# AN EFFICIENT HYBRID NUMERICAL METHOD FOR THE TWO-ASSET BLACK-SCHOLES PDE 

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#### Abstract

In this paper, an efficient hybrid numerical method for solving two-asset option pricing problem is presented based on the Crank-Nicolson and the radial basis function methods. For this purpose, the two-asset Black-Scholes partial differential equation is considered. Also, the convergence of the proposed method are proved and implementation of the proposed hybrid method is specifically studied on Exchange and Call on maximum Rainbow options. In addition, this method is compared to the explicit finite difference method as the benchmark and the results show that the proposed method can achieve a noticeably higher accuracy than the benchmark method at a similar computational time. Furthermore, the stability of the proposed hybrid method is numerically proved by considering the effect of the time step size to the computational accuracy in solving these problems.


## 1. Introduction

The financial markets are becoming more complex with trading many types of financial derivatives. A financial derivative is a contract with a value dependant on one or several underlying assets. The markets require updated values of these derivatives every second of the day, so pricing methods need to be more efficient.

Options are some of the most common derivatives. There are two main types of options: A call option gives its owner the right, but not the obligation, to buy some asset for a price referred to as the exercise price or strike price until a specified time called the expiration time of the option. The payoff of call option for strike price $E$ and expiration time $T$ is, payoff $=$ $\max (S-E, 0)$. A put option gives its owner the right, but not the obligation, to sell some asset for an exercise price until the expiration time of the option. The payoff of this option for strike price $E$ and expiration time $T$ defined by payoff $=\max (E-S, 0)$.

Furthermore, options are classified into European and American options. European options can be exercised only on the expiration time, but American options can be exercised any time on or before the expiration time.

[^0]Black-Scholes [1] and Merton [2] introduced a parabolic partial differential equation (PDE) that the price of the European option satisfies under certain assumption. During the last decades, researchers have been presenting some numerical methods in order to solve BlackScholes equation such as finite difference method $[3,4,5]$ and radial basis functions (RBFs) method $[6,7,8,9,10,11]$. See for more results $[12,13,14,15]$.

RBFs method is known as a powerful tool for interpolation of scattered data. The main advantage of radial basis functions method, is its meshless characteristic. The simplicity of the method and its ability to interpolate the scatterd data as well as its direct extension to higher dimension has made this method an important subject of numerical solution of PDEs. There are two kinds of RBFs, the picewise smooth and the infinitely smooth RBFs. Infinitely smooth RBFs have a shape parameter $c$, as the shape parameter has significant effect on the acurracy of the method, and the infinitely smooth RBFs can be spectrally accurate [16, 17]. Some wellknown RBFs are Gaussian (GA), Multiquadratic (MQ), Inverse Multiquadratic (IMQ) and Thin plate spline (TPS), table 1.

TABLE 1. Some well-known RBFs

| Name of function | Definition |
| :---: | :---: |
| Infinitely smooth RBFs |  |
| Gaussian (GA) | $\varphi(r)=e^{-c^{2} r^{2}}$ |
| Inverse quadratic (IQ) | $\varphi(r)=\left(c^{2}+r^{2}\right)^{-1}$ |
| Multiquadric (MQ) | $\varphi(r)=\sqrt{c^{2}+r^{2}}$ |
| Inverse Multiquadric (IMQ) | $\varphi(r)=\sqrt{c^{2}+r^{2}}-1$ |
| Piecewise smooth RBFs |  |
| Linear | $\varphi(r)=r$ |
| Cubic | $\varphi(r)=r^{3}$ |
| Thin plate spline (TSP) | $\varphi(r)=r^{2} \log (r)$ |

In this paper we applied RBFs in order to solve the two-asset Black-Scholes PDE for Exchange model and call on maximum model. In order to this purpose we use Multiquadratic radial basis functions.

The rest of this paper is organized as follows: In section 2 we discuss multi-asset BlackSholes, and in special case the two-asset Black-Scholes PDE. In section 3, a proposed method based on $\theta$ method and RBFs method for solving two-assets Black-Scholes PDE is presented. In addition, stability and convergence of the proposed method are proved in section 4. Exchange option and call on maximum Rainbow option are introduced in section 5. The proposed method is applied to solve these problems and their obtained numerical results are presented. In addition, the effect of the time step size $(\Delta t)$ and number of basis functions $(N)$ to the computational accuracy of proposed method are studied.

