Bull. Korean Math. Soc. **58** (2021), No. 5, pp. 1209–1219
https://doi.org/10.4134/BKMS.b200873
pISSN: 1015-8634 / eISSN: 2234-3016

(σ,σ)-DERIVATION AND (σ,τ)-WEAK AMENABILITY OF BEURLING ALGEBRA

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Abstract. Let G be a topological group with a locally compact and Hausdorff topology. Let ω be a diagonally bounded weight on G. In this paper, (σ,σ)-derivation and (σ,τ)-weak amenability of the Beurling algebra $L^1_\omega(G)$ are studied, where σ,τ are isometric automorphisms of $L^1_\omega(G)$. We prove that every continuous (σ,σ)-derivation from $L^1_\omega(G)$ into measure algebra $M_\omega(G)$ is (σ,σ)-inner and the Beurling algebra $L^1_\omega(G)$ is (σ,τ)-weakly amenable.

1. Introduction

Let $\mathcal{A}$ be an algebra and X be a Banach $\mathcal{A}$-bimodule. The dual X^* of X can be made into a dual Banach $\mathcal{A}$-bimodule, with module actions defined by

$$\langle x, a\cdot f\rangle = \langle x\cdot a, f\rangle,\ \langle x, f\cdot a\rangle = \langle a\cdot x, f\rangle,\quad a\in A, x\in X, f\in X^*.$$

A derivation δ from $\mathcal{A}$ into X is a bounded linear map which satisfies $\delta(ab) = \delta(a)\cdot b + a\cdot\delta(b)$ for all $a,b\in\mathcal{A}$. The derivation δ is said to be inner if there exists $x\in X$ such that $\delta(a) = \delta_x(a) = x\cdot a - a\cdot x$ for all $a\in\mathcal{A}$. When $X=\mathcal{A}$, we say that δ is a derivation on $\mathcal{A}$. Every derivation on a von-Neumann algebra [19] or a nest algebra [4] is inner. Johnson firstly raised the derivation question for group algebra, namely, whether every continuous derivation from group algebra $L^1(G)$ to the measure algebra $M(G)$ is inner and pursued it over the years in developing his theory of cohomology in Banach algebras [10].

The linear space of derivations from $\mathcal{A}$ into X is denoted by $Z^1(\mathcal{A},X)$ and the linear subspace of inner derivations is denoted by $N^1(\mathcal{A},X)$. We consider the quotient space $H^1(\mathcal{A},X) = Z^1(\mathcal{A},X)/N^1(\mathcal{A},X)$, called the first Hochschild cohomology group of $\mathcal{A}$ with coefficients in X. A Banach algebra $\mathcal{A}$ is amenable

Received October 14, 2020; Revised December 25, 2020; Accepted December 30, 2020.
2010 *Mathematics Subject Classification.* 47B49, 46K15.
Key words and phrases. (σ,σ)-derivation, (σ,τ)-weak amenability, Beurling algebras.
This work is supported by the National Natural Science Foundation of China (No. 12061018) and the Postdoctoral Science Foundation of China (No. 2018M633450). The first author is supported by Foundation of Educational Commission (No. KY[2017]092) and of Science and Technology department (No. [2018]1001) of Guizhou Province of China.

if every derivation from $\mathcal{A}$ into every dual Banach $\mathcal{A}$-bimodule is inner, equivalently if $H^1(\mathcal{A}, X^*) = 0$ for every Banach $\mathcal{A}$-bimodule X. This definition was introduced by Johnson in [10] where he proved that the group algebra $L^1(G)$ is amenable if and only if the G is amenable. A Banach algebra $\mathcal{A}$ is called weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = 0$. Johnson in [12] showed that group algebra $L^1(G)$ is weak amenability without any restriction on G, Despic and Ghahramani [6] gave a different proof by utilizing the lattice structure of $L^{\infty}_{\mathbb{R},1/\omega}(G)$. The first characterization of weights making $L^1_\omega(G)$ weakly amenable was given by Grønbæk for discrete Abelian groups in [8]. Extending the Johnson's celebrated result on $L^1(G)$, Grønbæk [9] characterized amenability for Beurling algebra $L^1_\omega(G)$, and proved that $L^1_\omega(G)$ is amenable if and only if the group G is amenable and the weight ω is diagonally bounded. Recently, the amenability and weak amenability of operator algebras has been extensively studied, see for example [1, 6, 11, 16, 20, 22].

The derivation problem for group algebra has been attempted by many researchers since Johnson raised it, and was completely solved affirmatively by Losert [14] in 2008. In 2012, a shorter proof was given using certain fixed point property for L-embedded Banach spaces [2]. Zhang [22] studied the weak amenability of Beurling algebra for locally compact abelian group, and mentioned that there is no result about the derivation problem on Beurling algebra.

