) + O(\log r) + S(r,F) \\
&\le 2T(r,G) - T(r,F) + O(\log r) + S(r,F),
\end{aligned}$$

i.e.,

$$T(r,F) = O(\log r) + S(r,F). \tag{3.34}$$

By (3.34) we deduce that F is a rational function. This together with (3.6) and the left equality of (3.17) and Lemma 2.2 implies that $T(r,f) = T(r,F) + O(1) = O(\log r)$. Combining this with (3.5), we have $T(r,L) = O(\log r)$, which contradicts (3.1). Theorem 1.1 is thus completely proved. □

Acknowledgements. The authors wish to express their thanks to the referee for his/her valuable suggestions and comments.

References

[1] E. Bombieri and A. Perelli, *Distinct zeros of L-functions*, Acta Arith., **83(3)**(1998), 271–281.

[2] G. G. Gundersen, *Meromorphic functions that share three or four values*, J. London Math. Soc., **s2-20(3)**(1979), 457–466.

[3] G. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc., **s2-37(1)**(1988), 88–104.

[4] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[5] W. K. Hayman and J. Miles, *On the growth of a meromorphic function and its derivatives*, Complex Var. Theory Appl., **12(1-4)**(1989), 245–260.

[6] P. C. Hu and B. Q. Li, *A simple proof and strengthening of a uniqueness theorem for L-functions*, Canadian math. Bull., **59(1)**(2016), 119–122.

[7] H. Ki, *A remark on the uniqueness of the Dirichlet series with a Riemann-type function equation*, Adv. Math. **231(5)**(2012), 2484-2490.

[8] I. Lahiri and A. Sarkar, *On uniqueness theorem of Tohge*, Arch. Math., **84(5)**(2005), 461–469.

[9] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

[10] B. Q. Li, *A result on value distribution of L-functions*, Proc. Amer. Math. Soc., **138(6)**(2009), 2071–2077.

[11] B. Q. Li , *A uniqueness theorem for Dirichlet series satisfying a Riemann type functional equation*, Adv. Math., **226(5)**(2011), 4198-4211.

[12] B. Q. Li, *On common zeros of L-functions*, Math. Z., **272(3–4)**(2012), 1097–1102.

[13] W.C. Lin and K. Tohge, *On shared-value properties of Painlevé Transcendents*, Comput. methods Funct.theory, **7(2)**(2007), 477–499.

[14] A. Z. Mokhon-ko, *On the Nevanlinna characteristics of some meromorphic functions*, in: Theory of Functions, Functional Analysis and Their Applications, vol.14, Izd-vo Khar'kovsk. Un-ta, 1971, pp. 83–87.

[15] R. Nevanlinna, *Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen*, Acta Math., **48(3)**(1926), 367–391.

[16] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), E. Bombieri et al. (eds.), Collected papers, Vol. II, Springer-Verlag, 1991, pp. 47–63.

[17] J. Steuding, *Value distribution of L-functions*, Lecture Notes in Mathematics, 1877, Springer-Verlag, Berlin, 2007.

[18] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[19] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin Heidelberg, 1993.

[20] H. X. Yi, *Unicity theorems for meromorphic functions that share three values*, Kodai Math. J., **18(2)**(1995), 300–314.

[21] Q. C. Zhang, *Meromorphic functions sharing three values*, Indian J. Pure Appl. Math., **30(7)**(1999), 667–682.

[22] Q. C. Zhang, *Meromorphic functions sharing values or sets*, Doctoral Thesis, Shandong University, 1999.