## 2. Multi-Asset Black-Scholes PDE

In the option market, various traded options are multi-asset options. The payoff of this group of options depends on more than one underlying assets.
Consider a portfolio consisting, $N$ underlying assets. Let $S_{i}$ be the price for asset $i, i=$ $1, \ldots, N$, in which each asset price follows a geometric Brownian motion

$$
\begin{equation*}
d S_{i}=S_{i}\left(\mu_{i} d t+\sigma_{i} d w_{i}\right) \quad i=1, \ldots, N \tag{2.1}
\end{equation*}
$$

where $\mu_{i}$ and $\sigma_{i}$ for $i=1, \ldots, N$ are denotes the average rate of growth and volatility of asset $S_{i}$, respectively. Each $d w_{i}$ for $i=1, \ldots, N$ satisfies

$$
E\left(d w_{i}\right)=0 \quad \text { and } \quad E\left(d w_{i}^{2}\right)=d t
$$

The $N$ Winner processes $w_{i}$ are correlated according to

$$
d w_{i} d w_{j}=\rho_{i j} d t
$$

where $\rho$ is the symmetric matrix

$$
\rho=\left[\begin{array}{ccccc}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1 N} \\
\rho_{12} & 1 & \rho_{23} & \cdots & \rho_{2 N} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\rho_{1 N} & \rho_{2 N} & \rho_{3 N} & \cdots & 1
\end{array}\right]
$$

So, we have

$$
d S_{i} d S_{j}=\sigma_{i} \sigma_{j} S_{i} S_{j} \rho_{i j} d t
$$

If the price for the option is $U=U\left(S_{1}, S_{2}, \ldots, S_{N}, t\right)$, the value $\Pi$ of the risk-free self-financing portfolio is given by

$$
\Pi=U-\sum_{i=1}^{N} \Delta_{i} S_{i}
$$

where $\Delta_{i}$ are the shares of each asset in the portfolio, then the increment is:

$$
\begin{equation*}
d \Pi=d U-\sum_{i=1}^{N} \Delta_{i} d S_{i} \tag{2.2}
\end{equation*}
$$

Applying Ito's lemma on the N -dimensional function $U\left(S_{1}, S_{2}, \ldots, S_{N}, t\right)$, we can write

$$
\begin{equation*}
d U=\sum_{i=1}^{N} \sigma_{i} S_{i} \frac{\partial U}{\partial S_{i}} d w_{i}+\left(\frac{\partial U}{\partial t}+\sum_{i=1}^{N} \mu_{i} S_{i} \frac{\partial U}{\partial S_{i}}+\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \frac{\partial^{2} U}{\partial S_{i} \partial S_{j}} \sigma_{i} \sigma_{j} \rho S_{i} S_{j}\right) d t \tag{2.3}
\end{equation*}
$$

Substituting (2.3) and (2.1) into (2.2) gives

$$
\begin{aligned}
& d \Pi=\sum_{i=1}^{N} \sigma_{i} S_{i}\left(\frac{\partial U}{\partial S_{i}}-\Delta_{i}\right) d w_{i}+\left(\frac{\partial U}{\partial t}+\sum_{i=1}^{N} \mu_{i} S_{i} \frac{\partial U}{\partial S_{i}}\right. \\
& \left.+\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \frac{\partial^{2} U}{\partial S_{i} \partial S_{j}} \sigma_{i} \sigma_{j} \rho S_{i} S_{j}-\sum_{i=1}^{N} \mu_{i} \Delta_{i} S_{i}\right) d t
\end{aligned}
$$

In order to eliminate the randomness of the portfolio $\Pi$, we set

$$
\Delta_{i}=\frac{\partial U}{\partial S_{i}}, \quad i=1, \ldots, N
$$

The return of this portfolio should be equal to the return from risk-free intrest rate, which means

$$
d \Pi=\left(\frac{\partial U}{\partial t}+\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \sigma_{i} \sigma_{j} \rho S_{i} S_{j} \frac{\partial^{2} U}{\partial S_{i} \partial S_{j}}\right) d t=r \Pi d t
$$

