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CONFORMABLE FRACTIONAL SENSE OF FOAM DRAINAGE EQUATION AND CONSTRUCTION OF ITS SOLUTIONS

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ABSTRACT. The modified F-expansion method is used to construct analytical solutions of the foam drainage equation with time- and space-fractional derivatives. The conformable derivatives are considered as spacial and temporal ones. As a result, some analytical exact solutions including kink, bright-dark soliton, periodic and rational solutions are obtained.

1. INTRODUCTION

Nonlinear partial differential equations (NLPDEs) model many real life problems as well. One of these problems is the drainage or creaming of liquid in a foam. An NLPDE for modelling the foam drainage procedure presents density of the foam as a function of time and its vertical position. One may encounter the presence of liquid foam in nature when liquid/gas systems are processed in industry. We can see foaming processes in distillation and absorption ones. Any interplay between gravity, viscous forces, and surface tension shows the drainage of liquid foams. As a matter of fact, application of foams is very widespread, from producing personal care creams and gels to car making processes (for more details see [1]). This means that foams have everyday utilizations in our real life. For example foams use in material and food processing, fire fighting, lotions and creams (as some personal care products), chemical industries, and structural material sciences [1, 2, 3]. The flow of a liquid through a foam can be described by foam drainage which is driven by gravity and capacity. During the last two decades, there are many efforts to progress in understanding the relevant physics which can be expressed by foam drainage in emulsion (cf. [4, 5, 6, 7] and references therein). Another aspect for a special foam is its stability. The stability is an important criterion for special foams and has an important relevance in the industry of detergents (cf. [8]). All of these show the range of applications of foam drainage or creaming procedure.

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The following time- and space-fractional version of the foam drainage equation is considered in the present work

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2}\, u\, \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + 2\, u^2\, \frac{\partial^\beta u}{\partial x^{2\beta}} + \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2, \; t > 0, \text{ and } \alpha,\; \beta \in (0,1]. \tag{1.1}$$

For integer boundary value 1 for α and β, the foam drainage Eq. (1.1) changes to the following well-known foam drainage equation

$$u_t = \frac{1}{2}\, u\, u_{xx} + 2\, u^2\, u_x + (u_x)^2, \tag{1.2}$$

where "$u(x,t)$ is the cross section of a channel formed where three films meet which usually indicated as *Plateau border*, x and t are scaled position and time coordinates, respectively [9]". Foams present hierarchical materials with complex and various structures without any limitation on their length. To have an example for a soft matter with a well defined structure, we can point out to a liquid foam. For the first time, J. Plateau clearly described such structures more than one-hundred years ago. Foam drainage Eq. (1.2) is investigated by many researchers. A semi analytical scheme, Adomian's decomposition algorithm, is used by Helal and Mehanna [10] to solve (1.2). Also, to obtain solutions of the foam drainage Eq. (1.2) they applied the $\tanh$ method which is an analytical method to solve PDEs. Some new analytical solutions for (1.2) were presented by Khani et al. [11] by using exp-function scheme. The homotopy analysis method is used by Darvishi and Khani [12] to obtain a series solution for (1.2). The Haar wavelets approach is used by Arbabi et al. [13] to present a semi analytical solution for (1.2). Kruglyakov et al. [14] presented a complete review focusing on works about foam drainage up to year 2008, their review paper contains the theoretical and experimental works on foam drainage models.

Foam materials are suitable for absorption and influencing into a spongy media. This in turn, causes that the foam materials have very substantial interests in industry and scientific researches. Hence they are ideal products in cosmetics and pharmaceutical products. Besides, because of their improved wetting behaviour, they are very useful in drug delivery processes. Recently, Trybala et al. [15] published an interesting review on researches on absorption of drainage of foams that placed on spongy substrates. That paper explains theoretical models, experimental observations and simulation studies of a foam drainage as well. Finally, Braun et al. [16] gave an overview on researches related to proteins, polyelectrolyte compositions, and imbibition microgels at single air or water media, in both foam films and a macroscopic foam.

Even though fractional differentiation (derivatives of arbitrary orders) has a 300-year history, recently because of applications of fractional differential equations (FDFs), these equations have received much attention. We may find applications of FDFs in different subjects in engineering (electricity, electromagentics, telecommunications lines, etc), physics (wave propagation, viscoelacity, etc), chemistry, biology, economics, etc. This is because of ability of FDEs in modelling real world problems. The most important property of the fractional derivative is its nonlocal property. As a matter of fact, a suitable modelling of natural phenomena in physics depends on both of the time instant and the prior time history. One can achieve this