Let (σ, τ) be a pair of continuous automorphisms of a Banach algebra $\mathcal{A}$. Naturally, we can define the notions of (σ, τ)-derivation and (σ, τ)-weak amenability. A bounded linear map $\delta : \mathcal{A} \to X$ is called a (σ, τ)-derivation if

$$\delta(ab) = \delta(a) \cdot \sigma(b) + \tau(a) \cdot \delta(b), \quad a, b \in \mathcal{A}.$$

A bounded linear map $\delta : \mathcal{A} \to X$ is called a (σ, τ)-inner derivation if there exists $x \in X$ such that

$$\delta(a) = \delta_x(a) = x \cdot \sigma(a) - \tau(a) \cdot x, \quad a \in \mathcal{A}.$$

We consider the following module actions on $\mathcal{A}$:

$$a \cdot x = \sigma(a)x, x \cdot a = x\tau(a), \quad a, x \in \mathcal{A}.$$

We denote the above $\mathcal{A}$-bimodule by $\mathcal{A}_{(\sigma,\tau)}$. Then $\mathcal{A}$ is called (σ, τ)-weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*{}_{(\sigma,\tau)}) = 0$. These notions are introduced and investigated in [3] and [15].

In this paper, we consider (σ, σ)-derivation and (σ, τ)-weak amenability of the Beurling algebra and prove that every continuous (σ, σ)-derivation from $L^1_\omega(G)$ into measure algebra $M_\omega(G)$ is (σ, σ)-inner and the Beurling algebra $L^1_\omega(G)$ is (σ, τ)-weakly amenable.

2. Preliminaries

Let G be a locally compact group with identity e and a fixed left Haar measure m. A weight on G is a continuous function $\omega : G \to (0, \infty)$ such that $\omega(st) \leq \omega(s)\omega(t)$, $\omega(e) = 1$ for all $s, t \in G$. Two weights ω_1 and ω_2 are equivalent if there exist constants $c_1 > 0$ and $c_2 > 0$ such that $c_1\omega_1(x) \leq \omega_2(x) \leq c_2\omega_1(x)$.

Let X be a Banach space of measures or of equivalence classes of functions on a locally compact group G, and let ω be a weight. We define the Banach space $X(\omega) := \{f \mid \omega f \in X\}$, where the norm of $X(\omega)$ is defined so that the map $f \to \omega f$ from $X(\omega)$ onto X is a linear isometry. In particular, we define

$$L^1_\omega(G) := L^1(G)(\omega) = \{f : \omega f \in L^1(G)\},$$

where $L^1(G)$ is the usual group algebra. Then the Banach space $L^1_\omega(G)$ is a Banach algebra with convolution product

$$f * g(t) = \int_G f(s)g(s^{-1}t)dm(s) \ (s, t \in G)$$

and the following norm

$$\|f\|_\omega = \int_G |f(s)|\omega(s)dm(s).$$

The Banach algebra $L^1_\omega(G)$ is called the Beurling algebra on G associated with the weight ω. It is known that $L^1_\omega(G)$ has a bounded approximate identity. Since $L^1_\omega(G)$ as a Banach space is isometrically isomorphic to $L^1(G)$, we can see that the dual of $L^1_\omega(G)$ is

$$L^\infty_{1/\omega}(G) := L^\infty(G)(1/\omega) = \{f : f/\omega \in L^\infty(G)\},$$

where $L^\infty(G)$ is the C^*-algebra of essentially bounded locally measurable functions on G. The duality is given by $f \mapsto \int_G f(s)g(s)dm(s)$, where $f \in L^1_\omega(G)$ and $g \in L^\infty_{1/\omega}(G)$. Then it can be seen that $L^\infty_{1/\omega}(G)$ is a C^*-algebra for the product $\cdot_\omega$ which is defined by

$$f \cdot_\omega g(s) = f(s)g(s)/\omega(s), \ (f, g \in L^\infty_{1/\omega}(G), \ s \in G),$$

the involution is the map $f \mapsto \bar{f}$ and the norm is defined by

$$\|f\|_{\infty,\omega} = \operatorname*{esssup}_{s\in G} |\frac{f(s)}{\omega(s)}|, \ (f \in L^\infty_{1/\omega}(G)).$$

Let

$$C_{0,1/\omega}(G) := C_0(G)(1/\omega) = \{f \in L^\infty_{1/\omega}(G) : f/\omega \in C_0(G)\},$$

where $C_0(G)$ is the C^*-algebra of all continuous functions vanishing at infinity on G. Then it can be seen that $C_{0,1/\omega}(G)$ with the norm $\| \cdot \|_{\infty,\omega}$ is a Banach space. When endowed with involution and product $\cdot_\omega$ of $L^\infty_{1/\omega}(G)$, $C_{0,1/\omega}(G)$ is a C^*-subalgebra of $L^\infty_{1/\omega}(G)$. Let

