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RELATIVE CLASS NUMBER ONE PROBLEM OF REAL QUADRATIC FIELDS AND CONTINUED FRACTION OF $\sqrt{m}$ WITH PERIOD 6

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Abstract. For a positive square-free integer m, let $K = \mathbb{Q}(\sqrt{m})$ be a real quadratic field. The relative class number $H_d(f)$ of K of discriminant d is the ratio of class numbers $\mathcal{O}_K$ and $\mathcal{O}_f$, where $\mathcal{O}_K$ is the ring of integers of K and $\mathcal{O}_f$ is the order of conductor f given by $\mathbb{Z} + f\mathcal{O}_K$. In 1856, Dirichlet showed that for certain m there exists an infinite number of f such that the relative class number $H_d(f)$ is one. But it remained open as to whether there exists such an f for each m. In this paper, we give a result for existence of real quadratic field $\mathbb{Q}(\sqrt{m})$ with relative class number one where the period of continued fraction expansion of $\sqrt{m}$ is 6.

1. Introduction

For a positive square-free integer m, let t_m and u_m be positive integers such that

$$\epsilon_m = \frac{t_m + u_m\sqrt{m}}{z} > 1$$

is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{m})$, where $z = 2$ if $m \equiv 1 \pmod 4$ and $z = 1$ otherwise. The discriminant d of K is m if $m \equiv 1 \pmod 4$, otherwise $d = 4m$. The ring of integers or maximal order $\mathcal{O}_K$ of K is $\mathbb{Z}[\frac{1+\sqrt{m}}{2}]$ when $m \equiv 1 \pmod 4$. Otherwise $\mathcal{O}_K = \mathbb{Z}[\sqrt{m}]$. For $f \in \mathbb{N}$, $\mathcal{O}_f = \mathbb{Z} + f\mathcal{O}_K$ is called an order in $\mathcal{O}_K$ of index f since $[\mathcal{O}_K : \mathcal{O}_f] = f$. In this case, the index f is called the conductor of $\mathcal{O}_f$ in $\mathcal{O}_K$. By Dirichlet's unit theorem, the units of $\mathcal{O}_K$ can be expressed by $\pm\epsilon_m^i (i \in \mathbb{Z})$ where ϵ_m is the fundamental unit of $\mathcal{O}_K$.

The relative class number $H_d(f)$ of K of discriminant d is the ratio of class numbers $\mathcal{O}_K$ and $\mathcal{O}_f$. Dirichlet[8] showed in 1856 that for certain m there exists an infinite number of f such that the relative class number $H_d(f)$ is one. But it remained open as to whether there exists such an f for each m [4]. Furness and

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Parker [9] proved that there exists a prime f such that the relative class number $H_d(f)$ of a real quadratic field $K = \mathbb{Q}(\sqrt{m})$ is one when $\sqrt{m}$ has a particular continued fraction form. In [7], Chakraborty and Saikia showed there exists a conductor f of relative class number one when the continued fraction of $\sqrt{m}$ is non-diagonal of period 4 or 5. In this paper, we give a result for existence of real quadratic field $\mathbb{Q}(\sqrt{m})$ with relative class number one where the period of continued fraction expansion of $\sqrt{m}$ is 6.

The following formula obtained from Dirichlet(cf. [4]) is very useful.

Theorem 1.1. *Let $\phi(f)$ be the smallest positive integer such that $\epsilon_m^{\phi(f)}$ belong to $\mathcal{O}_f$ and $\psi(f) = f\,\Pi_{q|f}\left(1 - \left(\frac{d}{q}\right)\frac{1}{q}\right)$ where $\left(\frac{d}{q}\right)$ is Kronecker Symbol of d modulo a prime q. Then*

$$H_d(f) = \frac{\psi(f)}{\phi(f)}. \tag{1}$$

The Kronecker symbol $\left(\frac{d}{q}\right)$ is the same as the Legendre symbol when q is an odd prime. For $q = 2$ and an odd integer d, $\left(\frac{d}{q}\right)$ is 1 if $d \equiv \pm 1 \pmod 8$ and -1 if $d \equiv \pm 3 \pmod 8$. From the fact that the relative class number $H_d(f)$ is an integer(cf. [5]), we observe that $\phi(f)$ always divides $\psi(f)$.

