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GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS WITH COEFFICIENTS HAVING THE SAME LOGARITHMIC ORDER

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Abstract. In this paper, we investigate the relations between the growth of meromorphic coefficients and that of meromorphic solutions of complex linear differential-difference equations with meromorphic coefficients of finite logarithmic order. Our results can be viewed as the generalization for both the cases of complex linear differential equations and complex linear difference equations.

1. Introduction and Preliminaries

We assume that the readers are familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions and the theory of complex linear differential equations in the complex plane $\mathbb{C}$ which are available in [9, 10]. Recently, the properties of meromorphic solutions of complex difference equations have become a subject of great interest from the viewpoint of Nevanlinna's theory and its difference analogues. Since then, many authors investigated the linear difference equations for example, [5, 11, 12]. In [11], Laine and Yang considered complex linear difference equations and obtained the following theorem.

Theorem 1.1. [11] *Let $A_0(z), ..., A_n(z)$ be entire functions of finite order such that among those having the maximal order $\sigma = \max_{0\leq k\leq n} \sigma(A_k)$, exactly one has its type strictly greater than the others. Then for any meromorphic solution $f\,(\not\equiv 0)$ of*

$$A_n(z) f(z+w_n) + ... + A_1(z) f(z+w_1) + A_0(z) f(z) = 0,$$

where $w_1, ..., w_n$ are distinct complex numbers, we have $\sigma(f) \geq \sigma + 1$.

Liu-Mao [12] considered the hyper-order of meromorphic solutions of the non-homogeneous linear difference equation

$$A_k(z)f(z+k) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = F(z), \tag{1.1}$$

where $k \in N_+$, and obtained the following theorem.

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Theorem 1.2. [12] *Let* $A_j\,(j=0,1,...,k)$ *and* $F(z)\,(\not\equiv 0)$ *be entire functions. If there exists an integer* $l\,(0\leq l\leq k)$ *such that*

$$\max\{\sigma_2(A_j);j=0,1,...,k,j\neq l\}\leq\sigma_2(A_l),\ \ (0<\sigma_2(A_l)<\infty)$$

and

$$\max\{\tau_2(A_j):\sigma_2(A_j)=\sigma_2(A_l);j=0,1,...,k,j\neq l\}<\tau_2(A_l),\ \ (0<\tau_2(A_l)<\infty),$$

then

(*i*) *If* $\sigma_2(F)<\sigma_2(A_l)$ *or* $\sigma_2(F)=\sigma_2(A_l)$ *and* $\tau_2(F)<\tau_2(A_l)$, *then every meromorphic solution* $f(z)\,(\not\equiv 0)$ *of Equation* (1.1) *satisfies* $\sigma(f)=\infty$ *and* $\sigma_2(f)\geq\sigma_2(A_l)$.

(*ii*) *If* $\sigma_2(F)>\sigma_2(A_l)$, *then every meromorphic solution* $f(z)\,(\not\equiv 0)$ *of Equation* (1.1) *satisfies* $\sigma(f)=\infty$ *and* $\sigma_2(f)\geq\sigma_2(F)$.

From above theorems, we deduce that when there is exactly one dominant coefficient among those coefficients having the same maximal order, we may obtain the growth relation between the solutions and the coefficients of the given complex linear difference equations or complex linear differential equations. Recently, many authors investigated the homogeneous and nonhomogeneous linear differential equations {cf. [2, 8, 13]}. Very recently many authors investigated the growth of meromorphic solutions of the linear differential-difference equations

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)\,f^{(j)}(z+c_i)=0 \tag{1.2}$$

and

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)\,f^{(j)}(z+c_i)=F(z), \tag{1.3}$$

and achieved many valuable results when the coefficients $A_{ij}(z)$ $(i=0,1,...,n;\ j=0,1,...,m)$ and $F(z)\,(\not\equiv 0)$ be entire or meromorphic functions of finite order and c_i $(i=0,1,...,n)$ are distinct complex constants {Cf. [6, 14, 15]}.