$$M_\omega(G) := M(G)(\omega) = \{\mu : \omega\mu \in M(G)\},$$

where $M(G)$ is the usual measure algebra consisting of regular Borel measures with finite total variation on G. Then $M_\omega(G)$ is a Banach space with respect to the norm

$$\|\mu\|_\omega = \int_G \omega(s)d|\mu|(s).$$

It can be seen that as a Banach space $M_\omega(G)$ is isometrically isomorphic to $C_{0,1/\omega}(G)^*$ with the duality given by $f \mapsto \int_G f(s)dm(s)$. The Banach space $M_\omega(G)$ is a Banach algebra under the convolution product

$$\mu * \nu(f) = \int_G \int_G f(xy)d\mu(x)d\nu(y) \quad (\mu, \nu \in M_\omega(G),\ f \in C_{0,1/\omega}(G)).$$

Moreover, we can identify each $f \in L^1_\omega(G)$, with a measure in $M_\omega(G) = C_{0,1/\omega}(G)^*$ via

$$h \mapsto \int_G h(s)f(s)dm(s), \quad (h \in C_{0,1/\omega}(G)^*),$$

and it can be seen that $L^1_\omega(G)$ is a closed two-sided ideal of $M_\omega(G)$. The space $C_c(G)$ of all compactly supported continuous functions is norm dense in $L^1_\omega(G)$. The closure of $C_c(G)$ in $L^\infty_{1/\omega}(G)$ is $C_{0,1/\omega}(G)$ which is a Banach $L^1_\omega(G)$-submodule of $L^\infty_{1/\omega}(G)$. For more details about Beurling algebra see [5].

Consider two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A}$ is a closed ideal of $\mathcal{B}$. For each $a \in \mathcal{A}$, put $p_a(b) = \|ba\| + \|ab\| (b \in \mathcal{B})$, the topology defined on $\mathcal{B}$ by these seminorms is called the strict topology.

A Banach space V is called an L-embedded Banach space if its bidual V^{**} admits a decomposition $V^{**} = V \oplus_1 V_0$ for some $V_0 \subset V^{**}$, where $\oplus_1$ indicates that the norm on V^{**} is the sum of the norms on V and V_0. According to [2], the predual of any von Neumann algebra and, in particular, the dual of any C^*-algebra is an L-embedded Banach space. We rewrite the fixed point theorem [2, Theorem A] in an L-embedded Banach space as follows:

Lemma 2.1. *Let A be a non-empty bounded subset of an L-embedded Banach space V. Then there is a point in V fixed for every isometry of V preserving A.*

Recall that a Banach $\mathcal{A}$-bimodule X is called neo-unital if $\{\mathcal{A} \cdot X \cdot \mathcal{A}\} = X$. The following lemma shows that $C_{0,1/\omega}(G)$ is a neo-unital $L^1_\omega(G)$-bimodule.

Lemma 2.2. $L^1_\omega(G) \cdot C_{0,\frac{1}{\omega}}(G) = C_{0,\frac{1}{\omega}}(G)$.

Proof. By [5, page 77], $C_{0,1/\omega}(G)$ is an $M_\omega(G)$-bimodule, with module actions defined by

$$\langle f, \mu \cdot \lambda \rangle = \langle f * \mu, \lambda \rangle, \quad \langle f, \lambda \cdot \mu \rangle = \langle \mu * f, \lambda \rangle,$$

$f \in L^1_\omega(G)$, $\lambda \in C_{0,1/\omega}(G)$, $\mu \in M_\omega(G)$,

$$\mu \cdot \lambda(t) = \int_G \lambda(ts)d\mu(s), \qquad \lambda \cdot \mu = \int_G \lambda(st)d\mu(s).$$

Thus $L^1_\omega(G)\cdot C_{0,1/\omega}(G)\subseteq C_{0,1/\omega}(G)$. It suffices to show that $C_{0,1/\omega}(G)\subseteq L^1_\omega(G)\cdot C_{0,1/\omega}(G)$. To this end, given $g\in C_c(G)$ we show that g belongs to the norm closure of $L^1_\omega(G)\cdot C_c(G)$. The result will then follow from the density of $C_c(G)$ in $C_{0,1/\omega}(G)$ and the Cohen factorization theorem [7, Theorem 16.1]. Let U be a fixed compact neighbourhood of the identity e in G and let $\{U_i\}_i$ be a base for the neighbourhood system at identity e, with each U_i contained in U. Observe that for each $f_i=\frac{\chi_{U_i}}{m(U_i)}\in L^1_\omega(G)$, $f_i\cdot g(t)=\int_G g(ts)f_i(s)dm(s)$. Let $K:=\operatorname{supp}(g)$ be the support of g. Then $\operatorname{supp}(f_i\cdot g-g)\subseteq KU^{-1}$. Let $M:=\sup\{\omega(x)^{-1}: x\in KU^{-1}\}$. As g is right uniformly continuous on G [13, Lemma 1.3.6], we can choose i_0 such that $\|R_sg-g\|_\infty\le\frac{\epsilon}{M}$ whenever $i\ge i_0$ and $s\in U_i$. Then for $i\ge i_0$ and any $t\in G$, we have