Now, we consider the continued fraction expansion of $\sqrt{m}$. Let l_m be the length of the period of the continued fraction of $\sqrt{m}$ and p_{l_m-1}/q_{l_m-1} the $(l_m - 1)$-th convergent of it. For the relation between the continued fraction of $\sqrt{m}$ and the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{m})$, it is well known as the following theorem(cf. [2, 14]).

Theorem 1.2. *Let m be a positive square-free integer and ϵ_m the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{m})$. Then*

$$\epsilon_m = p_{l_m-1} + q_{l_m-1}\sqrt{m}$$

or

$$\epsilon_m^3 = p_{l_m-1} + q_{l_m-1}\sqrt{m}$$

and the latter can only occur if $d \equiv 5 \pmod 8$.

Except for the case that $m \equiv 5 \pmod 8$, the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{m})$ is $\epsilon_m = p_{l_m-1} + q_{l_m-1}\sqrt{m}$. If m is a positive square-free integer congruent to 5 modulo 8, by Theorem 1.2, then $\epsilon_m = p_{l_m-1} + q_{l_m-1}\sqrt{m}$ or $\epsilon_m^3 = p_{l_m-1} + q_{l_m-1}\sqrt{m}$ which means that $u_m \leq 2q_{l_m-1}$.

2. Approach for the problem for existence of real quadratic field of relative class number one for conductor f

In this paper, we are interested in existence of a real quadratic field of relative class number one for conductor f. First, we remark how to approach this problem.

Remark 1. (1) If m does not divide u_m, one can find a prime f that divides m but does not divide u_m because m is square-free. That means that $\psi(f) = f$. On the other hand, by Theorem 1.1, $\phi(f) = 1$ or f. But, since ϵ_m does not belong to $\mathcal{O}_f$, $\phi(f)$ is equal to f and $H_d(f) = 1$. Therefore, if m does not divide u_m, there always exists a prime f such that $H_d(f) = 1$. In order to check the existence of a prime f where $H_d(f) = 1$, it is enough to consider only the case that m divides u_m.

(2) If m divides u_m, it is more complicated. If $m = 46$, the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is $\epsilon_{46} = 24335+3588\sqrt{46}$. In this case, m divides u_m and there does not exist a conductor f such that $H_d(f) = 1$(cf. [10, 15]). On the other hand, if $m = 1817$, we easily find that $H_d(2) = 1$.

But, generally, it is difficult to check if m divides u_m or not. For all $m < 10^7$, Stephen and Williams [17] found only 8 values where m divides u_m. They are 46, 430, 1817, 58254, 209991, 1752299, 3124318, and 4099215. In the case that m is prime, the problem as to whether if m does not divide u_m is closely related to two famous conjectures: One is Ankeny-Artin-Chowla conjecture[1], which says that for any prime p congruent to 1 modulo 4, $u_p \not\equiv 0 \pmod{p}$. The other one is the Mordell conjecture[16], which says that for any prime p congruent to 3 modulo 4, $u_p \not\equiv 0 \pmod{p}$. In [2] and [11], in order to show that two conjectures for special cases hold, authors proved that $p > u_p$. The inequality $p > u_p$ means that p does not divide u_p. Especially, when the length of period l_m of continued fraction of $\sqrt{m}$ is less than equal to 4, Byeon and Lee [2] proved that $m > u_m$ for every square-free positive integer m. Furthermore, for the case that $l_m = 5$, they also observed there are infinitely many m such that $u_m > m$ and proved $u_m \not\equiv 0 \pmod{m}$.

Remark 2. Note that if $m \equiv 2 \pmod{4}$ or $m \equiv 3 \pmod{4}$, then $u_m = q_{l_m-1}$ by Theorem 1.2. If $m \equiv 2 \pmod{4}$ and u_m is odd, we easily observe that there always exists a prime f such that $H_d(f) = 1$ by Remark 1 and parity between m and u_m. If $m \equiv 3 \pmod{4}$ and u_m is odd, there always exists a prime f such that $H_d(f) = 1$ by Theorem 4.1 of [6].

3. Relative class number one problem and continued fraction of $\sqrt{m}$ with period 6

In this paper, we consider the case that l_m is 6. In [12], for existence of continued fraction of $\sqrt{m}$ with period 6, the following theorem is proved.