situation using fractional derivatives as well because of its nonlocal property. By extending of an integer order differential equation, one may find an FDE. Fractional version of Riccati Eq. [17, 18], non-integer order Schrödinger Eqs. [19, 20], fractional nonlinear coupled Boussinesq-Burgers' Eq. [21], fractional Boussinesq Eq. [22] etc show some of these extensions.
There are some efforts to solve time- and/or space-fractional version of the foam drainage equations. Dahmani and Mesmoudi applied the Adomian method for obtaining solutions of a fractional version of the foam drainage Eq. [23]. Also Dahmani and Anber applied the variational iteration method for obtaining solutions of the equation. Singh et al. [25] used the technique of modified homotopy analysis transform method to solve time fractional version of the foam drainage model. In [26] the variational homotopy perturbation scheme is used to solve the fractional version of the equation with both time- and space-fractional derivatives. Iyiola et al. [27] applied the generalised homotopy analysis method to find solutions of time-fractional foam drainage equation. Hosseini Fadravi et al. [28] used the homotopy analysis method to present analytical solutions of the foam drainage equation with both space- and time-fractional differentiations. Gepreel and Omran [29] investigated analytical solutions for the time- and space-fractional derivative foam drainage model. By improving the $(\frac{G'}{G})$-expansion method, Akgul et al. [30] obtained some new traveling wave solutions of the time- and space-fractional foam drainage model. Alquran [31] solved the time-fractional foam drainage model by using the residual power series scheme. Mirzazadeh et al. [32] obtained some different types of solutions for the time- and space-fractional version of the foam drainage model. Also, a generalized form of time-fractional foam drainage model is studied by Wang et al. [33].

In the present work, the conformable derivative is considered as time- and space-derivative. The αth order conformable derivative is defined as

$$D_x^{\alpha}\left(f(x)\right) = \lim_{\theta \to 0} \frac{f(x + \theta\, x^{1-\alpha}) - f(x)}{\theta},$$

where $\alpha \in (0, 1]$, and f is defined for all positive x. This definition is presented by Khalil et al. [34]. Some properties for conformable fractional derivative can be found in [34]. One may find some recent applications of conformable fractional derivative in [35, 36, 37, 38, 39]. As solidification state of a foam in its creaming or drainage process can be modeled by fractional differentiation more exact than the integer order ones, hence we can conclude that the obtained results in this paper will be useful in any area of foam drainage problem.

2. THE MODIFIED F-EXPANSION ALGORITHM

The F-expansion algorithm is a step-wise procedure to solve NLPDEs. Here, we give the essential steps of the modified F-expansion method [40] to solve FDEs. They are extensions of the method for integer order PDEs.
Step i. A time- and space-fractional PDE can be considered as the following general form wherein the variable U depends on independent variables x, and t

$$\Gamma(U, D_t^{\alpha} U, D_x^{\beta} U, D_t^{2\alpha} U, \cdots) = 0, \tag{2.1}$$

where $\alpha, \beta \in (0,1]$. A fractional complex transform is presented by Li et al. [41] which converts an FDE to an ODE. This transformation is

$$U(x,t) = U(\mu), \qquad \mu = \frac{k}{\beta}\,x^{\beta} + \frac{w}{\alpha}\,t^{\alpha}, \tag{2.2}$$

wherein k, w are non zero arbitrary constants. Substituting (2.2) into (2.1) yields the following ODE

$$\Gamma(U, U^{'}, U^{''}, U^{'''}, \cdots) = 0, \tag{2.3}$$

where the prime shows differentiation with respect to μ.

Step ii. Now, we express $U(\mu)$ as

$$U(\mu) = a_0 + \sum_{i=-M}^{M} a_i\,F^{i}(\mu), \qquad (a_M \text{ is nonzero}), \tag{2.4}$$

where a_0, a_i are unknown constants that will be obtained, and $F(\mu)$ satisfies in the following Riccati equation

$$F^{'}(\mu) = \eta + \lambda\,F(\mu) + \gamma\,F^{2}(\mu). \tag{2.5}$$

For some special values of constants η, λ and γ function F will be obtained. Indeed, one must solve Riccati Eq. (2.5) to obtain function F which it is a simple and straightforward procedure. The solutions of Eq. (2.5) are well-known ones (see Table 1). One may determine the positive integer M by using the homogeneous balance approach between the highest order derivatives of $U(\mu)$ and the governing nonlinear term(s) in (2.3).

Step iii. In this step, we first substitute (2.4) into (2.3). After that, we use (2.5) to convert the left hand side of Eq. (2.3) to a finite series in $F^{i}(\mu)$, $i = -M, \cdots, -1, 0, 1, \cdots, M$. Next, by collecting all terms with the same order of $F^{i}(\mu)$ together, and finally setting each coefficient of the obtained polynomial equal to zero, a set of algebraic equations for $a_i, \eta, \lambda, \gamma, k$ and w is obtained.

Step iv. The solutions of (2.5) are well known, also by substituting $a_i, \eta, \lambda, \gamma, k$ and w into (2.4), the general form of exact solutions for (2.3) are obtained. Hence, as the general form of exact solutions of Eq. (2.5) are listed in Table 1, one may obtain a series of different types of solutions for (2.1).

3. Finding solutions

The modified F-expansion scheme is used for constructing the analytical solutions for the following time- and space-fractional foam drainage model

$$\frac{\partial^{\alpha} u}{\partial\, t^{\alpha}} = \frac{1}{2}\,u\,\frac{\partial^{2\,\beta} u}{\partial\, x^{2\,\beta}} + 2\,u^{2}\,\frac{\partial^{\beta} u}{\partial\, x^{2\,\beta}} + \left(\frac{\partial^{\beta} u}{\partial\, x^{\beta}}\right)^{2},\ t > 0,\ 0 < \alpha \le 1,\ 0 < \beta \le 1. \tag{3.1}$$

In order to achieve this goal, we use the following transformation

$$u(x,t) = u(\mu), \qquad \mu = \frac{k}{\beta}\,x^{\beta} + \frac{w}{\alpha}\,t^{\alpha}. \tag{3.2}$$

TABLE 1. Values of η, λ, γ and their relations with the corresponding function F in (2.5).