$$\begin{aligned}\Big|\frac{(f_i\cdot g-g)(t)}{\omega(t)}\Big| &\le M\big|(f_i\cdot g-g)(t)\big| = M\Big|\int_G g(ts)f_i(s)dm(s)-g(t)\Big|\\ &= M\Big|\frac{1}{m(U_i)}\int_{U_i}(g(ts)-g(t))dm(s)\Big|\\ &\le \frac{M}{m(U_i)}\int_{U_i}\big|(R_sg-g)(t)\big|dm(s)\\ &\le \frac{M}{m(U_i)}\int_{U_i}\|(R_sg-g)(t)\|_\infty dm(s)\\ &\le \frac{M}{m(U_i)}\int_{U_i}\frac{\epsilon}{M}dm(s)=\epsilon,\end{aligned}$$

which completes the proof by showing that $\|f_i\cdot g-g\|_{\infty,\frac{1}{\omega}}\to 0$. $\square$

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity which is contained as a closed ideal in a Banach algebra $\mathcal{B}$, let X be a neo-unital Banach $\mathcal{A}$-bimodule, and let σ,τ be isometric isomorphisms of $\mathcal{A}$. If D is a continuous (σ,τ)-derivation from $\mathcal{A}$ into X^*, $\tilde\sigma,\tilde\tau$ are extensions of σ and τ, respectively, then there is a unique $(\tilde\sigma,\tilde\tau)$-derivation $\tilde D$ from $\mathcal{B}$ into X^* such that $\tilde D|_{\mathcal{A}}=D$ and $\tilde D$ is continuous with respect to the strict topology on $\mathcal{B}$ and the w^*-topology on E^*.*

Proof. Since the image of a bounded approximate identity under an isometric isomorphism is again a bounded approximate identity, this lemma is proved as in [18, Proposition 2.1.6]. $\square$

3. (σ,σ)-derivationns of Beurling algebras

In this section we find conditions under which (σ,σ) derivations on the Beurling algebras are (σ,σ)-inner. The main result is as follows.

Theorem 3.1. *Let ω be a diagonally bounded weight on a locally compact group G and $\omega\ge 1$. Let σ be an isometric isomorphism of $L^1_\omega(G)$. Then every continuous (σ,σ)-derivation from $L^1_\omega(G)$ into $M_\omega(G)$ is (σ,σ)-inner.*

To complete the proof of Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let ω be a diagonally bounded weight on a locally compact group G and $\omega \geq 1$. Define*

$$\omega'(x) = \sup_{r\in G}\{\omega(rxr^{-1})\}, \quad x \in G.$$

Then ω' is a weight on G equivalent to ω, and $\omega'(x) = \omega'(rxr^{-1})$ for all $x, r \in G$.

Proof. Since ω is diagonally bounded, there is a constant $M > 0$ such that $\omega(r)\omega(r^{-1}) \leq M$ for every $r \in G$. Hence

$$\omega(rxr^{-1}) \leq \omega(x)\omega(r)\omega(r^{-1}) \leq M\omega(x), \quad x, r \in G.$$

This shows for each $x \in G$, $\omega'(x)$ is finite. By definition, ω' is a pointwise supremum of the set of continuous functions $\omega_r(x) = \omega(rxr^{-1})$. For any $a \in R$,

$$\{x \in G : \omega'(x) \leq a\} = \{x \in G : \omega_r(x) \leq a, r \in G\} = \bigcap_{r\in G}\{x \in G : \omega_r(x) \leq a\}.$$

Because for each $r \in G$, ω_r is continuous, each set $\{x \in G : \omega_r(x) \leq a\}$ is closed. Therefore, $\{x \in G : \omega'(x) \leq a\}$ is closed. This shows that ω' is a Borel measurable function. For $x, y \in G$,

$$\begin{aligned}\omega'(xy) &= \sup_{r\in G}\{\omega(rxyr^{-1})\} \leq \sup_{r\in G}\{\omega(rxr^{-1})\omega(ryr^{-1})\} \\ &\leq \sup_{r\in G}\{\omega(rxr^{-1})\} \cdot \sup_{r\in G}\{\omega(ryr^{-1})\} = \omega'(x)\omega'(y).\end{aligned}$$

Also, by definition of ω', it is easy to see $\omega'(e) = 1$, which shows ω' is a measurable weight. By [17, Theorem 3.7.5], any measurable weight bounded below by 1 is equivalent to a continuous weight, hence without loss of generality we may assume that ω' is continuous. We also have

$$\omega(x) = \omega(exe^{-1}) \leq \sup_{r\in G}\{\omega(rxr^{-1})\} = \omega'(x),$$

and so ω' is equivalent to ω. Finally, the last asserted equality is easy to verify. $\square$