The theory of meromorphic functions of finite positive order is fairly complete as compared to the theory of functions of order zero. Techniques that work well for functions of finite positive order often do not work for functions of order zero. In order to make some progress with functions of order zero many authors make use of the concept of logarithmic order {Cf. [3, 4]}. The logarithmic order of a meromorphic function f is defined as

$$\sigma_{\log}(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log\log r},$$

$f(z)$ is said to be of finite logarithmic order if the above limit superior is finite. It is clear that if a meromorphic function $f(z)$ has finite logarithmic order, then the order of $f(z)$ is zero. From the definition of logarithmic order, it is easily seen that the logarithmic order of a constant function is zero and of a non-constant rational function is 1. For a transcendental meromorphic function $f(z)$ the logarithmic order is at least 1. There is no meromorphic function with logarithmic order strictly between 0 and 1. Revisiting their ideas we would like to prove some results using the concepts of logarithmic order. In this connection, we recall the following definitions as follows.

We denote the linear measure for a set $E \subset [0, \infty)$, by $m(E) = \int_E dt$ and logarithmic measure for a set $E \subset (1, \infty)$, by $m_l(E) = \int_E \frac{dt}{t}$.

Definition 1.3. [4] The logarithmic order of a meromorphic function f is defined as

$$\sigma_{\log}(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log \log r}.$$

Definition 1.4. [4] The logarithmic type of a meromorphic function f with $1 \leq \sigma_{\log}(f) < \infty$ is defined as

$$\tau_{\log}(f) = \limsup_{r \to \infty} \frac{T(r, f)}{(\log r)^{\sigma_{\log}(f)}}.$$

Definition 1.5. [4] The logarithmic exponent of convergence of zeros and distinct zeros of f are defined by

$$\lambda_{\log}(f) = \limsup_{r \to \infty} \frac{\log n\left(r, \frac{1}{f}\right)}{\log \log r}$$

and

$$\overline{\lambda}_{\log}(f) = \limsup_{r \to \infty} \frac{\log \overline{n}\left(r, \frac{1}{f}\right)}{\log \log r}.$$

respectively, where $n\left(r, \frac{1}{f}\right)$ and $\overline{n}\left(r, \frac{1}{f}\right)$ denote the number of zeros and number of distinct zeros of f in $|z| \leq r$ respectively.

Definition 1.6. [9] For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f is defined as

$$\begin{aligned} \delta(a, f) &= \liminf_{r \to +\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}, \ a \neq \infty, \\ \delta(\infty, f) &= \liminf_{r \to +\infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N(r, f)}{T(r, f)}. \end{aligned}$$

A natural problem arises that: how to express the growth of solutions of the Equation (1.3) when the coefficients $A_{ij}(z)$ $(i = 0, 1, ..., n;\ j = 0, 1, ..., m)$ and $F(z)\,(\not\equiv 0)$ be meromorphic functions of finite logarithmic order. In the next section, we will give a partial answer to the above question.

The main purpose of this paper is to used the concept of logarithmic order in the complex plane to investigate the growth of solutions of linear differential-difference equations. In this direction we obtain the following results.

Theorem 1.7. *Let $A_{ij}(z)$ $(i = 0, 1, ..., n;\ j = 0, 1, ..., m)$ and $F(z)$ be meromorphic functions of finite logarithmic order. If there exists an integer $(0 \leq l \leq n)$ satisfying*

$$\max\{\sigma_{\log}(A_{ij}); (i, j) \neq (l, 0)\} < \sigma_{\log}(A_{l0}) < \infty \quad \text{and} \quad \delta(\infty, A_{l0}) > 0,$$

then

(i) If $\sigma_{\log}(F) < \sigma_{\log}(A_{l0})$, then every meromorphic solution f $(\not\equiv 0)$ of Equation (1.3) of finite logarithmic order satisfies $\sigma_{\log}(f) \geq \sigma_{\log}(A_{l0})$.

(ii) If $\sigma_{\log}(F) > \sigma_{\log}(A_{l0})$, then every meromorphic solution f $(\not\equiv 0)$ of Equation (1.3) of finite logarithmic order satisfies $\sigma_{\log}(f) \geq \sigma_{\log}(F)$.