Values of η, λ, γ	$F(\mu)$
$\eta = 0, \lambda = 1, \gamma = -1$	$\frac{1}{2} + \frac{1}{2}\tanh(\frac{\mu}{2})$
$\eta = 0, \lambda = -1, \gamma = 1$	$\frac{1}{2} - \frac{1}{2}\coth(\frac{\mu}{2})$
$\eta = \frac{1}{2}, \lambda = 0, \gamma = -\frac{1}{2}$	$\coth(\mu) \pm \mathrm{csch}(\mu), \tanh(\mu) \pm i\,\mathrm{sech}(\mu)$
$\eta = 1, \lambda = 0, \gamma = -1$	$\tanh(\mu), \coth(\mu)$
$\eta = \frac{1}{2}, \lambda = 0, \gamma = \frac{1}{2}$	$\sec(\mu) + \tan(\mu), \csc(\mu) - \cot(\mu)$
$\eta = -\frac{1}{2}, \lambda = 0, \gamma = -\frac{1}{2}$	$\sec(\mu) - \tan(\mu), \csc(\mu) + \cot(\mu)$
$\eta = 1(-1)$, $\lambda = 0, \gamma = 1(-1)$	$\tan(\mu)$, $\cot(\mu)$
$\eta = 0, \lambda = 0, \gamma \neq 0$	$-\frac{1}{\gamma\,\mu + m}$ (m, any arbitrary constant)
η, any arbitrary constant, $\lambda = \gamma = 0$	$\eta\,\mu$
η, any arbitrary constant, $\lambda \neq 0, \gamma = 0$	$\frac{\exp(\lambda) - \eta}{\lambda}$

Parameters k and w in (3.2) are constant values. Using (3.2) changes Eq. (3.1) to

$$-\,w u' + \frac{k^2\,u\,u''}{2} + 2\,k\,u^2\,u' + k^2\,(u')^2 = 0. \tag{3.3}$$

In (3.3) the prime represents differentiation with respect to variable μ.
The highest order derivative is balanced with the highest order nonlinear term in (3.3) which gives $M = 1$. Thus we obtain

$$u(\mu) = a_0 + a_{-1}\,F^{-1}(\mu) + a_1\,F(\mu), \tag{3.4}$$

where a_{-1}, a_0, a_1 are some constants which will be known later. Now, first we set u from (3.4) into (3.3). Next, we collect all terms of powers of $F^i(\mu)$ and set each coefficient equal to zero. All of these works result in a system of algebraic equations. Unknowns of this system are a_{-1}, a_0, a_1, k and w. We can use a symbolic package like Maple to solve this system which obtains the following cases for solutions of the system.
Case 1: For $\eta = 0$:

$$a_{-1} = 0, a_0 = \frac{-k\lambda}{2}, a_1 = -k\gamma, w = \frac{1}{4}k^3\lambda^2, \text{ and } k = k.$$

Case 2: For $\lambda = 0$:

$$a_{-1} = k\eta, a_0 = 0, a_1 = -k\gamma, w = -4\,k^3\eta\gamma, \text{ and } k = k$$

and

$$a_{-1} = 0, a_0 = 0, a_1 = -k\gamma, w = -k^3\eta\gamma, \text{ and } k = k.$$

Case 3: For $\eta = \lambda = 0$:

$$a_{-1} = 0, a_0 = 0, a_1 = -kc, w = w, \text{ and } k = k.$$

Substituting these cases in (3.4), and using information of Table 1, some new analytical solutions are obtained for time- and space-fractional model (3.1).

3.1. **Some new soliton like solutions.** (1) For $\eta = 0, \lambda = 1, \gamma = -1$, from the first case and data in Table 1, we obtain

$$u_1(x,t) = -\frac{1}{2}\,k + k\tanh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{4\,\alpha}\right]. \tag{3.5}$$

(2) For $\eta = 0, \lambda = -1, \gamma = 1$, from information of Table 1 and using the first case, the following solution is obtained

$$u_2(x,t) = \frac{1}{2}\,k - k\coth\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{4\,\alpha}\right]. \tag{3.6}$$

