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RELATIVE TWISTED KÄHLER-RICCI FLOWS ON FAMILIES OF COMPACT KÄHLER MANIFOLDS

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Abstract. Let $p : X \to \mathbf{D}$ be a proper surjective holomorphic submersion where X is a Kähler manifold and $\mathbf{D}$ is the unit disc in $\mathbb{C}$. Let Ω be a d-closed semi-positive real $(1,1)$-form on X. If each $X_s := p^{-1}(s)$ for $s \in \mathbf{D}$ satisfies $-c_1(X_s) + \Omega|_{X_s}$ is Kähler, then the Kähler-Ricci flow twisted by $\Omega|_{X_s}$ has a long time solution by Cao's theorem. This family of twisted Kähler-Ricci flows induces a relative Kähler form $\omega(t)$ on the total space X. In this paper, we prove that the positivity of $\omega(t)$ is preserved along the twisted Kähler-Ricci flow.

1. Introduction

Let $p : X \to \mathbf{D}$ be a proper surjective holomorphic submersion from a Kähler manifold X, equipped with a Kähler metric θ, to the unit disc $\mathbf{D}$ in $\mathbb{C}$. Then every fiber $X_s := p^{-1}(s)$ for $s \in \mathbf{D}$ is a compact Kähler manifold with the Kähler metric $\theta|_{X_s}$. Let Ω be a d-closed semi-positive $(1,1)$-form on X. Suppose that the Ricci curvature $-\mathrm{Ric}(\theta|_{X_s})$ of $\theta|_{X_s}$ satisfies that

$$\omega_s := -\mathrm{Ric}(\theta|_{X_s}) + \Omega|_{X_s} > 0 \tag{1}$$

on each fiber X_s. Then the twisted Kähler-Ricci flow on X_s is given as follows.

$$\frac{\partial}{\partial t}\omega_s(t) = -\omega_s(t) - \mathrm{Ric}(\omega_s(t)) + \Omega|_{X_s}$$
$$\omega_s(0) = \omega_s.$$

The celebrated theorem due to Cao implies that the above parabolic PDE has a long time solution ([2]). This family of twisted Kähler-Ricci flows induces a flow of relative Kähler metric $\omega(t)$ on X satisfying

$$\omega(t)|_{X_s} = \omega_s(t)$$

for $s \in \mathbf{D}$ and $t \in [0,\infty)$, which is a solution of the *relative twisted Kähler-Ricci flow*. (For the definition, see Section 3.2.) Here a relative Kähler form means a

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d-closed real $(1,1)$-form on X which is positive-definite on each fiber. Since the (twisted) Kähler-Ricci flow preserves the Kählerness, $\omega(t)|_{X_s}$ is always positive-definite for $t \in [0,\infty)$ on each fiber X_s. However it is not obvious that $\omega(t)$ is positive-definite along the horizontal direction in the total space X. In this paper, we prove the positivity of $\omega(t)$ on the total X provided that the initial relative Kähler form $\omega(0)$ is positive.

Theorem 1.1. *If $\omega(0)$ is semi-positive (resp. positive), then $\omega(t)$ is semi-positive (resp. positive) for all $t \in (0,\infty)$.*

Since Cao's theorem ([2]) also implies that the twisted Kähler Ricci flow $\omega_s(t)$ converges to the twisted Kähler-Einstein metric ρ_s on each fiber X_s satisfying

$$\rho_s = -\mathrm{Ric}(\rho_s) + \Omega|_{X_s},$$

we have the following corollary.

Corollary 1.2 (cf [5])**.** *The relative twisted Kähler-Einstein metric ρ on X is semi-positive.*

For the definition of the relative twisted Kähler-Einstein metric, see Section 4.

The positivity of the relative Kähler-Einstein metric is first studied by Schumacher ([7]). He proves that the relative Kähler-Einstein metric on a family of canonically polarized compact Kähler manifolds is positive on the total space. Păun generalizes it to the relative twisted Kähler-Einstein metric ([5]). On the other hand, Berman proves the parabolic version of Schumacher's result ([1]). More precisely, He proves that the positivity of the relative Kähler-Ricci flow is preserved along the flow. The main theorem of this paper is the parabolic version of Păun's result.