Proof of theorem 3.1. Since $L^1_\omega(G)$ has a bounded approximate identity, by Cohen's Factorization Theorem [7, Theorem 16.1], every $f \in L^1_\omega(G)$ can be written as $f = f_1 * f_2$, for some $f_1, f_2 \in L^1_\omega(G)$. Thus,

$$D(f) = D(f_1 * f_2) = f_1 \cdot D(f_2) + D(f_1) * f_2,$$

and since $L^1_\omega(G)$ is an ideal in $M_\omega(G)$, we get $D(f) \in L^1_\omega(G)$. Let ω' be the weight as in Lemma 3.2. Since ω' is equivalent to ω, we have $L^1_\omega(G) \cong L^1_{\omega'}(G)$ and $M_\omega(G) \cong M_{\omega'}(G)$, which means that we can view D as a continuous derivation from $L^1_{\omega'}(G)$ to $M_{\omega'}(G)$. Also $M_{\omega'}(G)$ is a dual of $C_{0,1/\omega'}(G)$, and by Lemma 2.2, $C_{0,1/\omega'}(G)$ is a neo-unital $L^1_\omega(G)$-bimodule. Thus by Lemma

2.3 we can extend D to a bounded $(\tilde{\sigma},\tilde{\sigma})$-derivation $\tilde{D} : M_{\omega'}(G) \to M_{\omega'}(G)$, which is strict-$w^*$ continuous.

Consider the function $b : G \to M_{\omega'}(G)$ defined by $b(t) = \widetilde{D}(\delta_t) * \widetilde{\sigma}(\delta_{t^{-1}})$, where $\delta_t \in M_{\omega'}(G)$ is the point mass at t. We claim that b is bounded. Indeed,

$$\|b(t)\| = \|\widetilde{D}(\delta_t) * \widetilde{\sigma}(\delta_{t^{-1}})\| \leq \|\widetilde{D}\|\|\delta_t\|_{\omega'}\|\delta_{t^{-1}}\|_{\omega'} = \|\widetilde{D}\|\omega'(t)\omega'(t^{-1}),$$

which is bounded as ω' is equivalent to ω, and ω is diagonally bounded. For $t \in G$, $\mu \in M_{\omega'}(G)$, define the action $t \cdot \mu = \widetilde{\sigma}(\delta_t) * \mu * \widetilde{\sigma}(\delta_{t^{-1}})$. Then

$$\begin{aligned} b(st) &= \widetilde{D}(\delta_{st}) * \widetilde{\sigma}(\delta_{(st)^{-1}}) = \widetilde{D}(\delta_s * \delta_t) * \widetilde{\sigma}(\delta_{t^{-1}} * \delta_{s^{-1}}) \\ &= (\widetilde{D}(\delta_s) * \widetilde{\sigma}(\delta_t) + \widetilde{\sigma}(\delta_s) * \widetilde{D}(\delta_t)) * \widetilde{\sigma}(\delta_{t^{-1}}) * \widetilde{\sigma}(\delta_{s^{-1}}) \\ &= (\widetilde{D}(\delta_s) * \widetilde{\sigma}(\delta_{s^{-1}}) + \widetilde{\sigma}(\delta_s) * (\widetilde{D}(\delta_t) * \widetilde{\sigma}(\delta_{t^{-1}})) * \widetilde{\sigma}(\delta_{s^{-1}}) = b(s) + s \cdot b(t). \end{aligned}$$

Using b we can define an action of G on $M_{\omega'}(G)$ as follows.

$$t(\mu) = t \cdot \mu + b(t) = \widetilde{\sigma}(\delta_t) * \mu * \widetilde{\sigma}(\delta_{t^{-1}}) + b(t), \quad t \in G,\ \mu \in M_{\omega'}(G).$$

We claim that this action is isometric. In fact, since $\widetilde{\sigma}$ is an isometric isomorphism of $M_{\omega'}(G)$, by [21, Theorem 2.4], there exist a continuous character $\gamma : G \to \mathbb{T}$ and an automorphism ϕ on G such that for each $t \in G$, we have

$$\widetilde{\sigma}(\delta_t) = \frac{\omega(t)\gamma(t)}{\omega(\phi(t))}\delta_{\phi(t)}.$$

