AN EFFICIENT HYBRID NUMERICAL METHOD FOR THE TWO-ASSET BLACK-SCHOLES PDE

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ABSTRACT. In this paper, an efficient hybrid numerical method for solving two-asset option pricing problem is presented based on the Crank-Nicolson and the radial basis function methods. For this purpose, the two-asset Black-Scholes partial differential equation is considered. Also, the convergence of the proposed method are proved and implementation of the proposed hybrid method is specifically studied on Exchange and Call on maximum Rainbow options. In addition, this method is compared to the explicit finite difference method as the benchmark and the results show that the proposed method can achieve a noticeably higher accuracy than the benchmark method at a similar computational time. Furthermore, the stability of the proposed hybrid method is numerically proved by considering the effect of the time step size to the computational accuracy in solving these problems.

1. INTRODUCTION

The financial markets are becoming more complex with trading many types of financial derivatives. A financial derivative is a contract with a value dependant on one or several underlying assets. The markets require updated values of these derivatives every second of the day, so pricing methods need to be more efficient.

Options are some of the most common derivatives. There are two main types of options: A call option gives its owner the right, but not the obligation, to buy some asset for a price referred to as the exercise price or strike price until a specified time called the expiration time of the option. The payoff of call option for strike price E and expiration time T is, payoff = max(S - E, 0). A put option gives its owner the right, but not the obligation, to sell some asset for an exercise price until the expiration time of the option. The payoff of this option for strike price E and expiration time T defined by payoff = max(E - S, 0).

Furthermore, options are classified into European and American options. European options can be exercised only on the expiration time, but American options can be exercised any time on or before the expiration time.

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Black-Scholes [1] and Merton [2] introduced a parabolic partial differential equation (PDE) that the price of the European option satisfies under certain assumption. During the last decades, researchers have been presenting some numerical methods in order to solve Black-Scholes equation such as finite difference method [3, 4, 5] and radial basis functions (RBFs) method [6, 7, 8, 9, 10, 11]. See for more results [12, 13, 14, 15].

RBFs method is known as a powerful tool for interpolation of scattered data. The main advantage of radial basis functions method, is its meshless characteristic. The simplicity of the method and its ability to interpolate the scatterd data as well as its direct extension to higher dimension has made this method an important subject of numerical solution of PDEs. There are two kinds of RBFs, the picewise smooth and the infinitely smooth RBFs. Infinitely smooth RBFs have a shape parameter *c*, as the shape parameter has significant effect on the acurracy of the method, and the infinitely smooth RBFs can be spectrally accurate [16, 17]. Some well-known RBFs are Gaussian (GA), Multiquadratic (MQ), Inverse Multiquadratic (IMQ) and Thin plate spline (TPS), table 1.

TABLE 1. Some well-known RBFs

Name of function	Definition	
Infinitely smooth RBFs		
Gaussian (GA)	$\varphi(r) = e^{-c^2 r^2}$	
Inverse quadratic (IQ)	$\varphi(r) = (c^2 + r^2)^{-1}$	
Multiquadric (MQ)	$\varphi(r) = \sqrt{c^2 + r^2}$	
Inverse Multiquadric (IMQ)	$\varphi(r) = \sqrt{c^2 + r^2}^{-1}$	
Piecewise smooth RBFs		
Linear	$\varphi(r) = r$	
Cubic	$\varphi(r) = r^3$	
Thin plate spline (TSP)	$\varphi(r) = r^2 \log(r)$	

In this paper we applied RBFs in order to solve the two-asset Black-Scholes PDE for Exchange model and call on maximum model. In order to this purpose we use Multiquadratic radial basis functions.