2. Preliminaries

Let $p : X^{n+1} \to \mathbf{D}$ is a proper surjective holomorphic submersion from a complex manifold X to the unit disc $\mathbf{D}$ in $\mathbb{C}$ such that every fiber $X_s := p^{-1}(s)$ is a Kähler complex manifold. We call this $p : X \to \mathbf{D}$ a *smooth family of compact Kähler manifolds.* If we denote the standard coordinate in $\mathbf{D}$ by s, one can take a local coordinate $(z^1,\dots,z^n)$ of a fixed fiber such that

- $(z^1,\dots,z^n,s)$ forms a local coordinate of X,
- $p(z^1,\dots,z^n,s) = s$ in the coordinate (z,s).

We call this an *admissible coordinate of p.*

Throughout this paper, small Greek letters $\alpha,\beta,\dots = 1,\dots,n$ stand for indices on $z = (z^1,\dots,z^n)$ unless otherwise specified. For a properly differentiable function f on X, we denote by

$$f_\alpha = \frac{\partial f}{\partial z^\alpha}, \quad \text{and} \quad f_{\bar\beta} = \frac{\partial f}{\partial z^{\bar\beta}}, \tag{2}$$

where $z^{\bar\beta}$ mean $\overline{z^\beta}$. If there is no confusion, we always use the Einstein convention.

2.1. Horizontal lifts and geodesic curvatures

Definition 1. Let $v := \partial/\partial s \in T'\mathbf{D}$ where $T'\mathbf{D}$ stands for the complex tangent space of type $(1,0)$ and let τ be a real $(1,1)$-form on X which is positive-definite on each fiber X_s.

1. A vector field v_τ of type $(1,0)$ is called a *horizontal lift* of v if v_τ satisfies that
   (i) $\langle v_\tau, W\rangle_\tau = 0$ for all $W \in T'X_s$,
   (ii) $dp(v_\tau) = v$.
2. The *geodesic curvature* $c(\tau)$ of τ is defined by the norm of v_τ with respect to the sesquilinear form $\langle\cdot,\cdot\rangle_\tau$ induced by τ, i.e.,

$$c(\tau) = \langle v_\tau, v_\tau\rangle_\tau .$$

Remark 1. Let $(z^1,\dots,z^n,s)$ be an admissible coordinate of p. Then τ is written as

$$\tau = \sqrt{-1}\left(\tau_{s\bar s}ds^i \wedge d\bar s + \tau_{s\bar\beta}ds \wedge dz^{\bar\beta} + \tau_{\alpha\bar s}dz^\alpha \wedge d\bar s + \tau_{\alpha\bar\beta}dz^\alpha \wedge dz^{\bar\beta}\right).$$

Since τ is positive-definite on each fiber X_s, the matrix $(\tau_{\alpha\bar\beta})$ is invertible. We denote the inverse matrix by $(\tau^{\bar\beta\alpha})$. Then the horizontal lift of $\partial/\partial s$ is given as

$$\left(\frac{\partial}{\partial s}\right)_\tau = \frac{\partial}{\partial s} - \tau_{s\bar\beta}\tau^{\bar\beta\alpha}\frac{\partial}{\partial z^\alpha}.$$

On the other hand, the geodesic curvature $c(\tau)$ is computed as

$$c(\tau) = \langle v_\tau, v_\tau\rangle_\tau = \tau_{s\bar s} - \tau_{s\bar\beta}\tau^{\bar\beta\alpha}\tau_{\alpha\bar s}.$$

It is well known that

$$\frac{\tau^{n+1}}{(n+1)!} = c(\tau)\cdot\frac{\tau^n}{n!} \wedge \sqrt{-1}ds \wedge d\bar s. \tag{3}$$

This says that if $c(\tau) > 0$ (resp. ≥ 0), then τ is a positive (resp. semi-positive) real $(1,1)$-form as τ is positive-definite when restricted to X_s.