(3) For $\eta = \frac{1}{2}, \lambda = 0, \gamma = -\frac{1}{2}$, using Case 2 and information of Table 1, the following solutions are obtained

$$u_3(x,t) = k\frac{\cosh^2\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right] - 1 + i\sinh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right]}{\cosh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right]\left(\sinh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right] + i\right)},$$

$$u_4(x,t) = -k\frac{-\cosh^2\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right] + 1 + i\sinh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right]}{\cosh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right]\left(\sinh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{\alpha}\right] - i\right)},$$

$$u_5(x,t) = \frac{k}{2}\,\coth\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{4\,\alpha}\right] \pm \frac{k}{2}\operatorname{csch}\left[\frac{\mathrm{k\,x}^\beta}{\beta} + \frac{\mathrm{k}^3\,\mathrm{t}^\alpha}{4\,\alpha}\right],$$

$$u_6(x,t) = \frac{k}{2}\,\tanh\left[\frac{k\,x^\beta}{\beta} + \frac{k^3\,t^\alpha}{4\,\alpha}\right] \pm \frac{k\,i}{2}\operatorname{sech}\left[\frac{\mathrm{k\,x}^\beta}{\beta} + \frac{\mathrm{k}^3\,\mathrm{t}^\alpha}{4\,\alpha}\right].$$

(4) For $\eta = 1, \lambda = 0, \gamma = -1$, and using information of Case 2, values of η, λ and γ in Table

1, yield

$$u_7(x,t) = k\tanh\left[\frac{k\,x^\beta}{\beta} + \frac{4k^3\,t^\alpha}{\alpha}\right] + k\coth\left[\frac{k\,x^\beta}{\beta} + \frac{4k^3\,t^\alpha}{\alpha}\right],$$

$$u_8(x,t) = k\tanh\left[\frac{k\,x^\beta}{\beta} + \frac{4k^3\,t^\alpha}{\alpha}\right],$$

$$u_9(x,t) = k\coth\left[\frac{k\,x^\beta}{\beta} + \frac{4k^3\,t^\alpha}{\alpha}\right].$$

3.2. **Some new trigonometric function solutions.** (i) Setting $\eta = \gamma = \frac{1}{2}, \lambda = 0$, using Table 1 and values of parameters in the third case, we obtain the followings

$$u_{10}(x,t) = -k\tan\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{\alpha}\right],$$

$$u_{11}(x,t) = k\cot\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{\alpha}\right],$$

$$u_{12}(x,t) = -\frac{k}{2}\left(\sec\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right] + \tan\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right]\right),$$

$$u_{13}(x,t) = -\frac{k}{2}\left(\csc\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right] - \cot\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right]\right).$$

(ii) With $\eta = \gamma = -\frac{1}{2}, \lambda = 0$, from Table 1 and values of parameters in the second case, we obtain

$$u_{14}(x,t) = \frac{k}{2}\left(\sec\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right] - \tan\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right]\right),$$

$$u_{15}(x,t) = \frac{k}{2}\left(\csc\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right] + \cot\left[\frac{k\,x^\beta}{\beta} - \frac{k^3\,t^\alpha}{4\alpha}\right]\right).$$

(iii) For $\eta = \gamma = 1, \lambda = 0$, from values of our parameters in Case 2 and data in Table 1, the following solutions are obtained

$$u_{16}(x,t) = k\left(\cot\left[\frac{k\,x^\beta}{\beta} - \frac{4k^3\,t^\alpha}{\alpha}\right] - \tan\left[\frac{k\,x^\beta}{\beta} - \frac{4k^3\,t^\alpha}{\alpha}\right]\right),$$

$$u_{17}(x,t) = k\left(\tan\left[\frac{k\,x^\beta}{\beta} - \frac{4k^3\,t^\alpha}{\alpha}\right] - \cot\left[\frac{k\,x^\beta}{\beta} - \frac{4k^3\,t^\alpha}{\alpha}\right]\right).$$

3.3. **Some new rational solutions.** For $\eta = \lambda = 0, \gamma \neq 0$, and using information of Table 1 and values of our parameters in the third case, we present the following rational solutions for (3.1)

$$u_{18}(x,t) = \frac{k\,\gamma}{\gamma\left[\frac{k\,x^\beta}{\beta} + \frac{w\,t^\alpha}{\alpha}\right] + m},$$

where m is an arbitrary constant.

4. A PHYSICAL VIEW ON THE PRESENTED RESULTS

It is well-known that two different types of envelope solitons, dark and bright, that can propagate in nonlinear dispersive media. A dark soliton is a solitary wave that is generated by cutting a portion of a continuous wave. Nonlinear chirping of the intensity dip in the continuous wave balances the group velocity dispersion of optical fiber in the normal region. Indeed a dark soliton is an amplitude dip in the continuous wave. It is well known that dark soliton is a localized surface, which is an amplitude dip, that causes a temporary decrease in wave amplitude, and usually can exist in normal dispersion region. Moreover, dark solitons are widely studied because of their unique features in inhomogeneous optical fibers. In other words, the intensity profile of the dark soliton exhibits a dip, or hole-soliton, in a uniform background. Dark solitons have shown to be stable and robust to losses. A bright soliton is defined as a localized surface soliton that makes a temporary increase in its wave amplitude, or simply bright soliton is a pulse on a zero-intensity background. This can be easily seen from the solutions of the KdV equation, where the bright soliton solution is represented by $u(x,t) = \text{sech}^2(x,t)$. However, the dark soliton is featured as a localized intensity dip below a continuous wave background. The dark solitons appear in the solutions of the Burgers equation, where the dark soliton solution is represented by $u(x,t) = \tanh(x,t)$. A combination of bright and dark solitons makes bright-dark solitons. Traveling kink solitons represent propagating clockwise twist, to complete this kind of solutions, it must be noted that traveling anti-kink solitons represent propagating counterclockwise twist. Finally, a rational solution can be recovered by taking a long wave limit of the soliton solutions.