For linear differential-difference Equation (1.3), we obtain the following theorem under somewhat different conditions from Theorem 1.7.

THEOREM 1.8. *Let $A_{ij}(z)$ $(i = 0, 1, ..., n;\ j = 0, 1, ..., m)$ and $F(z)$ be meromorphic functions of finite logarithmic order. If there exists an integer $(0 \leq l \leq n)$ such that $A_{l0}(z)$ satisfies*

$$\lambda_{\log}\left(\frac{1}{A_{l0}}\right) < \sigma_{\log}(A_{l0}) < \infty,$$
$$\max\{\sigma_{\log}(A_{ij});(i,j) \neq (l,0)\} \leq \sigma_{\log}(A_{l0})$$

and

$$\sum_{\substack{\sigma_{\log}(A_{ij})=\sigma_{\log}(A_{l0}) \\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}) < \tau_{\log}(A_{l0}) < \infty.$$

then

(*i*) *If $\sigma_{\log}(F) < \sigma_{\log}(A_{l0})$ or $\sigma_{\log}(F) = \sigma_{\log}(A_{l0})$ and*

$$\sum_{\substack{\sigma_{\log}(A_{ij})=\sigma_{\log}(A_{l0}) \\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}) + \tau_{\log}(F) < \tau_{\log}(A_{l0})$$

or $\sigma_{\log}(F) = \sigma_{\log}(A_{l0})$ and $\sum_{\substack{\sigma_{\log}(A_{ij})=\sigma_{\log}(A_{l0}) \\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}) < \tau_{\log}(F)$, then every meromorphic solution f $(\not\equiv 0)$ of Equation (1.3) of finite logarithmic order satisfies $\sigma_{\log}(f) \geq \sigma_{\log}(A_{l0})$.

(*ii*) *If $\sigma_{\log}(F) > \sigma_{\log}(A_{l0})$, then every meromorphic solution f $(\not\equiv 0)$ of Equation (1.3) of finite logarithmic order satisfies $\sigma_{\log}(f) \geq \sigma_{\log}(F)$.*

2. Preliminary Lemmas

To prove the above theorems, we need some lemmas as follows.

LEMMA 2.1. [1] *Let f be a meromorphic function with finite logarithmic order $1 \leq \sigma_{\log}(f) < \infty$ and finite logarithmic type $0 < \tau_{\log}(f) < \infty$, then for any given $\beta < \tau_{\log}(f)$ there exists a subset $E \subset [1,\infty)$ of infinite logarithmic measure such that for all $r \in E$, we have*

$$T(r,f) > \beta(\log r)^{\sigma_{\log}(f)}.$$

LEMMA 2.2. [1] *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$, and let f be a meromorphic function of finite logarithmic order. Let σ be the logarithmic order of $f(z)$. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left((\log r)^{\sigma-1+\varepsilon}\right).$$

REMARK 2.3. It is shown that { Cf. [7], p.66}, for an arbitrary complex number $c \neq 0$, the following inequality

$$(1+o(1))\,T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))\,T(r+|c|, f(z)),$$

holds as $r \to \infty$ for an arbitrary meromorphic function $f(z)$. Therefore, it is easy to obtain that

$$\sigma_{\log}(f(z+c)) = \sigma_{\log}(f),\ \mu_{\log}(f(z+c)) = \mu_{\log}(f).$$

3. Proof of Main Results

Proof of Theorem 1.7. (i) We suppose that $f(z)$ has finite logarithmic order. We divide (1.3) by $f(z+c_l)$ to obtain

(4.1)
$$-A_{l0}(z)=\sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m}A_{ij}(z)\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\frac{f(z+c_i)}{f(z+c_l)}+\sum_{j=1}^{m}A_{lj}(z)\frac{f^{(j)}(z+c_l)}{f(z+c_l)}-\frac{F(z)}{f(z+c_l)}.$$