2.2. Hermitian metrics on the relative canonical line bundle

The relative canonical line bundle $K_{X/\mathbf{D}}$ is defined by

$$K_{X/\mathbf{D}} = K_X \otimes (p^*K_{\mathbf{D}})^{-1}.$$

For a given relative Kähler form τ on X, which is a d-closed real $(1,1)$ form, which is positive-definite on each fiber X_s, there exists a hermitiain metric $h^\tau_{X/\mathbf{D}}$ on $K_{X/\mathbf{D}}$ as follows:

Let (z,s) be an admissible coordinate in X so that $(\tau_{\alpha\bar\beta})$ is positive-definite on each fiber X_s. Then $\sum \tau_{\alpha\bar\beta}(z,s)dz^\alpha\wedge dz^{\bar\beta}$ gives a hermitian metric on each fiber X_s. It follows that

$$\left(\det\left(\tau_{\alpha\bar\beta}(z,s)_{1\le\alpha,\beta\le n}\right)\right)^{-1} \tag{4}$$

gives a hermitian metric on the relative canonical line bundle $K_{X/\mathbf{D}}$, which is denoted by $h^\tau_{X/\mathbf{D}}$. The curvature form $\Theta_\tau := \Theta_{h^\tau_{X/\mathbf{D}}}(K_{X/\mathbf{D}})$ of $h^\tau_{X/\mathbf{D}}$ on $K_{X/\mathbf{D}}$ is given by

$$\Theta_{h^\tau_{X/\mathbf{D}}}(K_{X/\mathbf{D}}) = dd^c\log\det(\tau_{\alpha\bar\beta}(z,s)),$$

where d^c is the real operator defined by $d^c=\frac{\sqrt{-1}}{2}(\bar\partial-\partial)$, so thst $dd^c=\sqrt{-1}\partial\bar\partial$. It is obvious that the curvature form also can be written as

$$\Theta_{h^\tau_{X/\mathbf{D}}}(K_{X/\mathbf{D}}) = dd^c\log\det\left(\tau^n\wedge p^*dV_s\right),$$

where $dV_s:=\sqrt{-1}ds\wedge d\bar s$ is the Euclidean volume form on $\mathbf{D}$. Then it immediately follows from the definition that if τ is a relative Kähler form, then

$$\Theta_{h^\tau_{X/\mathbf{D}}}(K_{X/\mathbf{D}})|_{X_s} = -\mathrm{Ric}(\tau|_{X_s}).$$

3. Twisted Kähler-Ricci flow

In this section, we recall the twisted Kähler-Ricci flow on a compact Kähler manifold and define the relative twisted Kähler-Ricci flow on a family of compact Kähler manifolds.

3.1. Twisted Kähler-Ricci flow

Let (X,θ) be a compact Kähler manifold and Ω be a d-closed semi-positive $(1,1)$ form on X such that

$$\widehat\omega := -\mathrm{Ric}(\theta)+\Omega>0.$$

The twisted Kähler-Ricci flow is given as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\omega(t) &= -\omega(t)-\mathrm{Ric}(\omega(t))+\Omega,\\ \omega(0)&=\widehat\omega.\end{aligned}$$

If we write $\omega(t)=\widehat\omega+dd^c\varphi(t)$, then the above equation is equivalent to the following equation:

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(t) &= \log\frac{(\widehat\omega+dd^c\varphi(t))^n}{\widehat\omega^n}-\varphi-f,\\ \varphi(0)&=0\end{aligned}$$

where f is the smooth function on X which is defined by $f=-\log(\widehat\omega^n/\theta^n)$. It is easy to see that f satisfies that

$$dd^cf=\widehat\omega+\mathrm{Ric}(\widehat\omega)-\Omega.$$

Cao proves the long time existence of the (twisted) Kähler-Ricci flow and the convergence to the (twisted) Kähler-Einstein metric ([2]).