In this section, we have a physical view on the constructed solutions of the conformable sense of space- and time-fractional foam drainage Eq. (3.1). Our solutions can be categorized to four different types, namely, kink solutions, soliton or more precisely, bright-dark soliton solutions, periodic solutions and rational ones. In fact, solutions $u_1, u_3, u_4,$ and $u_8(x,t)$ are kink solutions for (3.1). As they had similar forms, only graph of one of them is plotted. Three-dimensional plot of u_1 in (3.5) is figured out in part (a) of Fig. 1, part (b) of this figure shows the solution (3.5) in two dimensional case for $t = 0, 1, \text{and } 2$. Also solutions u_2, u_5, u_6, u_7, u_9 and $u_{15}(x,t)$ are bright-dark soliton solutions of (3.1). Since plots of these solitonic solutions are different from each other, we have potted all of them. Parts (a) of Figs. 2, 3, 4, 5, 6, and 12 show 3D plots of these soliton solutions for some special values of their parameters. Parts (b) of these figures show 2D plots of these solutions for $t = 0, 1, \text{and } 2$. We can easily see the soliton property for all of these solutions from two-dimensional plots. Seven solutions, namely, $u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{16}$ and $u_{17}(x,t)$ are periodic solutions. Parts (a) of Figs. 7, 8, 9, 10, 11, 13, and 14 show these periodic solutions for some special values of their parameters in three-dimensional case while their 2D plots are depicted in parts (b) of these figures. Finally, $u_{18}(x,t)$ is a rational solution of (3.1). Three- and two-dimensional cases of this solutions are depicted in Fig. 15(a), and 15(b). In addition, to investigate the consequence of the non-integer order of space- and time-differentiation of the constructed solutions, two dimensional plots of solutions are depicted. Parts (c) of all figures show that the nature of the solution is changed by changing of the fractional order. Indeed, when $\alpha \to 1$, and $\beta \to 1$, the solutions close to the solution of the integer order of the foam drainage Eq. (1.2). This confirms that, our constructed solutions continuously depend on the non-integer order of derivatives. By these figures and solutions, we will have new physical explanations for the time- and space-fractional version of the foam drainage model. It must be noted that, the foam is a porous medium therfore

5. Conclusions

Four different types analytical solutions have derived for the time- and space-fractional foam drainage model. This is done by applying the modified F-expansion approach. The modified F-expansion scheme is a useful mathematical algorithm to obtain more new exact hyperbolic, rational and trigonometric function solutions for many nonlinear FDEs in physical and mathematical sciences. The modified F-expansion algorithm is a powerful direct and step-wise solution procedure which it is modified to solve fractional PDEs. One can apply this method to solve another nonlinear FDEs that have trigonometric function solutions. We claim that, the presented analytical solutions in this paper were not presented previously in the other published works and all of them are new ones. All presented solutions in this paper are checked to satisfy in their related equations. Finally, the reported analytical solutions showed the powerful application of the modified F-expansion method in solving time- and space-fractional foam drainage equation.

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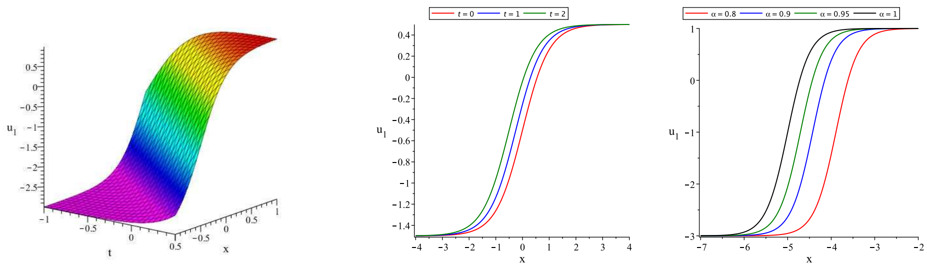
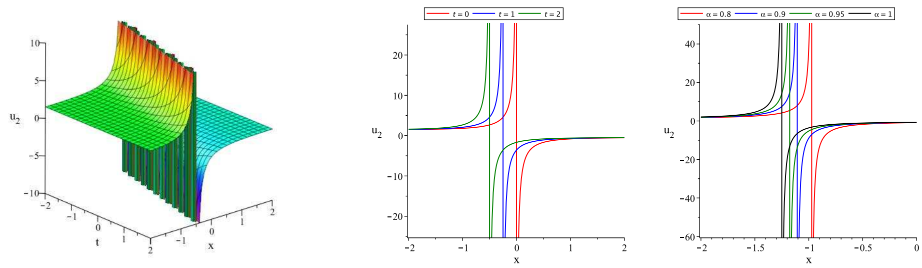
FIGURE 1. (a) Three-dimensional plot of kink solution (3.5) with $k = 2, \alpha = 1$ and $\beta = 1$. (b) Two-dimensional graphs of the kink solution (3.5) for $t = 0, 1, \text{and } 2$. (c) Plots of u_1 in (3.5) for $\beta = 1$ and different values α.