The rest of this paper is organized as follows: In section 2 we discuss multi-asset Black-Sholes, and in special case the two-asset Black-Scholes PDE. In section 3, a proposed method based on θ method and RBFs method for solving two-assets Black-Scholes PDE is presented. In addition, stability and convergence of the proposed method are proved in section 4. Exchange option and call on maximum Rainbow option are introduced in section 5. The proposed method is applied to solve these problems and their obtained numerical results are presented. In addition, the effect of the time step size (Δt) and number of basis functions (N) to the computational accuracy of proposed method are studied.

2. Multi-Asset Black-Scholes PDE

In the option market, various traded options are multi-asset options. The payoff of this group of options depends on more than one underlying assets.

Consider a portfolio consisting, N underlying assets. Let S_i be the price for asset i, i = 1, ..., N, in which each asset price follows a geometric Brownian motion

$$dS_i = S_i(\mu_i dt + \sigma_i dw_i) \quad i = 1, ..., N.$$
(2.1)

where μ_i and σ_i for i = 1, ..., N are denotes the average rate of growth and volatility of asset S_i , respectively. Each dw_i for i = 1, ..., N satisfies

$$E(dw_i) = 0$$
 and $E(dw_i^2) = dt$

The N Winner processes w_i are correlated according to

$$dw_i dw_j = \rho_{ij} dt,$$

where ρ is the symmetric matrix

$$\rho = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \rho_{23} & \cdots & \rho_{2N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{1N} & \rho_{2N} & \rho_{3N} & \cdots & 1 \end{bmatrix}.$$

So, we have

$$dS_i dS_j = \sigma_i \sigma_j S_i S_j \rho_{ij} dt.$$

If the price for the option is $U = U(S_1, S_2, ..., S_N, t)$, the value Π of the risk-free self-financing portfolio is given by

$$\Pi = U - \sum_{i=1}^{N} \Delta_i S_i,$$

where Δ_i are the shares of each asset in the portfolio, then the increment is:

$$d\Pi = dU - \sum_{i=1}^{N} \Delta_i dS_i.$$
(2.2)

Applying Ito's lemma on the N-dimensional function $U(S_1, S_2, ..., S_N, t)$, we can write

$$dU = \sum_{i=1}^{N} \sigma_i S_i \frac{\partial U}{\partial S_i} dw_i + \left(\frac{\partial U}{\partial t} + \sum_{i=1}^{N} \mu_i S_i \frac{\partial U}{\partial S_i} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \frac{\partial^2 U}{\partial S_i \partial S_j} \sigma_i \sigma_j \rho S_i S_j\right) dt.$$
(2.3)

Substituting (2.3) and (2.1) into (2.2) gives

$$d\Pi = \sum_{i=1}^{N} \sigma_i S_i (\frac{\partial U}{\partial S_i} - \Delta_i) dw_i + (\frac{\partial U}{\partial t} + \sum_{i=1}^{N} \mu_i S_i \frac{\partial U}{\partial S_i} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \frac{\partial^2 U}{\partial S_i \partial S_j} \sigma_i \sigma_j \rho S_i S_j - \sum_{i=1}^{N} \mu_i \Delta_i S_i) dt.$$

In order to eliminate the randomness of the portfolio Π , we set

$$\Delta_i = \frac{\partial U}{\partial S_i}, \quad i = 1, ..., N.$$

The return of this portfolio should be equal to the return from risk-free intrest rate, which means

$$d\Pi = \left(\frac{\partial U}{\partial t} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \sigma_i \sigma_j \rho S_i S_j \frac{\partial^2 U}{\partial S_i \partial S_j}\right) dt = r \Pi dt.$$

Therefore

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \sigma_i \sigma_j \rho S_i S_j \frac{\partial^2 U}{\partial S_i \partial S_j} = r(U - \sum_{i=1}^{N} S_i \frac{\partial U}{\partial S_i})$$

Rearrenging the terms to get the N-dimensional Black-Scholes PDE,

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \sigma_i \sigma_j \rho S_i S_j \frac{\partial^2 U}{\partial S_i \partial S_j} + \sum_{i=1}^{N} r S_i \frac{\partial U}{\partial S_i} - rU = 0$$