Therefore

$$
\frac{\partial U}{\partial t}+\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \sigma_{i} \sigma_{j} \rho S_{i} S_{j} \frac{\partial^{2} U}{\partial S_{i} \partial S_{j}}=r\left(U-\sum_{i=1}^{N} S_{i} \frac{\partial U}{\partial S_{i}}\right)
$$

Rearrenging the terms to get the N -dimensional Black-Scholes PDE,

$$
\frac{\partial U}{\partial t}+\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \sigma_{i} \sigma_{j} \rho S_{i} S_{j} \frac{\partial^{2} U}{\partial S_{i} \partial S_{j}}+\sum_{i=1}^{N} r S_{i} \frac{\partial U}{\partial S_{i}}-r U=0
$$

2.1. Two-asset Black-Scholes PDE. A two-asset option is a special case of multi-asset options, where the number of underlying asset is two. The two-asset Black-Scholes equation is a two-dimensional parabolic PDE:

$$
\frac{\partial U}{\partial t}+\frac{1}{2} \sigma_{1}^{2} S_{1}^{2} \frac{\partial^{2} U}{\partial S_{1}^{2}}+\sigma_{1} \sigma_{2} \rho S_{1} S_{2} \frac{\partial^{2} U}{\partial S_{1} \partial S_{2}}+\frac{1}{2} \sigma_{2}^{2} S_{2}^{2} \frac{\partial^{2} U}{\partial S_{2}^{2}}+r S_{1} \frac{\partial U}{\partial S_{1}}+r S_{2} \frac{\partial U}{\partial S_{2}}-r U=0
$$

where $\sigma_{1}$ and $\sigma_{2}$ are the volatility of assets $S_{1}$ and $S_{2}$, respectively, $\rho$ is the correlation coefficient between $S_{1}$ and $S_{2}$ and $r$ is the risk-free rate. $E$ is the strike price and $T$ is the expiration time.
The solution domain is $\left\{S_{1} \in[0, \infty), S_{2} \in[0, \infty), t \in[0, T]\right\}$ and the final condition is:

$$
\begin{equation*}
U\left(S_{1}, S_{2}, T\right)=\text { payoff }\left(S_{1}, S_{2}\right) \tag{2.4}
\end{equation*}
$$

From the change of variable $\tau=T-t$, we obtain

$$
\begin{equation*}
\frac{\partial U}{\partial \tau}-\frac{1}{2} \sigma_{1}^{2} S_{1}^{2} \frac{\partial^{2} U}{\partial S_{1}^{2}}-\sigma_{1} \sigma_{2} \rho S_{1} S_{2} \frac{\partial^{2} U}{\partial S_{1} \partial S_{2}}-\frac{1}{2} \sigma_{2}^{2} S_{2}^{2} \frac{\partial^{2} U}{\partial S_{2}^{2}}-r S_{1} \frac{\partial U}{\partial S_{1}}-r S_{2} \frac{\partial U}{\partial S_{2}}+r U=0 \tag{2.5}
\end{equation*}
$$

Now, the final condition (2.4) becomes initial condition:

$$
U\left(S_{1}, S_{2}, 0\right)=\text { payoff }\left(S_{1}, S_{2}\right)
$$

Two-asset options are usually classified into:

- Exchange option
- Spread option
- Basket option
- Rainbow option

In this paper we focus on Exchange and Rainbow options which will be described in section 5 .

## 3. PROPOSED METHOD FOR TWO-ASSET OPTION PRICING

In this paper we consider two-asset Black-Scholes PDE

$$
\begin{align*}
& \frac{\partial U}{\partial t}(x, y, t)-\frac{1}{2} \sigma_{1}^{2} x^{2} \frac{\partial^{2} U}{\partial x^{2}}(x, y, t)-\sigma_{1} \sigma_{2} \rho x y \frac{\partial^{2} U}{\partial x \partial y}(x, y, t)-\frac{1}{2} \sigma_{2}^{2} y^{2} \frac{\partial^{2} U}{\partial y^{2}}(x, y, t) \\
& -r x \frac{\partial U}{\partial x}(x, y, t)-r y \frac{\partial U}{\partial y}(x, y, t)+r U(x, y, t)=0 \tag{3.1}
\end{align*}
$$