By (4.1) and Remark 2.3, for sufficiently large r, we have

$$\begin{aligned}m(r,A_{l0}) \leq\ & \sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m}m(r,A_{ij})+\sum_{j=1}^{m}m(r,A_{lj})+\sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m}m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)\\ &+\sum_{\substack{i=0\\i\neq l}}^{n}m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)+m\left(r,\frac{F(z)}{f(z+c_l)}\right)+O(1)\\ \leq\ & \sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m}T(r,A_{ij})+\sum_{j=1}^{m}T(r,A_{lj})+\sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m}m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)\end{aligned}$$

$$(4.2)\qquad +\sum_{\substack{i=0\\i\neq l}}^{n}m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)+T(r,F)+(1+o(1))T(r+|c_l|,f)+O(1).$$

In view of Lemma 2.2 it follows that for any given $\varepsilon>0$, we have

$$(4.3)\qquad m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)=O\left((\log r)^{\sigma_{\log}(f)-1+\varepsilon}\right),\quad i=0,1,\ldots,n,\ i\neq l.$$

By the logarithmic derivative lemma and Remark 2.3, for sufficiently large r, we have

$$(4.4)\qquad m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)=O(\log r),\quad i=0,1,\ldots,n,\ j=1,2,\ldots,m.$$

Let us set $\delta=\delta(\infty,A_{l0})>0$, then for sufficiently large r, we have

$$(4.5)\qquad m(r,A_{l0})\geq\frac{\delta}{2}T(r,A_{l0}).$$

Substituting (4.3), (4.4) and (4.5) into (4.2), we get for sufficiently large r that

$$\frac{\delta}{2}T(r,A_{l0})\leq\sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m}T(r,A_{ij})+\sum_{j=1}^{m}T(r,A_{lj})+T(r,F)+O(\log r)$$

$$(4.6)\qquad +O\left((\log r)^{\sigma_{\log}(f)-1+\varepsilon}\right)+2T(2r,f).$$

Then from (4.6), we obtain that

$$(4.7)\qquad \sigma_{\log}(A_{l0})\leq\max_{(i,j)\neq(l,0)}\{\sigma_{\log}(f),\sigma_{\log}(f)-1+\varepsilon,\sigma_{\log}(A_{ij}),\sigma_{\log}(F)\}.$$

If $\sigma_{\log}(F)<\sigma_{\log}(A_{l0})$, then by (4.7) and the fact $\sigma_{\log}(A_{ij})<\sigma_{\log}(A_{l0}),(i,j)\neq(l,0)$, we have $\sigma_{\log}(f)\geq\sigma_{\log}(A_{l0})$.

(ii) If $\sigma_{\log}(F) > \sigma_{\log}(A_{l0})$, then on the contrary we may suppose that $\sigma_{\log}(f) < \sigma_{\log}(F)$. By Equation (1.3) and Remark 2.3, we obtain that

$$\sigma_{\log}\left(\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z+c_i)\right) < \sigma_{\log}(F),$$

which is a contradiction. Hence, we have $\sigma_{\log}(f) \geq \sigma_{\log}(F)$.

This proves the theorem. □

Proof of Theorem 1.8. (i) We suppose that $f(z)$ has finite logarithmic order. If $\sigma_{\log}(F) < \sigma_{\log}(A_{l0})$, or $\sigma_{\log}(F) = \sigma_{\log}(A_{l0})$ and

$$\sum_{\substack{\sigma_{\log}(A_{ij})=\sigma_{\log}(A_{l0})\\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}) + \tau_{\log}(F) < \tau_{\log}(A_{l0}),$$

then by (4.1) and Remark 2.3, we have for sufficiently large r,

$$\begin{aligned} & T(r, A_{l0}) \\ = \; & m(r, A_{l0}) + N(r, A_{l0}) \\ \leq \; & \sum_{\substack{i=0\\ i\neq l}}^{n}\sum_{j=0}^{m} T(r, A_{ij}) + \sum_{j=1}^{m} T(r, A_{lj}) + \sum_{\substack{i=0\\ i\neq l}}^{n}\sum_{j=0}^{m} m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) \end{aligned}$$

$$+\sum_{\substack{i=0\\ i\neq l}}^{n} m\left(r, \frac{f(z+c_i)}{f(z+c_l)}\right) + T(r,F) + (1+o(1))\, T(r+|c_l|, f) + N(r, A_{l0}) + O(1). \tag{4.8}$$