2.1. **Two-asset Black-Scholes PDE.** A two-asset option is a special case of multi-asset options, where the number of underlying asset is two. The two-asset Black-Scholes equation is a two-dimensional parabolic PDE:

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 U}{\partial S_1^2} + \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 U}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 U}{\partial S_2^2} + rS_1 \frac{\partial U}{\partial S_1} + rS_2 \frac{\partial U}{\partial S_2} - rU = 0,$$

where σ_1 and σ_2 are the volatility of assets S_1 and S_2 , respectively, ρ is the correlation coefficient between S_1 and S_2 and r is the risk-free rate. E is the strike price and T is the expiration time.

The solution domain is $\{S_1 \in [0, \infty), S_2 \in [0, \infty), t \in [0, T]\}$ and the final condition is:

$$U(S_1, S_2, T) = payoff(S_1, S_2).$$
(2.4)

From the change of variable $\tau = T - t$, we obtain

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 U}{\partial S_1^2} - \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 U}{\partial S_1 \partial S_2} - \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 U}{\partial S_2^2} - rS_1 \frac{\partial U}{\partial S_1} - rS_2 \frac{\partial U}{\partial S_2} + rU = 0.$$
(2.5)

Now, the final condition (2.4) becomes initial condition:

$$U(S_1, S_2, 0) = payoff(S_1, S_2).$$

Two-asset options are usually classified into:

- Exchange option
- Spread option
- Basket option
- Rainbow option

In this paper we focus on Exchange and Rainbow options which will be described in section 5.

3. PROPOSED METHOD FOR TWO-ASSET OPTION PRICING

In this paper we consider two-asset Black-Scholes PDE

$$\frac{\partial U}{\partial t}(x,y,t) - \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 U}{\partial x^2}(x,y,t) - \sigma_1 \sigma_2 \rho x y \frac{\partial^2 U}{\partial x \partial y}(x,y,t) - \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 U}{\partial y^2}(x,y,t) - rx \frac{\partial U}{\partial x}(x,y,t) - ry \frac{\partial U}{\partial y}(x,y,t) + rU(x,y,t) = 0.$$
(3.1)

The domain for each asset price is $[0, +\infty)$, but in a numerical method, we usually truncate the domain to $[0, L_1]$ and $[0, L_2]$ respectively. The choice of L_1 and L_2 usually depends on the evaluation area we are interested in.

We consider (3.1) with initial condition:

$$U(x, y, 0) = payoff(x, y), \qquad (3.2)$$

and boundary conditions:

$$U(0, y, t) = \alpha(y, t), U(L_1, y, t) = \beta(y, t), U(x, 0, t) = \gamma(x, t), U(x, L_2, t) = \delta(x, t),$$

where $\alpha(y,t)$, $\beta(y,t)$, $\gamma(x,t)$ and $\delta(x,t)$ functions are consistent to the exact solution of (3.1). We discretize the domain with N division in x-axis and y-axis, not necessarily uniform as

$$\{x_i\}_{i=1}^N and\{y_i\}_{i=1}^N,\tag{3.3}$$

and M time steps, so interval [0, T] is discretized with $\Delta t = \frac{T}{M}$, that T denotes the expiration time.

Now, we approximate function U with RBF method according to:

$$U(x, y, t) = \sum_{i=1}^{N^2} \lambda_i(t)\phi_i(x, y),$$
(3.4)

where $\phi(x, y) = [\varphi(r_{1,1}), ..., \varphi(r_{1,N}), \varphi(r_{2,1}), ..., \varphi(r_{2,N}), ..., \varphi(r_{N,1}), ..., \varphi(r_{N,N})],$ $r_{i,j} = \sqrt{(x - x_i)^2 + (y - y_j)^2}$ and φ is a radial basis function. By defining the operator