The domain for each asset price is $[0,+\infty)$, but in a numerical method, we usually truncate the domain to $\left[0, L_{1}\right]$ and $\left[0, L_{2}\right]$ respectively. The choice of $L_{1}$ and $L_{2}$ usually depends on the evaluation area we are interested in.
We consider (3.1) with initial condition:

$$
\begin{equation*}
U(x, y, 0)=\text { payoff }(x, y) \tag{3.2}
\end{equation*}
$$

and boundary conditions:

$$
\begin{aligned}
& U(0, y, t)=\alpha(y, t) \\
& U\left(L_{1}, y, t\right)=\beta(y, t) \\
& U(x, 0, t)=\gamma(x, t) \\
& U\left(x, L_{2}, t\right)=\delta(x, t)
\end{aligned}
$$

where $\alpha(y, t), \beta(y, t), \gamma(x, t)$ and $\delta(x, t)$ functions are consistent to the exact solution of (3.1). We discretize the domain with $N$ division in x-axis and y-axis, not necessarily uniform as

$$
\begin{equation*}
\left\{x_{i}\right\}_{i=1}^{N} \operatorname{and}\left\{y_{i}\right\}_{i=1}^{N}, \tag{3.3}
\end{equation*}
$$

and $M$ time steps, so interval $[0, T]$ is discretized with $\Delta t=\frac{T}{M}$, that $T$ denotes the expiration time.
Now, we approximate function $U$ with RBF method according to:

$$
\begin{equation*}
U(x, y, t)=\sum_{i=1}^{N^{2}} \lambda_{i}(t) \phi_{i}(x, y) \tag{3.4}
\end{equation*}
$$

where $\phi(x, y)=\left[\varphi\left(r_{1,1}\right), \ldots, \varphi\left(r_{1, N}\right), \varphi\left(r_{2,1}\right), \ldots, \varphi\left(r_{2, N}\right), \ldots, \varphi\left(r_{N, 1}\right), \ldots, \varphi\left(r_{N, N}\right)\right]$, $r_{i, j}=\sqrt{\left(x-x_{i}\right)^{2}+\left(y-y_{j}\right)^{2}}$ and $\varphi$ is a radial basis function. By defining the operator

$$
D=-\frac{1}{2} \sigma_{1}^{2} x^{2} \frac{\partial^{2}}{\partial x^{2}}-\sigma_{1} \sigma_{2} \rho x y \frac{\partial^{2}}{\partial x \partial y}-\frac{1}{2} \sigma_{2}^{2} y^{2} \frac{\partial^{2}}{\partial y^{2}}-r x \frac{\partial}{\partial x}-r y \frac{\partial}{\partial y}+r
$$

we can rewrite (3.1) to:

$$
\frac{\partial U}{\partial t}(x, y, t)+D U(x, y, t)=0
$$

Using the $\theta$ method

$$
\begin{equation*}
\left(\frac{U(x, y, t+\Delta t)-U(x, y, t)}{\Delta t}+O(\Delta t)\right)+(1-\theta) D U(x, y, t+\Delta t)+\theta D U(x, y, t)=0 \tag{3.5}
\end{equation*}
$$

where the parameter $\theta$ is chosen in interval $[0,1]$. By rearranging (3.5) we have

$$
[1+(1-\theta) \Delta t D] U^{n+1}=[1-\theta \Delta t D] U^{n}
$$

where $U^{n}=U\left(x, y, t^{n}\right)$. Defining

$$
A=1+(1-\theta) \Delta t D \quad \text { and } \quad B=1-\theta \Delta t D
$$

we obtain

$$
\begin{equation*}
A U^{n+1}=B U^{n} \tag{3.6}
\end{equation*}
$$

By using RBF approximation, we find

$$
\begin{align*}
& U^{n+1}=\sum_{i=1}^{N^{2}} \lambda_{i}^{n+1} \phi_{i}(x, y)  \tag{3.7}\\
& U^{n}=\sum_{i=1}^{N^{2}} \lambda_{i}^{n} \phi_{i}(x, y) \tag{3.8}
\end{align*}
$$