Theorem 3.1 (Cao)**.** *The twisted Kähler-Ricci flow has a long time solution. Moreover,* $\rho = \lim_{t\to\infty} \omega(t)$ *satisfies that*

$$\mathrm{Ric}(\rho) = -\rho + \Omega.$$

The Kähler form ρ is called a *twisted Kähler-Einstein metric* on X.

3.2. Relative twisted Kähler-Ricci flow

Let $p : X \to \mathbf{D}$ be a smooth family of compact Kähler manifolds over the unit disc $\mathbf{D}$ in $\mathbb{C}$. Suppose that X is Kähler with a Kähler form θ on X. We denote by $\Theta_\theta := \Theta_{h^\theta_{X/\mathbf{D}}}(K_{X/\mathbf{D}})$ the curvature of the hermitian metric $h^\theta_{X/\mathbf{D}}$ on $K_{X/\mathbf{D}}$ induced by the Kähler form θ. Let Ω be a d-closed semi-positive real $(1,1)$ form on X and $\widehat{\omega} := \Theta_\theta + \Omega$. Suppose that $\widehat{\omega}$ satisfies

$$\widehat{\omega}|_s = (\Theta_\theta + \Omega)|_{X_s} > 0 \tag{5}$$

on each fiber X_s. Note that Equation (5) is equivalent to Equation (1) as $\Theta|_{X_s} = -\mathrm{Ric}(\theta|_{X_s})$. If we define the smooth function $f \in C^\infty(X)$ by

$$f = -\log \frac{\widehat{\omega}^n \wedge p^* dV_s}{\theta^n \wedge p^* dV_s},$$

then it is obvious that

$$dd^c f = -\Theta_{\widehat{\omega}} + \Theta_\theta.$$

In particular, $dd^c f|_{X_s} = \mathrm{Ric}(\widehat{\omega}|_{X_s}) + \widehat{\omega}|_{X_s} - \Omega|_{X_s}$. Hence Theorem 3.1 implies that for each fiber X_s, there exists a smooth function $\varphi_s(t)$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_s(t) &= \log \frac{(\widehat{\omega}|_{X_s} + dd^c \varphi_s(t))^n}{(\widehat{\omega}|_{X_s})^n} - \varphi_s - f|_{X_s}, \\ \varphi_s(0) &= 0 \end{aligned}$$

for all $t \in [0, \infty)$. If we define $\varphi : X \to \mathbb{R}$ by $\varphi(x,t) = \varphi_s(x,t)$ for $p(x) = s$, then for each $t > 0$, φ is smooth on the total space X by the standard argument using the implicit function theorem (cf [1, 3]).

Now we define a d-closed real $(1,1)$ form $\omega(t)$ on the total space X by

$$\omega(t) = \Theta_\theta + \Omega + dd^c \varphi.$$

Then one can easily see that

$$\begin{aligned} \frac{\partial}{\partial t}\omega(t) &= \frac{\partial}{\partial t}(\Theta_\theta + \Omega + dd^c\varphi) = dd^c\left(\frac{\partial \varphi}{\partial t}\right) \\ &= dd^c\left(\log \frac{(\widehat{\omega} + dd^c\varphi(t))^n \wedge p^* dV_s}{\widehat{\omega}^n \wedge p^* dV_s} - \varphi - f\right) \\ &= \Theta_{\omega(t)} - \Theta_{\widehat{\omega}} - dd^c\varphi - dd^c f \\ &= \Theta_{\omega(t)} - \Theta_{\widehat{\omega}} - dd^c\varphi + \Theta_{\widehat{\omega}} - \Theta_\theta \\ &= \Theta_{\omega(t)} - \omega(t) + \Omega. \end{aligned}$$

Hence we have the following proposition.