Theorem 3.1. *Let l_m be 6. If rs is even or rs is odd and t is even, there always exists a positive integer a_0 such that $\sqrt{m} = [a_0, \overline{r, s, t, s, r, 2a_0}]$. For the other cases, there does not exist such m.*

Suppose that if $m \equiv 2 \pmod{4}$ or $m \equiv 3 \pmod{4}$. Then $u_m = q_{l_m-1}$. If the period of continued fraction expansion of $\sqrt{m}$ is 6, the expansion has the form of $\sqrt{m} = [a_0, \overline{r, s, t, s, r, 2a_0}]$. By Theorem 3.1, there exists the expansion only for the cases that rs is even or rs is odd and t is even. By simple computation, we have $u_m = q_5 = r^2s^2t + 2r^2s + 2rst + t + 2r$. If rs is even and t is odd, we easily observe that q_5 is odd. That means that there exists a prime f such that $H_d(f) = 1$ by Remark 2.

On the other hand, if rs is even and t is even or rs is odd and t is even, q_5 is even. For these cases, let's consider for more concrete expression of m. Suppose the continued fraction expansion of $\sqrt{m}$ has the form $[a_0, \overline{r, s, t, s, r, 2a_0}]$. Then the positive integer m can be expressed as follows:

$$m = a_0^2 + b, \quad b = \frac{2s + s^2t + 2a_0(rs^2t + st + 2rs + 1)}{2r + 2r^2s + t + 2rst + r^2s^2t}.$$

Putting $2a_0 = kr + u$ where $0 \le u < r$, we have

$$b = k + A, \quad A = \frac{u(rs^2t + st + 2sr + 1) + s^2t + 2s - k(rst + t + r)}{r^2s^2t + 2r^2s + 2rst + t + 2r}.$$

We can easily check that $A < 1$ by noting that r, s, and t are less than equal to a_0(cf. [13]). First, we consider the case $A = 0$, which means that

$$u(rs^2t + st + 2sr + 1) + s^2t + 2s = k(rst + t + r).$$

That is,

$$k = us + \frac{urs + u + s^2t + 2s}{rst + t + r}.$$

If $u = 0$, then $2a_0 = kr$ and $s^2t + 2s = krst + kt + kr = 2a_0st + 2a_0 + kt$. But it does not happen by the fact that r, s, and t are not greater than a_0. Therefore, u is a positive integer. In fact, there exists the case that $A = 0$ and $u \neq 0$. For example, we consider the case that $m = 966$. The continued fraction expansion of $\sqrt{966}$ is $[31; \overline{12, 2, 2, 2, 2, 12, 62}]$. In this case, we have $A = 0$, $u = 2$, and $k = 5$. Moreover, $u_{966} = 1850$, which means that $m < u_m$. But one can see that m does not divide u_m. In general, for an even integer s, putting $r = 2s^2 + 2s$, $t = 2$, $k = 2s+1$, and $u = 2$, we have $m = 4s^6 + 12s^5 + 13s^4 + 10s^3 + 7s^2 + 4s + 2$, $u_m = q_5 = 8s^6 + 24s^5 + 24s^4 + 16s^3 + 12s^2 + 4s + 2$. In this case, we can observe that $m < u_m < 2m$ for every even positive integer s (Note that if s is an odd integer, then m is not square-free since m is divided by 4). It means that m can not divide u_m. Therefore, there exists a prime f such that $H_d(f) = 1$ for an infinite real quadratic field family. Combining our results, we have the following theorem.

Theorem 3.2. *Let m be a square-free positive integer such that the continued fraction expansion of $\sqrt{m}$ have $[a_0, \overline{r, s, t, s, r, 2a_0}]$. If t is odd, there exists a prime f such that $H_d(f) = 1$. If rs is even and t is even, there are infinitely many m satisfying $m < u_m$. Moreover, in that case, there exist infinitely many real quadratic fields with relative class number one.*

Remark 3. For the case that $A \neq 0$, it is more complicated to find an infinite real quadratic fields family with relative class number one. For example, we consider the case that $m = 418$. The continued fraction expansion of $\sqrt{418}$ is $[20; \overline{2, 4, 20, 4, 2, 40}]$. In this case, we have $A = -2$, $u = 0$, and $k = 20$.

Moreover, $u_{418} = 1656$, which means that $m < u_m$. But m does not divide u_m. Therefore there also exists a prime f such that $H_d(f) = 1$ for $m = 418$.

Remark 4. If $m > u_m$, we can easily see that m does not divide u_m. But, by Theorem 3.2, there are infinitely many m satisfying $m < u_m$. Nevertheless, if the period of the length of continued fraction expansion of $\sqrt{m}$ is 6, we expect that there always exists a prime f such that $H_d(f) = 1$. For a positive square-free integer m less than 10^7, our statement is true.

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