Substituting values from (3.7) and (3.8) into (3.6) for all interial and boundary points of collocation points (3.3), we get the scheme in matrix form:

$$
\begin{equation*}
A \Phi \lambda^{n+1}=B \Phi \lambda^{n}+g^{n+1} \tag{3.9}
\end{equation*}
$$

where $\Phi=\left[\varphi\left(r_{i, j}\right)\right]_{i, j=1}^{N}$ and $g^{n+1}$ is a $N^{2} \times 1$ vector, such that according to interial points its components are equal to zero and its other componets are obtained by substituting boundary points into their boundary conditions.
Subsequently (3.9) can be written as

$$
\lambda^{n+1}=(A \Phi)^{-1}(B \Phi) \lambda^{n}+(A \Phi)^{-1} g^{n+1}
$$

So,

$$
\begin{equation*}
\lambda^{n+1}=H \lambda^{n}+G \tag{3.10}
\end{equation*}
$$

where $H=(A \Phi)^{-1}(B \Phi)$ and $G=(A \Phi)^{-1} g^{n+1}$,
from (3.4) and (3.10) it follows that

$$
\begin{equation*}
\mathcal{U}^{n+1}=\Phi H \Phi^{-1} \mathcal{U}^{n}+\Phi G^{n+1} \tag{3.11}
\end{equation*}
$$

In above relation $\mathcal{U}^{0}$ vector is obtained using initial condition (3.2).

## 4. THE CONVERGENCE OF THE PROPOSED METHOD

In this section, we prove the convergence of the scheme (3.11).
We define matrix $E=\Phi H \Phi^{-1}$. The components of the matrix $E$ depends on the constant $\gamma=\frac{\Delta t}{h^{s}}$, where $h$ is the distance between any two nodes, and $s$ is the highest order of partial differential operator, where $s$ is equal to 2 for mentioned problem (3.1).
We know that $\left|U_{t}(x, y)-u_{t}(x, y)\right| \leq \beta_{l} h^{l-1}|u|_{\mathcal{N}_{\varphi}(\Omega)}$, where $l \in \mathbb{N}, \mathcal{N}_{\varphi}(\Omega)$ is a native space of $\operatorname{RBF} \varphi$ and $u^{n}(x, y)$ is the exact solution of (3.1) at time $n \Delta t$. [18]
We assume that (3.11) is accurate of order $p$, it yields that

$$
\begin{equation*}
\mathrm{u}^{n+1}=\Phi H \Phi^{-1} \mathrm{u}^{n}+\Phi G^{n+1}+O\left((\Delta t)+h^{p}\right), \quad \Delta t \rightarrow 0, h \rightarrow 0, \forall n \tag{4.1}
\end{equation*}
$$

Now we define $e^{n}(x, y)=u^{n}(x, y)-U^{n}(x, y)$. By subtracting (3.11) from (4.1) we get:

$$
\mathrm{e}^{n+1}=E \mathrm{e}^{n}+O\left((\Delta t)+h^{p}\right), \quad \Delta t \rightarrow 0, h \rightarrow 0
$$

By Lax-Richtmyer definiton of convergency the scheme in (3.11) is convergent if

$$
\begin{equation*}
\|E\| \leq 1 \tag{4.2}
\end{equation*}
$$

hence, there exist a constant $\eta$ such that

$$
\left\|\mathrm{e}^{n+1}\right\| \leq\|E\|\left\|\mathrm{e}^{n}\right\|+\eta\left((\Delta t)+h^{p}\right)
$$

It is seen that $e^{0}=0$, using the initial condition. So we have

$$
\left\|e^{n+1}\right\| \leq\left(1+\|E\|+\|E\|^{2}+\ldots+\|E\|^{n}\right) \eta\left((\Delta t)+h^{p}\right)
$$

By considering the convergency condition (4.2), we obtain

$$
\left\|\mathrm{e}^{n+1}\right\| \leq(n+1) \eta\left((\Delta t)+h^{p}\right)
$$

So convergence of the scheme is proved.

## 5. IMPLEMENTATION OF THE PROPOSED METHOD

In this section we introduce Exchange and Rainbow options. The numerical solutions for them are further considered using the proposed method.
5.1. Exchange option. Exchange option is usually used in energy market.The payoff of this option is:

$$
\begin{equation*}
\text { payoff }=\max \left(S_{1}-S_{2}, 0\right) \tag{5.1}
\end{equation*}
$$

Since there is no strike price term, the classification of call and put option is not used for this option.