Therefore, for any $\mu_1, \mu_2 \in M_{\omega'}(G)$,

$$\begin{aligned} &\|t(\mu_1 - \mu_2)\|_{M_{\omega'}(G)} \\ &= \|t \cdot (\mu_1 - \mu_2)\|_{M_{\omega'}(G)} \\ &= \|\widetilde{\sigma}(\delta_t) * (\mu_1 - \mu_2) * \widetilde{\sigma}(\delta_{t^{-1}})\|_{M_{\omega'}(G)} \\ &= \int_G \omega'(x) d|\widetilde{\sigma}(\delta_t) * (\mu_1 - \mu_2) * \widetilde{\sigma}(\delta_{t^{-1}})|(x) \\ &= |\frac{\omega(t)\gamma(t)}{\omega(\phi(t))}||\frac{\omega(t^{-1})\gamma(t^{-1})}{\omega(\phi(t^{-1}))}| \int_G \omega'(x) d|\delta_{\phi(t)} * (\mu_1 - \mu_2) * \delta_{\phi(t^{-1})}|(x) \\ &= \int_G \omega'(\phi(t)x\phi(t^{-1})) d|\mu_1 - \mu_2|(x) \\ &= \int_G \omega'(x) d|\mu_1 - \mu_2|(x) \\ &= \|\mu_1 - \mu_2\|_{M_{\omega'}(G)}. \end{aligned}$$

We will now apply the fixed point theorem (Lemma 2.1) to the bounded set $A = b(G)$ in the Banach space $M_{\omega'}(G)$. Note that, since $M_{\omega'}(G)$ is the dual of the $C_{0,1/\omega'}(G)$, hence it is an L-embedded Banach space. We already know that $b(G)$ is a non-empty bounded subset of $M_{\omega'}(G)$, and we only need to observe that $b(G)$ is invariant under the isometric action of G on $M_{\omega'}(G)$ defined

$$t(b(G)) = \{t(b(s)) : s \in G\} = \{t \cdot b(s) + b(t) : s \in G\} = \{b(ts) : s \in G\} = b(G).$$

By the fixed point theorem there is a measure $\mu \in M_{\omega'}(G)$ such that $t(\mu) = \mu$ for all $t \in G$. By definition of b,

$$\widetilde{D}(\delta_t) * \widetilde{\sigma}(\delta_{t^{-1}}) = b(t) = t(\mu) - t \cdot \mu = \mu - t \cdot \mu = \mu - \widetilde{\sigma}(\delta_t) * \mu * \widetilde{\sigma}(\delta_{t^{-1}}),$$

and convoluting this equality with $\tilde{\sigma}(\delta_t)$ on the right, we obtain

$$\widetilde{D}(\delta_t) = \mu * \widetilde{\sigma}(\delta_t) - \widetilde{\sigma}(\delta_t) * \mu.$$

Next we show that

$$D(f) = \widetilde{D}(f) = \mu * \tilde{\sigma}(f) - \tilde{\sigma}(f) * \mu, \quad f \in L^1_{\omega'}(G).$$

Let $f \in L^1_{\omega'}(G)$. Then, by [22, Lemma 2.1], we can find a net $\{f_\alpha\}$ in $\operatorname{lin}\{\delta_t : t \in G\}$ such that $f_\alpha \xrightarrow[\text{strict}]{} f$. Then for each f_α,

$$\widetilde{D}(f_\alpha) = \mu * \widetilde{\sigma}(f_\alpha) - \widetilde{\sigma}(f_\alpha) * \mu.$$

It suffices to show that $\widetilde{D}(f_\alpha) \xrightarrow[\text{strict}]{} D(f)$ and $\mu * \widetilde{\sigma}(f_\alpha) - \widetilde{\sigma}(f_\alpha) * \mu \xrightarrow[\text{strict}]{} \mu * \widetilde{\sigma}(f) - \widetilde{\sigma}(f) * \mu$.

For each $h \in L^1_{\omega'}(G)$, there exists $g \in L^1_{\omega'}(G)$ such that $\sigma(g) = h$. As $\tilde{\sigma}|_{L^1_{\omega'}(G)} = \sigma$, we get $\widetilde{\sigma}(g) = h$. Since $f_\alpha \xrightarrow[\text{strict}]{} f$, we get that $h * f_\alpha \xrightarrow[\text{norm}]{} h * f$. Thus $\widetilde{D}(g * f_\alpha) \xrightarrow[\text{norm}]{} \widetilde{D}(g * f)$ as $\widetilde{D}$ is continuous. Also, since $\widetilde{D}(g) = D(g) \in L^1_{\omega'}(G)$ and $\widetilde{\sigma}(f_\alpha) \xrightarrow[\text{strict}]{} \widetilde{\sigma}(f)$, we get $\widetilde{D}(g) * \widetilde{\sigma}(f_\alpha) \xrightarrow[\text{norm}]{} \widetilde{D}(g) * \widetilde{\sigma}(f)$. Hence, since $\tilde{D}$ is a derivation,

$$\begin{aligned} h * \widetilde{D}(f_\alpha) &= \widetilde{\sigma}(g) * \widetilde{D}(f_\alpha) = \widetilde{D}(g * f_\alpha) - \widetilde{D}(g) * \widetilde{\sigma}(f_\alpha) \\ &\xrightarrow[\text{norm}]{} \widetilde{D}(g * f) - \widetilde{D}(g) * \widetilde{\sigma}(f) = \widetilde{\sigma}(g) * \widetilde{D}(f) = h * \widetilde{D}(f). \end{aligned}$$

Similarly we can prove $\widetilde{D}(f_\alpha) * h \xrightarrow[\text{norm}]{} \widetilde{D}(f) * h$ for every $h \in L^1_{\omega'}(G)$. This means that $\tilde{D}(f_\alpha) \xrightarrow[\text{strict}]{} \tilde{D}(f)$.