$$D = -\frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2}{\partial x^2} - \sigma_1 \sigma_2 \rho x y \frac{\partial^2}{\partial x \partial y} - \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2}{\partial y^2} - rx \frac{\partial}{\partial x} - ry \frac{\partial}{\partial y} + r,$$

we can rewrite (3.1) to:

$$\frac{\partial U}{\partial t}(x, y, t) + DU(x, y, t) = 0.$$

Using the θ method

$$\left(\frac{U(x,y,t+\Delta t) - U(x,y,t)}{\Delta t} + O(\Delta t)\right) + (1-\theta)DU(x,y,t+\Delta t) + \theta DU(x,y,t) = 0,$$
(3.5)

where the parameter θ is chosen in interval [0, 1]. By rearranging (3.5) we have

$$[1 + (1 - \theta)\Delta tD]U^{n+1} = [1 - \theta\Delta tD]U^n,$$

where $U^n = U(x, y, t^n)$. Defining

$$A = 1 + (1 - \theta)\Delta tD$$
 and $B = 1 - \theta\Delta tD$,

we obtain

$$AU^{n+1} = BU^n. aga{3.6}$$

By using RBF approximation, we find

$$U^{n+1} = \sum_{i=1}^{N^2} \lambda_i^{n+1} \phi_i(x, y), \qquad (3.7)$$

$$U^{n} = \sum_{i=1}^{N^{2}} \lambda_{i}^{n} \phi_{i}(x, y).$$
(3.8)

Substituting values from (3.7) and (3.8) into (3.6) for all interial and boundary points of collocation points (3.3), we get the scheme in matrix form:

$$A\Phi\lambda^{n+1} = B\Phi\lambda^n + g^{n+1},\tag{3.9}$$

where $\Phi = [\varphi(r_{i,j})]_{i,j=1}^N$ and g^{n+1} is a $N^2 \times 1$ vector, such that according to interial points its components are equal to zero and its other components are obtained by substituting boundary points into their boundary conditions.

Subsequently (3.9) can be written as

$$\lambda^{n+1} = (A\Phi)^{-1}(B\Phi)\lambda^n + (A\Phi)^{-1}g^{n+1}.$$

So,

$$\lambda^{n+1} = H\lambda^n + G, \tag{3.10}$$

where $H = (A\Phi)^{-1}(B\Phi)$ and $G = (A\Phi)^{-1}g^{n+1}$, from (3.4) and (3.10) it follows that

$$\mathcal{U}^{n+1} = \Phi H \Phi^{-1} \mathcal{U}^n + \Phi G^{n+1}.$$
(3.11)

In above relation \mathcal{U}^0 vector is obtained using initial condition (3.2).

4. The convergence of the proposed method

In this section, we prove the convergence of the scheme (3.11). We define matrix $E = \Phi H \Phi^{-1}$. The components of the matrix E depends on the constant $\gamma = \frac{\Delta t}{h^s}$, where h is the distance between any two nodes, and s is the highest order of partial differential operator, where s is equal to 2 for mentioned problem (3.1). We know that $|U_t(x, y) - u_t(x, y)| \leq \beta_l h^{l-1} |u|_{\mathcal{N}_{\varphi}(\Omega)}$, where $l \in \mathbb{N}$, $\mathcal{N}_{\varphi}(\Omega)$ is a native space of RBF φ and $u^n(x, y)$ is the exact solution of (3.1) at time $n\Delta t$. [18] We assume that (3.11) is accurate of order p, it yields that

$$\mathbf{u}^{n+1} = \Phi H \Phi^{-1} \mathbf{u}^n + \Phi G^{n+1} + O((\Delta t) + h^p), \quad \Delta t \to 0, h \to 0, \forall n$$
(4.1)

Now we define $e^n(x, y) = u^n(x, y) - U^n(x, y)$. By subtracting (3.11) from (4.1) we get:

$$\mathbf{e}^{n+1} = E\mathbf{e}^n + O((\Delta t) + h^p), \quad \Delta t \to 0, h \to 0$$