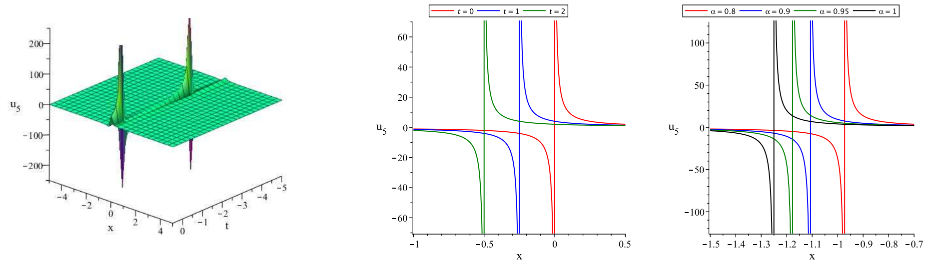
(a) (b) (c)

FIGURE 2. (a) Soliton solution (3.6) with $k = 1, \alpha = 1$ and $\beta = 1$ in three-dimensional case. (b) Plots of the soliton solution (3.6) for $t = 0, 1, 2$ in two-dimensional form. (c) Figures of solution (3.6) for $\beta = 1$ and some values of α.



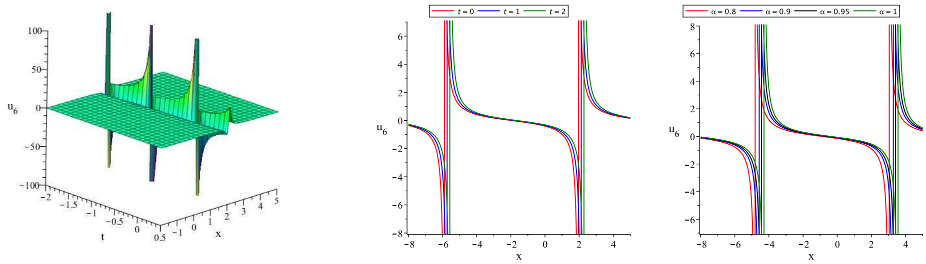
(a) (b) (c)

FIGURE 3. (a) Soliton solution u_5 with $k = 0.5, \alpha = 1$ and $\beta = 1$. (b) Plots of the soliton solution u_5 for $t = 0, 1, 2$. (c) Plots of u_5 for special values of α and β.



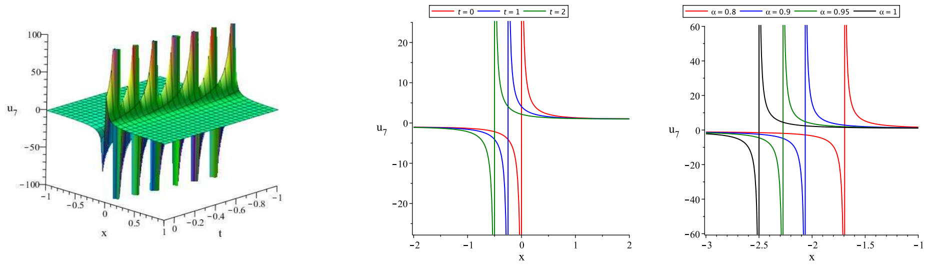
(a) (b) (c)

FIGURE 4. (a) Soliton solution u_6 with $k = 0.8\,i, \alpha = 1$ and $\beta = 1$. (b) Soliton solution u_6 for $t = 0, 1, 2$. (c) Plots of u_6 for some values of α and $\beta = 1$.



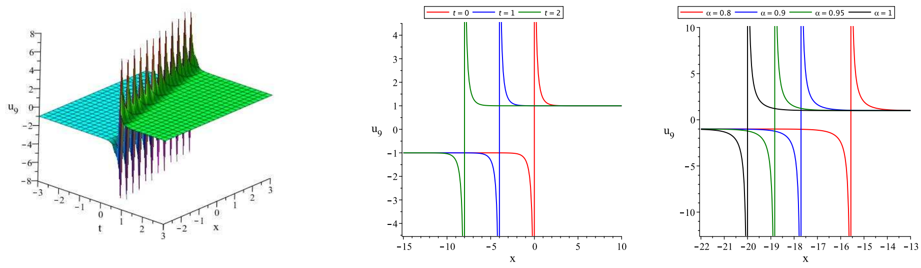
(a) (b) (c)

FIGURE 5. (a) Soliton solution of u_7 with $k = 0.5$, and $\alpha = \beta = 1$. (b) Soliton solution u_7 for $t = 0, 1, 2$. (c) Graphs of u_7 for some special values of α and $\beta = 1$.