Also (4.3) and (4.4) hold. From Lemma 2.1 it follows that for the above ε, there exists a subset $E \subset [1,\infty)$ with infinite logarithmic measure such that for all $r \in E$ and $r \to \infty$, and so we have

$$T(r, A_{l0}) > (\tau_{\log}(A_{l0}) - \varepsilon)(\log r)^{\sigma_{\log}(A_{l0})}. \tag{4.9}$$

Let us denote

$$\begin{aligned} \sigma_2 &= \max\{\sigma_{\log}(A_{ij}) : \sigma_{\log}(A_{ij}) < \sigma_{\log}(A_{l0}); (i,j) \neq (l,0)\}, \\ \text{and } \tau_2 &= \sum_{\substack{\sigma_{\log}(A_{ij})=\sigma_{\log}(A_{l0})\\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}). \end{aligned}$$

If $\sigma_{\log}(A_{ij}) < \sigma_{\log}(A_{l0})$, then for the above ε and sufficiently large r, we have

$$T(r, A_{ij}) \leq (\log r)^{\sigma_2+\varepsilon}. \tag{4.10}$$

If $\sigma_{\log}(A_{ij}) = \sigma_{\log}(A_{l0})$, $(i,j) \neq (l,0)$, then for the above ε and sufficiently large r, we have

$$T(r, A_{ij}) \leq (\tau_{\log}(A_{ij}) + \varepsilon)(\log r)^{\sigma_{\log}(A_{l0})}, \quad (i,j) \neq (l,0). \tag{4.11}$$

By the definition of $\lambda_{\log}\left(\frac{1}{A_{l0}}\right)$, for the above ε and for sufficiently large r, we get that

$$N(r, A_{l0}) < (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+\varepsilon}. \tag{4.12}$$

If $\sigma_{\log}(F) < \sigma_{\log}(A_{l0})$, then for the above ε and sufficiently large r, we have

$$T(r,F) \le (\log r)^{\sigma_{\log}(F)+\varepsilon}. \tag{4.13}$$

Now, we may choose sufficiently small ε satisfying

$$\begin{aligned} 0 &< (k+2)\varepsilon \\ &< \min\left\{\sigma_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right), \sigma_{\log}(A_{l0}) - \sigma_2, \sigma_{\log}(A_{l0}) - \sigma_{\log}(F), \tau_{\log}(A_{l0}) - \tau_2\right\}, \end{aligned}$$

Substitute(4.3), (4.4) and (4.9)-(4.13) into (4.8), for $r \in E$ and $r \to \infty$, we obtain that

$$\begin{aligned} &(\tau_{\log}(A_{l0}) - \tau_2 - (k+1)\varepsilon)(\log r)^{\sigma_{\log}(A_{l0})} \\ <\ & O\left((\log r)^{\sigma_2+\varepsilon}\right) + (\log r)^{\sigma_{\log}(F)+\varepsilon} + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+\varepsilon} \\ & + O\left((\log r)^{\sigma_{\log}(f)+\varepsilon}\right) + O(\log r). \end{aligned} \tag{4.14}$$

From (4.14), we get that $\sigma_{\log}(f) \ge \sigma_{\log}(A_{l0})$.

If $\sigma_{\log}(F) = \sigma_{\log}(A_{l0})$ and $\tau_2 + \tau_{\log}(F) < \tau_{\log}(A_{l0})$, then for the above ε and sufficiently large r, we have

$$T(r,F) \le (\tau_{\log}(F) + \varepsilon)(\log r)^{\sigma_{\log}(A_{l0})}. \tag{4.15}$$

Now, we may choose sufficiently small ε satisfying

$$\begin{aligned} 0 &< (k+3)\varepsilon \\ &< \min\left\{\sigma_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right), \sigma_{\log}(A_{l0}) - \sigma_2, \tau_{\log}(A_{l0}) - \tau_{\log}(F) - \tau_2\right\}. \end{aligned}$$

Substitute(4.3), (4.4) and (4.9)-(4.12) and (4.15) into (4.8), for $r \in E$ and $r \to \infty$, we obtain that

$$\begin{aligned} &(\tau_{\log}(A_{l0}) - \tau_{\log}(F) - \tau_2 - (k+2)\varepsilon)(\log r)^{\sigma_{\log}(A_{l0})} \\ <\ & O\left((\log r)^{\sigma_2+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+\varepsilon} \\ & + O\left((\log r)^{\sigma_{\log}(f)+\varepsilon}\right) + O(\log r). \end{aligned} \tag{4.16}$$

From (4.16), we get that $\sigma_{\log}(f) \ge \sigma_{\log}(A_{l0})$.