By Lax-Richtmyer definiton of convergency the scheme in (3.11) is convergent if

$$\|E\| \le 1,\tag{4.2}$$

hence, there exist a constant η such that

$$\|\mathbf{e}^{n+1}\| \le \|E\| \|\mathbf{e}^n\| + \eta((\Delta t) + h^p).$$

It is seen that $e^0 = 0$, using the initial condition. So we have

$$\|\mathbf{e}^{n+1}\| \le \left(1 + \|E\| + \|E\|^2 + \dots + \|E\|^n\right) \eta((\Delta t) + h^p).$$

By considering the convergency condition (4.2), we obtain

$$\|\mathbf{e}^{n+1}\| \le (n+1)\eta((\Delta t) + h^p).$$

So convergence of the scheme is proved.

5. IMPLEMENTATION OF THE PROPOSED METHOD

In this section we introduce Exchange and Rainbow options. The numerical solutions for them are further considered using the proposed method.

5.1. Exchange option. Exchange option is usually used in energy market. The payoff of this option is:

$$payoff = max(S_1 - S_2, 0).$$
 (5.1)

Since there is no strike price term, the classification of call and put option is not used for this option.

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There are various types of boundary conditions, in this paper we consider the following:

$$U(0, S_2, t) = 0, \qquad 0 \le S_2 \le L_2, 0 < t \le T, \qquad (5.2)$$

$$U(L_1, S_2, t) = max(L_1 - S_2 e^{-r(T-t)}, 0), \qquad 0 \le S_2 \le L_2, 0 < t \le T, \qquad (5.3)$$

$$U(S_1, 0, t) = S_1, 0 < S_1 < L_1, 0 < t \le T, (5.4)$$

$$U(S_1, L_2, t) = max(S_1 - L_2 e^{-r(T-t)}, 0), \qquad 0 < S_1 < L_1, 0 < t \le T.$$
(5.5)

An analytical solution formula to two-asset Black-Scholes equation for Exchange option was introduced by Margrabe[19]. The exact solution is as bellow:

$$C(S_1, S_2, t) = S_1 N(d_1) - S_2 N(d_2),$$

where

$$d_{1} = \frac{\ln(\frac{S_{1}}{S_{2}}) + (r + \frac{1}{2}\sigma^{2})(T - t)}{\sigma\sqrt{T - t}}, \quad d_{2} = \frac{\ln(\frac{S_{2}}{S_{1}}) - (r + \frac{1}{2}\sigma^{2})(T - t)}{\sigma\sqrt{T - t}} = d_{1} - \sigma\sqrt{T - t},$$
$$\sigma = \sqrt{\sigma_{1}^{2} + \sigma_{2}^{2} - 2\rho\sigma_{1}\sigma_{2}} \qquad N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{z^{2}}{2}} dz.$$

The numerical solutions of Exchange option using proposed method is presented in next section.

5.1.1. An example of Exchange option. In this section, we consider PDE (2.5) with initial condition (5.1) and boundary conditions (5.2)-(5.5), when

$$\sigma_1 = \sigma_2 = 0.2, T = 0.5, \rho = 0.1, r = 0.1, L_1 = L_2 = 40$$

In order to use the proposed method, we suppose N = 10, M = 30 and $\theta = 0.5$. This problem is solved by the proposed method by MQ RBF and appropriate shape parameter. The results of the proposed method and the explicit finite difference method as benchmark, are shown in table 2. It can be seen that the proposed method has higher accuracy than the explicit finite difference method. Interestingly, the computational time of the proposed method (0.1S) is less than the benchmark method (3.6S).