There are various types of boundary conditions, in this paper we consider the following:

$$
\begin{array}{lc}
U\left(0, S_{2}, t\right)=0, & 0 \leq S_{2} \leq L_{2}, 0<t \leq T \\
U\left(L_{1}, S_{2}, t\right)=\max \left(L_{1}-S_{2} e^{-r(T-t)}, 0\right), & 0 \leq S_{2} \leq L_{2}, 0<t \leq T \\
U\left(S_{1}, 0, t\right)=S_{1}, & 0<S_{1}<L_{1}, 0<t \leq T \\
U\left(S_{1}, L_{2}, t\right)=\max \left(S_{1}-L_{2} e^{-r(T-t)}, 0\right), & 0<S_{1}<L_{1}, 0<t \leq T
\end{array}
$$

An analytical solution formula to two-asset Black-Scholes equation for Exchange option was introduced by Margrabe[19]. The exact solution is as bellow:

$$
C\left(S_{1}, S_{2}, t\right)=S_{1} N\left(d_{1}\right)-S_{2} N\left(d_{2}\right),
$$

where

$$
\begin{array}{ll}
d_{1}=\frac{\ln \left(\frac{S_{1}}{S_{2}}\right)+\left(r+\frac{1}{2} \sigma^{2}\right)(T-t)}{\sigma \sqrt{T-t}}, & d_{2}=\frac{\ln \left(\frac{S_{2}}{S_{1}}\right)-\left(r+\frac{1}{2} \sigma^{2}\right)(T-t)}{\sigma \sqrt{T-t}}=d_{1}-\sigma \sqrt{T-t}, \\
\sigma=\sqrt{\sigma_{1}^{2}+\sigma_{2}^{2}-2 \rho \sigma_{1} \sigma_{2}} & N(d)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{d} e^{-\frac{z^{2}}{2}} d z .
\end{array}
$$

The numerical solutions of Exchange option using proposed method is presented in next section.
5.1.1. An example of Exchange option. In this section, we consider PDE (2.5) with initial condition (5.1) and boundary conditions (5.2)-(5.5), when

$$
\sigma_{1}=\sigma_{2}=0.2, T=0.5, \rho=0.1, r=0.1, L_{1}=L_{2}=40
$$

In order to use the proposed method, we suppose $N=10, M=30$ and $\theta=0.5$. This problem is solved by the proposed method by MQ RBF and appropriate shape parameter. The results of the proposed method and the explicit finite difference method as benchmark, are shown in table 2. It can be seen that the proposed method has higher accuracy than the explicit finite difference method. Interestingly, the computational time of the proposed method $(0.1 S)$ is less than the benchmark method (3.6S).

Table 2. Results for Exchange option example

| $S_{1}$ | $S_{2}$ | Approx by proposed method <br> $N=10, \theta=0.5$ | Approx by explicite FDM <br> $\Delta S=0.4, \Delta t=0.005$ | Exact solution |
| :---: | :---: | :---: | :---: | :---: |
| 4 | 8 | $9.404495358567 \times 10^{-5}$ | $7.588127497265 \times 10^{-5}$ | $9.404893887087 \times 10^{-5}$ |
| 8 | 16 | $1.880976487882 \times 10^{-4}$ | $5.650924067118 \times 10^{-5}$ | $1.8809787777417 \times 10^{-4}$ |
| 10 | 4 | 6.195082338527 | 6.000000541463 | 6.195082339727 |
| 16 | 16 | 1.610562395741 | 1.207223681520 | 1.610562395729 |
| 20 | 16 | 4.891152587957 | 4.198831239879 | 4.891152587955 |

In case of $S_{1}=16, S_{2}=16$ the effect of the time step size ( $\Delta t$ ) to the computational accuracy is shown in Fig. 1, and the results, numerically, confirm the stability of the proposed method.


Figure 1. Variation of the absolute error with $\Delta t$ in Exchange option example

The exact solution and the approximate solution in the case of $N=10, S_{1}=16$ and different values of $S_{2}$ are shown in Fig. 2. It can be seen that the exact solution and the approximate solution are almost the same.


Figure 2. The exact solution and the approximate solution in the case of $N=10, S_{1}=16$

Absolute errors in the case of $S_{1}=20$ and different values of $S_{2}$ for $N=10, N=8, N=6$ are shown in Fig. 3.