Now, let us show that $\mu * \widetilde{\sigma}(f_\alpha) - \widetilde{\sigma}(f_\alpha) * \mu \xrightarrow[\text{strict}]{} \mu * \widetilde{\sigma}(f) - \widetilde{\sigma}(f) * \mu$. For any $h \in L^1_{\omega'}(G)$,

$$h * (\mu * \widetilde{\sigma}(f_\alpha) - \widetilde{\sigma}(f_\alpha) * \mu) = h * \mu * \widetilde{\sigma}(f_\alpha) - h * \widetilde{\sigma}(f_\alpha) * \mu.$$

Since $\widetilde{\sigma}(f_\alpha) \xrightarrow[\text{strict}]{} \widetilde{\sigma}(f)$, $h * \widetilde{\sigma}(f_\alpha) \xrightarrow[\text{norm}]{} h * \widetilde{\sigma}(f)$ in $L^1_{\omega'}(G)$. Thus, we obtain that $h * \widetilde{\sigma}(f_\alpha) * \mu \xrightarrow[\text{norm}]{} h * \widetilde{\sigma}(f) * \mu$. Since $L^1_{\omega'}(G)$ is an ideal in $M_{\omega'}(G)$, we get that $h * \mu \in M_{\omega'}(G)$, and again using the fact that $\widetilde{\sigma}(f_\alpha) \xrightarrow[\text{strict}]{} \widetilde{\sigma}(f)$, $h * \mu * \widetilde{\sigma}(f_\alpha) \xrightarrow[\text{norm}]{} h * \mu * \widetilde{\sigma}(f)$. Hence,

$$h * (\mu * \widetilde{\sigma}(f_\alpha) - \widetilde{\sigma}(f_\alpha) * \mu) \xrightarrow[\text{norm}]{} h * (\mu * \widetilde{\sigma}(f) - \widetilde{\sigma}(f) * \mu)$$

for every $h \in L^1_{\omega'}(G)$. Similarly,

$$(\mu * \widetilde{\sigma}(f_\alpha) - \widetilde{\sigma}(f_\alpha) * \mu) * h \xrightarrow[\text{norm}]{} (\mu * \widetilde{\sigma}(f) - \widetilde{\sigma}(f) * \mu) * h$$

for every $h \in L^1_{\omega'}(G)$. This means that $\mu * \widetilde{\sigma}(f_\alpha) - \widetilde{\sigma}(f_\alpha) * \mu \xrightarrow[\text{strict}]{} \mu * \widetilde{\sigma}(f) - \widetilde{\sigma}(f) * \mu$. This completes the proof. $\square$

Corollary 3.3. *Let G be a locally compact group and $\omega \geq 1$ be a diagonally bounded weight on G. Then every continuous derivation $D : L^1_\omega(G) \to M_\omega(G)$ is inner.*

4. (σ,τ)-weak amenability of the Beurling algebra

In this section we prove the (σ,τ)-weak amenability of Beurling algebra $L^1_\omega(G)$. The idea of proof is borrowed from [6].

Theorem 4.1. *Let G be a locally compact group and ω be a diagonally bounded weight on G. Then the Beurling algebra $L^1_\omega(G)$ is (σ,τ)-weakly amenable.*

Proof. Let D be a bounded (σ,τ)-derivation from $L^1_\omega(G)$ to $L^\infty_{1/\omega}(G)$. Since $L^\infty_{1/\omega}(G)$ is the dual of the neo-unital $L^1_\omega(G)$-bimodule $L^1_\omega(G)$, by Lemma 2.3, we can extend D to a bounded $(\tilde{\sigma},\tilde{\tau})$-derivation $\tilde{D} : M_\omega(G) \to L^\infty_{\frac{1}{\omega}}(G)$, which is continuous in strict-w^* topology. If we show that $\tilde{D}$ is $(\tilde{\sigma},\tilde{\tau})$-inner, then so is D . Consider the set

$$S = \{\mathrm{Re}(\tilde{\tau}(\delta_{t^{-1}}) \cdot \tilde{D}(\delta_t)) : t \in G\},$$

where δ_t denotes the point mass at $t \in G$, and $\mathrm{Re}(\phi)$ stands for the real part of the function $\phi \in L^\infty_{1/\omega}(G)$. Then S is a bounded subset of the vector lattice $L^\infty_{\mathbb{R},1/\omega}(G)$ of real-valued functions in $L^\infty_{1/\omega}(G)$. Indeed,