If $\sigma_{\log}(F) = \sigma_{\log}(A_{l0})$ and

$$\sum_{\sigma_{\log}(A_{ij}) = \sigma_{\log}(A_{l0})} \tau_{\log}(A_{ij}) < \tau_{\log}(F),$$

then by Equation (1.3), Remark 2.3 and $T\left(r, f^{(n)}\right) \le (n+1)T(r,f) + S(r,f), n \in N_+$, we have that for sufficiently large r,

$$\begin{aligned} T(r,F) &\le \sum_{(i,j)\ne(l,0)} T(r,A_{ij}) + T(r,A_{l0}) + \sum_{i=0}^{n}\sum_{j=0}^{m} T(r, f^{(j)}(z+c_i)) \\ &\le \sum_{(i,j)\ne(l,0)} T(r,A_{ij}) + T(r,A_{l0}) + O(T(2r,f)) + S(r,f). \end{aligned} \tag{4.17}$$

If $\sigma_{\log}(F) = \sigma_{\log}(A_{l0})$ and $\tau_2 + \tau_{\log}(A_{l0}) < \tau_{\log}(F)$, then by Lemma 2.1, for the above ε, there exists a subset $E \subset [1, \infty)$ with infinite logarithmic measure such that for all $r \in E$ and $r \to \infty$, we have

$$T(r, F) > (\tau_{\log}(F) - \varepsilon) (\log r)^{\sigma_{\log}(A_{l0})}. \tag{4.18}$$

By the definition of $\tau_{\log}(A_{l0})$, we have for the above ε and sufficiently large r,

$$T(r, A_{l0}) \leq (\tau_{\log}(A_{l0}) + \varepsilon) (\log r)^{\sigma_{\log}(A_{l0})}. \tag{4.19}$$

Now, we may choose sufficiently small ε satisfying

$$0 < (k+3)\varepsilon < \min\{\sigma_{\log}(A_{l0}) - \sigma_2, \tau_{\log}(F) - \tau_{\log}(A_{l0}) - \tau_2\}.$$

Substituting (4.10), (4.11), (4.18) and (4.19) into (4.17), for $r \in E$ and $r \to \infty$, we obtain that

$$\begin{aligned}&(\tau_{\log}(F) - \tau_{\log}(A_{l0}) - \tau_2 - (k+2)\varepsilon) (\log r)^{\sigma_{\log}(A_{l0})} \\ &< O((\log r)^{\sigma_2+\varepsilon}) + O\left((\log r)^{\sigma_{\log}(f)+\varepsilon}\right).\end{aligned} \tag{4.20}$$

Hence it follows by (4.20) that $\sigma_{\log}(f) \geq \sigma_{\log}(A_{l0})$.

(ii) If $\sigma_{\log}(F) > \sigma_{\log}(A_{l0})$, then on the contrary we may suppose that $\sigma_{\log}(f) < \sigma_{\log}(F)$. By Equation (1.4) and Remark 2.3, we obtain that

$$\sigma_{\log}\left(\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z+c_i)\right) < \sigma_{\log}(F),$$

which is a contradiction. Hence, we have $\sigma_{\log}(f) \geq \sigma_{\log}(F)$.

The proves the theorem. □

4. Future aspects

Keeping in mind the results already established, one may explore for analogous theorems in which the coefficients of differential-difference equations are bi-complex valued meromorphic functions of finite logarithmic order. Furthermore, the case in which the coefficients of differential-difference equations are meromorphic functions of finite logarithmic order in a sector of the unit disc is still a virgin domain for the new researchers and therefore it may be posed as an open problem to the future workers of this branch.

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