Figure 3. Absolute errors in the case of $S_{1}=20$ for $N=10, N=8, N=6$
5.2. Rainbow option. Rainbow option is based on a combination of various assets like a rainbow is a combination of various colors. There are different forms of Rainbow option. Some of the typical models of Rainbow options and their payoffs are listed in table 3.

TABLE 3. Examples of Rainbow option

| Name | payoff |
| :---: | :---: |
| Multi-asset Rainbow option | $\max \left(S_{1}-E_{1}, S_{2}-E_{2}, 0\right)$ |
| Pyramid Rainbow option | $\max \left(\left\|S_{1}-E_{1}\right\|+\left\|S_{2}-E_{2}\right\|-E, 0\right)$ |
| Max option | $\max \left(S_{1}, S_{2}\right)$ |
| Call on maximum option | $\max \left(\max \left(S_{1}, S_{2}\right)-E, 0\right)$ |
| Put on maximum option | $\max \left(E-\max \left(S_{1}, S_{2}\right), 0\right)$ |
| Call on minimum option | $\max \left(\min \left(S_{1}, S_{2}\right)-E, 0\right)$ |
| Put on minimum option | $\max \left(E-\min \left(S_{1}, S_{2}\right), 0\right)$ |

For more details refer to[20, 21, 22].
In this paper we consider call on maximum option. The boundary conditions are: $\square C(0,0, t)=0$.If $S_{1}=0$ and $S_{2} \neq 0$, the option value $C$ depends only on $S_{2}$ and $t$ :

$$
\frac{\partial C}{\partial t}\left(S_{2}, t\right)-\frac{1}{2} \sigma_{2}^{2} S_{2}^{2} \frac{\partial^{2} C}{\partial S_{2}^{2}}\left(S_{2}, t\right)-r S_{2} \frac{\partial C}{\partial S_{2}}\left(S_{2}, t\right)+r C\left(S_{2}, t\right)=0
$$If $S_{1} \neq 0$ and $S_{2}=0$, the option value $C$ depends only on $S_{1}$ and $t$ :

$$
\frac{\partial C}{\partial t}\left(S_{1}, t\right)-\frac{1}{2} \sigma_{1}^{2} S_{1}^{2} \frac{\partial^{2} C}{\partial S_{1}^{2}}\left(S_{1}, t\right)-r S_{1} \frac{\partial C}{\partial S_{1}}\left(S_{1}, t\right)+r C\left(S_{1}, t\right)=0
$$If $S_{1} \rightarrow \infty$ and $S_{2} \rightarrow \infty$, the option value is approximately equal to $S_{1}$ or $S_{2}$.If $S_{1} \rightarrow \infty$ and $S_{2}$ is finite, the option value is approximately equal to $S_{1}$.If $S_{1}$ is finite and $S_{2} \rightarrow \infty$, the option value is approximately equal to $S_{2}$.

The exact solution is:

$$
\begin{aligned}
& C\left(S_{1}, S_{2}, t\right)=S_{1}\left[N\left(\delta_{1}\right)-N^{\prime}\left(-d_{1}, \delta_{1}, \rho_{1}\right)\right]+S_{2}\left[N\left(\delta_{2}\right)-N^{\prime}\left(-d_{2}, \delta_{2}, \rho_{2}\right)\right] \\
& +E e^{-r(T-t)} N^{\prime}\left(-d_{1}+\sigma_{1} \sqrt{T-t},-d_{2}+\sigma_{2} \sqrt{T-t}, \rho\right)-E e^{-r(T-t)}
\end{aligned}
$$