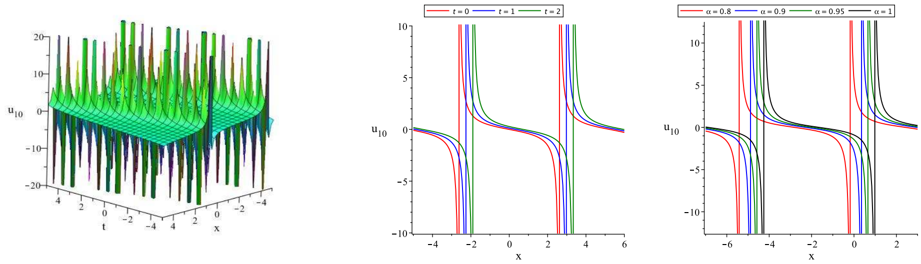
(a) (b) (c)

FIGURE 6. (a) Soliton solution u_9 with $k = \alpha = \beta = 1$. (b) Figures of the soliton solution u_9 for $t = 0, 1, 2$. (c) Plots of u_9 for varying α and special case $\beta = 1$.



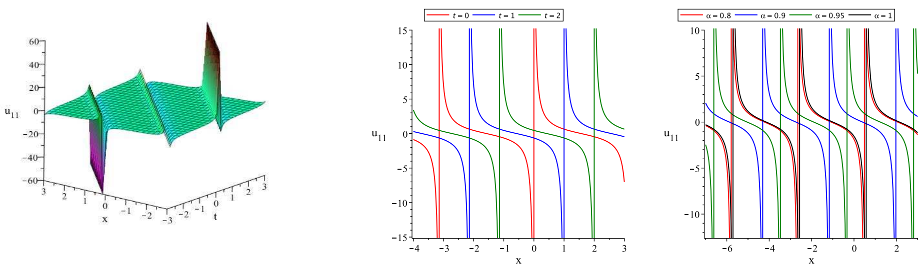
(a) (b) (c)

FIGURE 7. (a) Periodic solution u_{10} with $k = 0.6, \alpha = \beta = 1$. (b) Plots of the periodic solution u_{10} for $t = 0, 1, 2$. (c) Graphs of u_{10} for $\beta = 1$ and special values α.



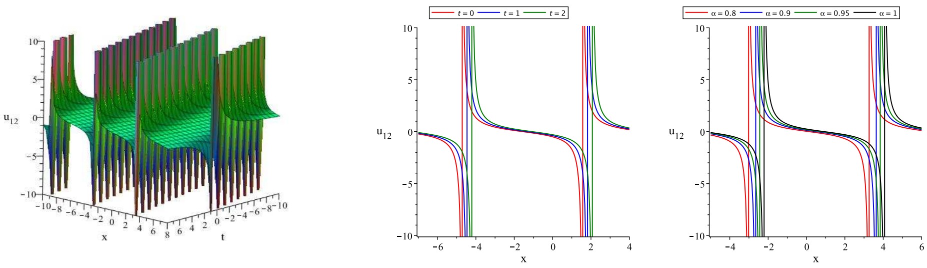
(a) (b) (c)

FIGURE 8. (a) Periodic solution u_{11} with $k = \alpha = \beta = 1$. (b) Plots of the periodic solution u_{11} for $t = 0, 1, 2$. (c) Plots of u_{11} for $\beta = 1$ and varying α.



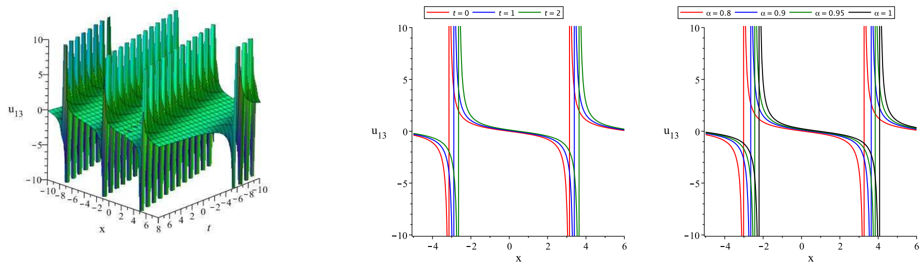
(a) (b) (c)

FIGURE 9. (a) Periodic solution u_{12} with $\alpha = \beta = k = 1$. (b) Figures of the periodic solution u_{12} for $t = 0, 1, 2$. (c) Graphs of u_{12} for $\beta = 1$ and different values of α.



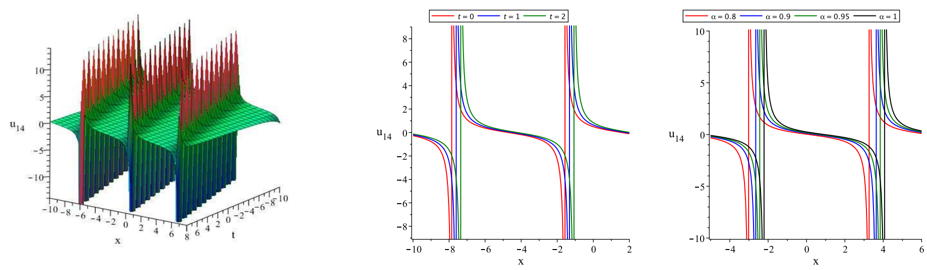
(a) (b) (c)

FIGURE 10. (a) Periodic solution u_{13} with $\beta = k = \alpha = 1$. (b) Plots of the periodic solution u_{13} for $t = 0, 1, 2$. (c) Figures of u_{13} for $\beta = 1$ and $\alpha = 0.8, 0, 9, 0.95, 1$.