TABLE 2. Results for Exchange option example

S_1	S_2	Approx by proposed method	Approx by explicite FDM	Exact solution
		$N = 10, \theta = 0.5$	$\Delta S = 0.4, \Delta t = 0.005$	
4	8	$9.404495358567 \times 10^{-5}$	$7.588127497265 \times 10^{-5}$	$9.404893887087 \times 10^{-5}$
8	16	$1.880976487882 \times 10^{-4}$	$5.650924067118 \times 10^{-5}$	$1.8809787777417 \times 10^{-4}$
10	4	6.195082338527	6.000000541463	6.195082339727
16	16	1.610562395741	1.207223681520	1.610562395729
20	16	4.891152587957	4.198831239879	4.891152587955

In case of $S_1 = 16$, $S_2 = 16$ the effect of the time step size (Δt) to the computational accuracy is shown in Fig. 1, and the results, numerically, confirm the stability of the proposed method.



FIGURE 1. Variation of the absolute error with Δt in Exchange option example

The exact solution and the approximate solution in the case of $N = 10, S_1 = 16$ and different values of S_2 are shown in Fig. 2. It can be seen that the exact solution and the approximate solution are almost the same.



FIGURE 2. The exact solution and the approximate solution in the case of $N = 10, S_1 = 16$

Absolute errors in the case of $S_1 = 20$ and different values of S_2 for N = 10, N = 8, N = 6 are shown in Fig. 3.



FIGURE 3. Absolute errors in the case of $S_1 = 20$ for N = 10, N = 8, N = 6

5.2. **Rainbow option.** Rainbow option is based on a combination of various assets like a rainbow is a combination of various colors. There are different forms of Rainbow option. Some of the typical models of Rainbow options and their payoffs are listed in table 3.

Name	payoff	
Multi-asset Rainbow option	$max(S_1 - E_1, S_2 - E_2, 0)$	
Pyramid Rainbow option	$max(S_1 - E_1 + S_2 - E_2 - E, 0)$	
Max option	$max(S_1, S_2)$	
Call on maximum option	$max(max(S_1, S_2) - E, 0)$	
Put on maximum option	$max(E - max(S_1, S_2), 0)$	
Call on minimum option	$max(min(S_1, S_2) - E, 0)$	
Put on minimum option	$max(E - min(S_1, S_2), 0)$	

TABLE 3. Examples of Rainbow option

For more details refer to [20, 21, 22].

In this paper we consider call on maximum option. The boundary conditions are: $\Box C(0,0,t) = 0.$

 \Box If $S_1 = 0$ and $S_2 \neq 0$, the option value C depends only on S_2 and t:

$$\frac{\partial C}{\partial t}(S_2,t) - \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 C}{\partial S_2^2}(S_2,t) - rS_2 \frac{\partial C}{\partial S_2}(S_2,t) + rC(S_2,t) = 0.$$

 \Box If $S_1 \neq 0$ and $S_2 = 0$, the option value C depends only on S_1 and t:

$$\frac{\partial C}{\partial t}(S_1,t) - \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 C}{\partial S_1^2}(S_1,t) - rS_1 \frac{\partial C}{\partial S_1}(S_1,t) + rC(S_1,t) = 0$$

 \Box If $S_1 \to \infty$ and $S_2 \to \infty$, the option value is approximately equal to S_1 or S_2 . \Box If $S_1 \to \infty$ and S_2 is finite, the option value is approximately equal to S_1 . \Box If S_1 is finite and $S_2 \to \infty$, the option value is approximately equal to S_2 . The exact solution is:

$$C(S_1, S_2, t) = S_1[N(\delta_1) - N'(-d_1, \delta_1, \rho_1)] + S_2[N(\delta_2) - N'(-d_2, \delta_2, \rho_2)] + Ee^{-r(T-t)}N'(-d_1 + \sigma_1\sqrt{T-t}, -d_2 + \sigma_2\sqrt{T-t}, \rho) - Ee^{-r(T-t)},$$