where

$$
\begin{aligned}
& d_{1}=\frac{\ln \left(\frac{S_{1}}{E}\right)+\left(r+\frac{1}{2} \sigma_{1}^{2}\right)(T-t)}{\sigma_{1} \sqrt{T-t}}, \quad d_{2}=\frac{\ln \left(\frac{S_{2}}{E}\right)+\left(r+\frac{1}{2} \sigma_{2}^{2}\right)(T-t)}{\sigma_{2} \sqrt{T-t}}, \\
& \delta_{1}=\frac{\ln \left(\frac{S_{1}}{S_{2}}\right)+\left(\frac{1}{2} \sigma^{2}\right)(T-t)}{\sigma \sqrt{T-t}}, \quad \delta_{2}=\frac{\ln \left(\frac{S_{2}}{S_{1}}\right)+\left(\frac{1}{2} \sigma^{2}\right)(T-t)}{\sigma \sqrt{T-t}}, \\
& \rho_{1}=\frac{\rho \sigma_{2}-\sigma_{1}}{\sigma} \rho_{2}=\frac{\rho \sigma_{1}-\sigma_{2}}{\sigma} \\
& N(\delta)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\delta} e^{-\frac{z^{2}}{2} d z .} \\
& N^{\prime}(d, \delta, \rho)=\frac{\sqrt{\sigma_{1}^{2}+\sigma_{2}^{2}-2 \rho \sigma_{1} \sigma_{2}}}{2 \pi \sqrt{1-\rho^{2}}} \int_{-\infty}^{d} \int_{-\infty}^{\delta} e^{-\frac{x^{2}-2 \rho x y+y^{2}}{2\left(1-\rho^{2}\right)}} d x d y
\end{aligned}
$$

The numerical solutions for call on maximum option are discussed in rest of this paper.
5.2.1. An example of call on maximum option. Here we consider PDE (2.5) with initial and boundary conditions consistent with Call on maximum Rainbow option when

$$
E=10, \sigma_{1}=\sigma_{2}=0.2, T=0.5, \rho=0.1, r=0.1, L_{1}=L_{2}=40
$$

In order to use the proposed method we suppose $N=10, M=30$ and $\theta=0.5$. This problem is solved by the proposed method by MQ RBF and appropriate shape parameter. The results of the proposed method and the explicit finite difference method as benchmark, are shown in table 4, which is shown the high accuracy of the proposed method. The computational time of the proposed method $(1.4 S)$ is less than the benchmark method (3.6S).
In case of $S_{1}=20, S_{2}=8$ the effect of the time step size $(\Delta t)$ to the computational accuracy is shown in Fig. 4.

TABLE 4. Results for a Call on maximum option example

| $S_{1}$ | $S_{2}$ | Approx by proposed method <br> $N=10, \theta=0.5$ | Approx by explicite FDM <br> $\Delta S=0.4, \Delta t=0.005$ | Exact solution |
| :---: | :---: | :---: | :---: | :---: |
| 4 | 8 | 0.065720085208 | 0.065795946004 | 0.065720085211 |
| 8 | 16 | 6.487819019705 | 6.487863495039 | 6.487819019515 |
| 10 | 4 | 0.827780396052 | 0.822091299757 | 0.827780396011 |
| 16 | 16 | 7.696995177509 | 7.694959078947 | 7.696995177078 |
| 20 | 8 | 10.487706098473 | 10.487772099135 | 10.487706094291 |
| 20 | 16 | 10.687059187097 | 10.686378201908 | 10.687059187049 |



Figure 4. Variation of the absolute error with $\Delta t$ in Call on maximum option example

The exact solution and the approximate solution in the case of $N=10, S_{2}=20$ and different values of $S_{1}$ are shown in Fig. 5. It can be observed that our numerical result are in excellent alignment with the corresponding exact solution.
Absolute errors in the case of $S_{2}=16$ and different values of $S_{1}$ for $N=10, N=8, N=6$ are shown in Fig. 6.

## 6. CONCLUSION

Two-asset options whose payoff depends on two underlying assets, usually, are categorized into: Exchange, Spread, Basket and Rainbow options. An important way for evaluating these options is to solve two-asset Black-Scholes PDE (3.1). In this paper, an efficient hybrid numerical method for solving PDE (3.1) was introduced based on the Crank-Nicolson and the radial basis functions methods. Furthermore, the convergence of the proposed method were proved. The proposed method were used for pricing of Exchange and Rainbow options. The merit of


Figure 5. The exact solution and the approximate solution in the case of $N=10, S_{2}=20$


Figure 6. Absolute errors in the case of $S_{2}=16$ for $N=10, N=8, N=6$
the proposed hybrid method is its ability to achieve high accuracy without the need to use high computational cost. Furthermore, the effect of the time step size of this method on the computational accuracy was studied and the results confirmed that by reducing the length of the time step size, the error of the method decreases rapidly. Also, it can be observed that our numerical result are in excellent alignment with the corresponding exact solution. The mentioned method can easily be extended to obtain the price of the other two-asset options.

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