Proposition 3.2. *$\omega(t)$ satisfies the following.*

$$\begin{aligned}\frac{\partial}{\partial t}\omega(t) &= \Theta_{\omega(t)} - \omega(t) + \Omega, \\ \omega(0) &= \widehat{\omega}.\end{aligned} \tag{6}$$

Remark 2. Equation (6) is called the *relative twisted Kähler-Ricci flow* since if we restrict $\omega(t)$ on a fiber X_s, then it satisfies that

$$\begin{aligned}\frac{\partial}{\partial t}\omega(t)|_{X_s} &= -\mathrm{Ric}(\omega(t)|_{X_s}) - \omega(t)|_{X_s} + \Omega|_{X_s}, \\ \omega(0)|_{X_s} &= \widehat{\omega}|_{X_s}.\end{aligned}$$

4. Proof of the main theorem

In this section, we prove Theorem 1.1 and Corollary 1.2.

Let $p : X \to \mathbf{D}$ be a family of compact Kähler manifolds over the unit disc $\mathbf{D}$ in $\mathbb{C}$ and θ be a Kähler metric on X. Then the geodesic curvature $c(\omega(t))$ of the solution $\omega(t)$ of the relative twisted Kähler-Ricci flow satisfies a parabolic PDE on each fiber. This is a twisted version of the PDE which is first introduced by Berman (Theorem 4.5 in [1]). The proof is essentially the same as the one in [1], so we omit the detailed proof.

Proposition 4.1. *$c(\omega(t))$ satisfies the following parabolic PDE on each fiber X_s.*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) c(\omega(t)) = -c(\omega(t)) + \left|\bar{\partial} v_{\omega(t)}\right|^2_{\omega(t)} + \Omega\left(v_{\omega(t)}, \overline{v_{\omega(t)}}\right), \tag{7}$$

where $\Delta_{\omega(t)}$ is the Laplacian with respect to the Kähler metric $\omega(t)$ and $|\cdot|_{\omega(t)}$ is the pointwise norm with respect to $\omega(t)$.

Proof. For the proof, we refer to see the proof of Theorem 4.2 in [3]. The only difference is that we make use of (6) instead of Equation (3.5) in [3]. In particular, (6) implies the following:

$$\left(\log\det(g_{\alpha\overline{\beta}})\right)_{i\bar{j}} = \left(\frac{\partial}{\partial t} g\right)_{i\bar{j}} + g_{i\bar{j}} - \Omega_{i\bar{j}}$$

where $i, j = s, \alpha, \beta, \ldots$. □

Remark 3. In ([5]), Păun introduces the relative twisted Kähler-Einstein metric and shows that its geodesic curvature satisfies a certain elliptic PDE. Equation (7) is the parabolic version of the elliptic PDE.

Now we prove the main theorem.

Theorem 4.2 (Theorem 1.1)**.** *If $\omega(0)$ is semi-positive (resp. positive), then $\omega(t)$ is semi-positive (resp. positive) for all $t \in (0,\infty)$. More precisely, if $\Omega\left(v_{\omega(0)}, \overline{v_{\omega(0)}}\right)$ or $\left|\bar{\partial} v_{\omega(0)}\right|^2_{\omega(0)}$ does not vanish identically on X_s, then $\omega(t)$ is strictly positive on X_s for all $t \in [0,\infty)$.*

Proof. Applying the weak parabolic maximum principle to Equation (7), one can conclude that if $c(\omega(0))$ is semi-positive then $c(\omega(t))$ is semi-positive for all $t \in [0,\infty)$ (cf. [4]). Thus $\omega(t)$ is semi-positive on X by (3).

Now suppose that $\Omega\left(v_{\omega(0)}, \overline{v_{\omega(0)}}\right)$ or $\left|\bar{\partial} v_{\omega(0)}\right|^2_{\omega(0)}$ does not vanish identically on a fixed fiber X_s. If we let $h(t) = e^{-t} c(\omega(t))$, then $h(t)$ is non-negative. Moreover, $h(t)$ satisfies that

$$\frac{\partial}{\partial t} h(t) = \Delta_{\omega(t)} h(t) + e^{-t}\left(\left|\bar{\partial} v_{\omega(t)}\right|^2_{\omega(t)} + \Omega\left(v_{\omega(t)}, \overline{v_{\omega(t)}}\right)\right) \geq \Delta_{\omega(t)} h(t)$$