$$\begin{aligned}
\|\mathrm{Re}(\tilde{\tau}(\delta_{t^{-1}}) \cdot \tilde{D}(\delta_t))\|_{L^\infty_{\frac{1}{\omega}}(G)} &\leq \|\tilde{D}\| \|\tilde{\tau}(\delta_{t^{-1}})\|_{M_\omega(G)} \|\delta_t\|_{M_\omega(G)} \\
&= \|\tilde{D}\| \|\delta_{t^{-1}}\|_{M_\omega(G)} \|\delta_t\|_{M_\omega(G)} \\
&\leq \|\tilde{D}\| \omega(t^{-1})\omega(t) \leq M\|\tilde{D}\|
\end{aligned}$$

because τ is an isometric isomorphism and ω is diagonally bounded say bound with M. Since D is a derivation, for every $x, t \in G$,

$$\begin{aligned}
\tilde{\tau}(\delta_{t^{-1}}) \cdot \tilde{D}(\delta_t) &= \tilde{\tau}(\delta_{t^{-1}}) \cdot \tilde{D}(\delta_{tx^{-1}} * \delta_x) \\
&= \tilde{\tau}(\delta_{t^{-1}})\tilde{\tau}(\delta_{tx^{-1}}) \cdot \tilde{D}(\delta_x) + \tilde{\tau}(\delta_{t^{-1}}) \cdot \tilde{D}(\delta_{tx^{-1}}) \cdot \tilde{\sigma}(\delta_x) \\
&= \tilde{\tau}(\delta_{x^{-1}}) \cdot \tilde{D}(\delta_x) + \tilde{\tau}(\delta_{x^{-1}}) \cdot \tilde{\tau}(\delta_{(tx^{-1})^{-1}}) \cdot \tilde{D}(\delta_{tx^{-1}}) \cdot \tilde{\sigma}(\delta_x).
\end{aligned}$$

Then, since $L^\infty_{\mathbb{R},1/\omega}(G)$ is a complete vector lattice, $\phi_1 = \sup_{t \in G}(S)$ exists in $L^\infty_{\mathbb{R},1/\omega}(G)$. Hence

$$\begin{aligned}
\tilde{\tau}(\delta_x) \cdot \phi_1 &= \tilde{\tau}(\delta_x) \cdot \sup_{t \in G}\{\mathrm{Re}(\tilde{\tau}(\delta_{t^{-1}}) \cdot \tilde{D}(\delta_t))\} \\
&= \tilde{\tau}(\delta_x)\tilde{\tau}(\delta_{x^{-1}}) \cdot \mathrm{Re}\{\tilde{D}(\delta_x)\} + \sup_{t \in G}\{\mathrm{Re}(\tilde{\tau}(\delta_{(tx^{-1})^{-1}}) \cdot \tilde{D}(\delta_{tx^{-1}}) \cdot \tilde{\sigma}(\delta_x)\} \\
&= \mathrm{Re}\{\tilde{D}(\delta_x)\} + \sup_{tx^{-1} \in G}\{\mathrm{Re}(\tilde{\tau}(\delta_{(tx^{-1})^{-1}}) \cdot \tilde{D}(\delta_{tx^{-1}})\} \cdot \tilde{\sigma}(\delta_x)
\end{aligned}$$

$$= \mathrm{Re}\{\tilde{D}(\delta_x)\} + \phi_1 \cdot \tilde{\sigma}(\delta_x).$$

It follows that

$$\mathrm{Re}\{\tilde{D}(\delta_x)\} = \tilde{\tau}(\delta_x) \cdot \phi_1 - \phi_1 \cdot \tilde{\sigma}(\delta_x), \quad x \in G.$$

Similarly, by considering imaginary part, we obtain $\phi_2 \in L^\infty_{\mathbb{R},\frac{1}{\omega}}(G)$ such that

$$\mathrm{Im}\{\tilde{D}(\delta_x)\} = \tilde{\tau}(\delta_x) \cdot \phi_2 - \phi_2 \cdot \tilde{\sigma}(\delta_x), \quad x \in G.$$

Therefore,

$$\tilde{D}(\delta_x) = \tilde{\tau}(\delta_x) \cdot \phi - \phi \cdot \tilde{\sigma}(\delta_x), \quad x \in G,$$

where $\phi = \phi_1 + i\phi_2$. By [22, Lemma 2.1], every measure $\mu \in M_\omega(G)$ is the strict topology limit of a net of linear combinations of point masses and $\tilde{D}$ is strict-w^*-continuous. Therefore, we obtain

$$\tilde{D}(\mu) = \tilde{\tau}(\mu) \cdot \phi - \phi \cdot \tilde{\sigma}(\mu)$$

for every $\mu \in M_\omega(G)$. This means that $\tilde{D}$ is $(\tilde{\sigma}, \tilde{\tau})$-inner, which completes the proof. □

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