where

$$\begin{split} d_1 &= \frac{\ln(\frac{S_1}{E}) + (r + \frac{1}{2}\sigma_1^2)(T - t)}{\sigma_1\sqrt{T - t}}, \quad d_2 &= \frac{\ln(\frac{S_2}{E}) + (r + \frac{1}{2}\sigma_2^2)(T - t)}{\sigma_2\sqrt{T - t}}, \\ \delta_1 &= \frac{\ln(\frac{S_1}{S_2}) + (\frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \quad \delta_2 &= \frac{\ln(\frac{S_2}{S_1}) + (\frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \\ \rho_1 &= \frac{\rho\sigma_2 - \sigma_1}{\sigma} \qquad \rho_2 &= \frac{\rho\sigma_1 - \sigma_2}{\sigma} \qquad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \\ N(\delta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} e^{-\frac{z^2}{2}} dz. \\ N'(d, \delta, \rho) &= \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{d} \int_{-\infty}^{\delta} e^{-\frac{x^2 - 2\rhoxy + y^2}{2(1 - \rho^2)}} dxdy. \end{split}$$

The numerical solutions for call on maximum option are discussed in rest of this paper.

5.2.1. An example of call on maximum option. Here we consider PDE (2.5) with initial and boundary conditions consistent with Call on maximum Rainbow option when

$$E = 10, \sigma_1 = \sigma_2 = 0.2, T = 0.5, \rho = 0.1, r = 0.1, L_1 = L_2 = 40$$

In order to use the proposed method we suppose N = 10, M = 30 and $\theta = 0.5$. This problem is solved by the proposed method by MQ RBF and appropriate shape parameter. The results of the proposed method and the explicit finite difference method as benchmark, are shown in table 4, which is shown the high accuracy of the proposed method. The computational time of the proposed method (1.4S) is less than the benchmark method (3.6S).

In case of $S_1 = 20, S_2 = 8$ the effect of the time step size (Δt) to the computational accuracy is shown in Fig. 4 .

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TABLE 4. Results for a Call on maximum option example

S_1	S_2	Approx by proposed method	Approx by explicite FDM	Exact solution
		$N = 10, \theta = 0.5$	$\Delta S = 0.4, \Delta t = 0.005$	
4	8	0.065720085208	0.065795946004	0.065720085211
8	16	6.487819019705	6.487863495039	6.487819019515
10	4	0.827780396052	0.822091299757	0.827780396011
16	16	7.696995177509	7.694959078947	7.696995177078
20	8	10.487706098473	10.487772099135	10.487706094291
20	16	10.687059187097	10.686378201908	10.687059187049



FIGURE 4. Variation of the absolute error with Δt in Call on maximum option example

The exact solution and the approximate solution in the case of $N = 10, S_2 = 20$ and different values of S_1 are shown in Fig. 5. It can be observed that our numerical result are in excellent alignment with the corresponding exact solution.

Absolute errors in the case of $S_2 = 16$ and different values of S_1 for N = 10, N = 8, N = 6 are shown in Fig. 6.

6. CONCLUSION

Two-asset options whose payoff depends on two underlying assets, usually, are categorized into: Exchange, Spread, Basket and Rainbow options. An important way for evaluating these options is to solve two-asset Black-Scholes PDE (3.1). In this paper, an efficient hybrid numerical method for solving PDE (3.1) was introduced based on the Crank-Nicolson and the radial basis functions methods. Furthermore, the convergence of the proposed method were proved. The proposed method were used for pricing of Exchange and Rainbow options. The merit of



FIGURE 5. The exact solution and the approximate solution in the case of $N = 10, S_2 = 20$



FIGURE 6. Absolute errors in the case of $S_2 = 16$ for N = 10, N = 8, N = 6

the proposed hybrid method is its ability to achieve high accuracy without the need to use high computational cost. Furthermore, the effect of the time step size of this method on the computational accuracy was studied and the results confirmed that by reducing the length of the time step size, the error of the method decreases rapidly. Also, it can be observed that our numerical result are in excellent alignment with the corresponding exact solution. The mentioned method can easily be extended to obtain the price of the other two-asset options.

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