on $X_s \times [0,\infty)$. The strong maximum principle can be invoked to say that $h(t) > 0$ for $t > 0$ or $h(0) \equiv 0$. Suppose that $h(0) \equiv 0$. Then it is easy to see that

$$\left.\frac{\partial}{\partial t} c(\omega(t))\right|_{t=0} = \Omega\left(v_{\omega(0)}, \overline{v_{\omega(0)}}\right) + \left|\bar{\partial} v_{\omega(0)}\right|^2_{\omega(0)}.$$

Sine the right-hand-side is not identically zero, there exists $x_0 \in X_s$ such that $\left.\frac{\partial}{\partial t} c(\omega(t))\right|_{t=0}(x_0) > 0$. So there exists a $\varepsilon > 0$ such that $c(\omega(t))(x_0) > 0$ for $t \in (0,\varepsilon)$. Again the strong maximum principle yields that $c(\omega(t)) > 0$ for $t > 0$. This completes the proof. □

Remark 4. It is well known that $\bar{\partial} v_{\omega(t)}$ is the harmonic representative of the Kodaira-Spencer class of the family $p : X \to \mathbf{D}$ with respect to $\omega(t)$ ([7]). Hence the family is not locally trivial, then $\bar{\partial} v_{\omega(t)} \neq 0$.

For Corollary 1.2, first we recall that the relative twisted Kähler-Einstein metric. On each fiber X_s, Yau's theorem implies that the following complex Monge-Ampère equation has a unique solution ([6]):

$$(\widehat{\omega}|_{X_s} + dd^c \phi_s)^n = e^{\phi_s} (\theta|_{X_s})^n,$$
$$\widehat{\omega}|_{X_s} + dd^c \phi_s > 0.$$

Again the implicit function theorem implies that the function $\phi : X \to \mathbb{R}$ defined by $\phi(x) = \phi_s(x)$ where $p(x) = s$ is smooth on X. The relative twisted Kähler-Einstein metric ρ, which is a d-closed real $(1,1)$ form on X defined by

$$\rho = \widehat{\omega} + dd^c \phi.$$

Then one can see that ρ satisfies that

$$\Theta_\theta = \rho - \Omega.$$

Again it is proved by Cao that $\varphi(t)$ converges locally uniformly to ϕ as $t \to \infty$ in $C^\infty(X)$-topology ([2]). (See also [1]). It also implies that $\omega(t)$ converges to ρ as $t \to \infty$. Therefore $c(\rho)$ is semi-positive since $c(\omega(t))$ converges to $c(\rho)$ as $t \to \infty$. This completes the proof.

References

[1] Berman, R. J., *Relative Kähler-Ricci flows and their quantization*, Anal. PDE 6 (2013), 131–180.
[2] Cao, H.-D., *Deformation of Kähler metrics to Kähler Einstein metrics on compact Kähler manifolds*, Invent. Math. 81(2) (1985), 359–372.
[3] Choi, Y.-J., Yoo, S., *Fiberwise Kähler-Ricci flows on families of bounded strongly pseudoconvex domains*, arXiv:2005.12645.
[4] Chow, B., Lu, P., Ni, L., *Hamilton's Ricci flow*, Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006. xxxvi+608 pp
[5] Păun, M., *Relative adjoint transcendental classes and Albanese map of compact Kähler manifolds with nef Ricci curvature*, Higher Dimensional Algebraic Geometry: In honour of Professor Yujiro Kawamata's sixtieth birthday. Mathematical Society of Japan, 2017.
[6] Yau, S.-T., *On the Ricci curvature of a compact Kähler Manifold and the complex Monge-Ampere equation I*, Comm. Pure Appl. Math. 31 (1978), 339–411.
[7] Schumacher, G., *Positivity of relative canonical bundles and applications*, Invent. Math. 190 (2012), no. 1, 1–56.

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