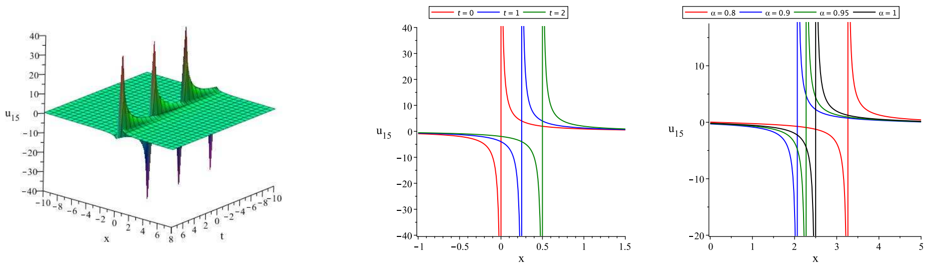
(a) (b) (c)

FIGURE 11. (a) Periodic solution u_{14} with $k = \beta = \alpha = 1$. (b) Plots of the periodic solution u_{14} for $t = 0, 1, 2$. (c) Plots of u_{14} for $\beta = 1$ and $\alpha = 0.8, 0, 9, 0.95, 1$.



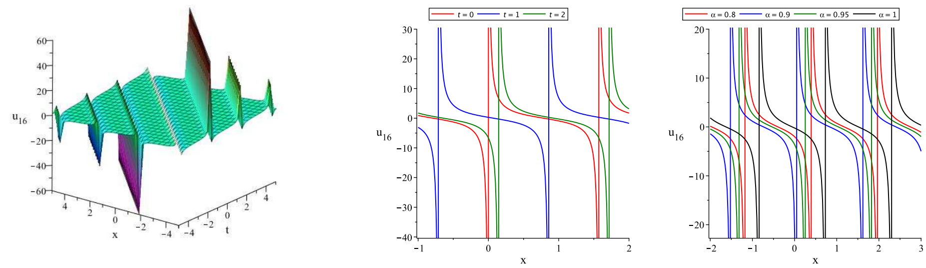
(a) (b) (c)

FIGURE 12. (a) Soliton solution u_{15} with $k = 0.5, \alpha = \beta = 1$. (b) Plots of the soliton solution u_{15} for $t = 0, 1, 2$. (c) Plots of u_{15} for $\beta = 1$ and some special values of α.



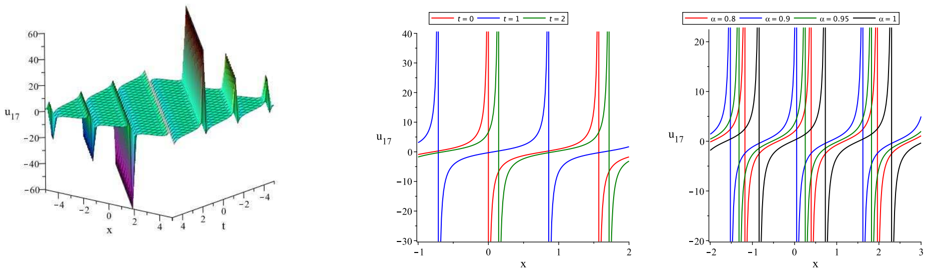
(a) (b) (c)

FIGURE 13. (a) Periodic solution u_{16} with $k = 0.5, \alpha = \beta = 1$. (b) Plots of the periodic solution u_{16} for $t = 0, 1, 2$. (c) Plots of u_{16} for $\beta = 1$ and $\alpha = 0.8, 0, 9, 0.95, 1$.



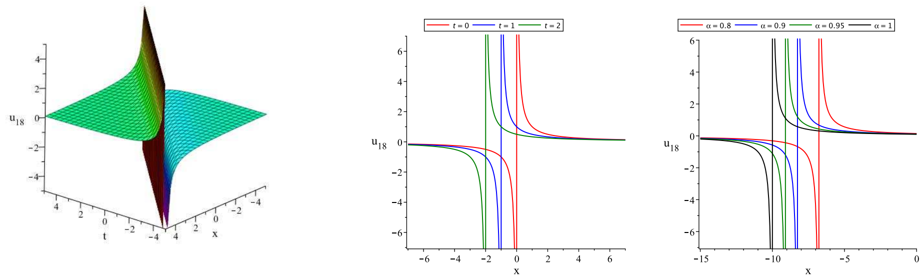
(a) (b) (c)

FIGURE 14. (a) Periodic solution u_{17} with $k = 0.5, \alpha = \beta = 1$. (b) Graphs of the periodic solution u_{17} for $t = 0, 1, 2$. (c) Plots of u_{17} for $\beta = 1$ and $\alpha = 0.8, 0, 9, 0.95, 1$.



(a) (b) (c)

FIGURE 15. (a) Rational solution u_{18} with $k = w = c = m = 1, \alpha = \beta = 1$. (b) Plots of the periodic solution u_{18} for $t = 0, 1, 2$. (c) Graphs of u_{18} for $\beta = 1$ and $\alpha = 0.8, 0, 9, 0.95, 1$.



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