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ON MAXIMAL COMPACT FRAMES

Jayaprasad P N*, Madhavan Namboothiri N M, Santhosh P K, and Varghese Jacob

Abstract. Every closed subset of a compact topological space is compact. Also every compact subset of a Hausdorff topological space is closed. It follows that compact subsets are precisely the closed subsets in a compact Hausdorff space. It is also proved that a topological space is maximal compact if and only if its compact subsets are precisely the closed subsets. A locale is a categorical extension of topological spaces and a frame is an object in its opposite category. We investigate to find whether the closed sublocales are exactly the compact sublocales of a compact Hausdorff frame. We also try to investigate whether the closed sublocales are exactly the compact sublocales of a maximal compact frame.

1. Introduction

Garrett Birkfoff, in 1936, pointed out the notion of the comparison of two different topologies on the same basic set. He had done this by ordering these topologies as a lattice under set inclusion. A topological space (X, T) with property R is said to be maximal R if T is a maximal element in the set $R(X)$ of all topologies on the set X having property R with the partial ordering of set inclusions. The set of all topologies sharing a given property may not have a greatest element, but it may have maximal elements.

In topological spaces, a closed subspace of a compact space is compact and a compact subspace of a Hausdorff space is closed. Thus in a compact Hausdorff space, closed subspaces coincide with compact subspaces. A topological space is maximal compact if and only if its compact subsets are precisely the closed sets [1]. Norman Levine named those spaces in which closed subsets coincide with compact subsets as *C-C Spaces*. A detailed analysis of its properties are discussed in [9]. It seems worthwhile to study these results in the context of the category of frames which in turn is the opposite category of locales, a categorical extension of topological spaces. We extend these results into the case of frames. A characterization for a frame which exhibits these analogous properties is also formulated and the association with topological spaces is also discussed in this paper.

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∗ Corresponding author.

2. Preliminaries

The term frame was coined by C.H. Dowker and studied by D. Strauss [3]. A *frame* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$ holds for all $a \in L, S \subseteq L$. A map between frames that preserves arbitrary joins and finite meets is called a *frame homomorphism*. Associated with a frame homomorphism $h : M \to L$ is its right adjoint $h_* : L \to M$ given by $h_*(b) = \bigvee \{x \in M : h(x) \leq b\}$. We denote the top element and the bottom element of a frame by 1 and 0 respectively. The category of frames and frame homomorphisms is denoted by **Frm**. The dual category $\mathbf{Frm}^{op}$ is referred to as the category of locales denoted by **Loc**. The morphisms in **Loc**, called *localic maps*, are given by the right adjoints of frame homomorphisms between two objects. A frame is said to be *spatial*, if it is isomorphic to the topology ΩX of a topological space $(X, \Omega X)$.

A subset of a frame which is closed under arbitrary joins and finite meets in that frame is called a *subframe*. A *sublocale* M of a locale L can be represented in terms of an onto frame homomorphism $h : L \to M$ in the sense that the image of M under the right adjoint $h_* : M \to L$ will represent that sublocale. For a locale L, denote $\uparrow a = \{x \in L : x \geq a\}$ and $\downarrow b = \{x \in L : x \leq b\}$. Then the sublocale given by the frame homomorphism $j : L \to \uparrow a$ defined by $x \to a \vee x$ for any $a \in L$ is called a *closed sublocale* of L. A *cover* in a frame L is a subset S of L with $\bigvee S = 1_L$. A frame L is said to be *compact* if each cover A of L has a finite subcover. For a detailed reading concerning frames we refer to [7].

Definition 2.1. [2] A frame M is called a *singly generated extension* of a frame A if A is a subframe of M, and M is generated by A and some $b \in M$. We write $M = A[b]$.

Let L be any frame and A be a subframe. An element $b \in L$ is said to be *compact relative to the subframe* A if for every $S \subseteq A$ with $b \leq \bigvee S$, there exists $F \subseteq S$ with F finite and $b \leq \bigvee F$. We state some results used in proving some of the results in this paper.

Theorem 2.2. [5] *Let A be a subframe of the frame L and $b \in L - A$ be complemented in L. Consider the following statements about $A[b]$.*
(1) $A[b]$ is compact.
(2) b^c is compact relative to $A[b]$.
Then, the following statements hold.
(a) Statement (1) implies statement (2).
(b) If A is compact, then (1) and (2) are equivalent.

Theorem 2.3. [7] *An image of a compact sublocale $S \subseteq L$ under a localic map $f : L \to M$ is compact.*

The following results are proved in [9].

Theorem 2.4. *If (X, τ) is a compact Hausdorff space, then τ is M.R.C.*

Theorem 2.5. *Suppose that (X, τ) is a topological space. Then (X, τ) is C-C if and only if τ is M.R.C.*

Theorem 2.6. *Let (X, τ) be a topological space. If (X, τ) is C-C, then it is compact and T_1.*

3. Maximal Compact Frames

It is known that [10] every closed sublocale of a compact locale is compact and every compact sublocale of a regular locale is closed. Hence in compact regular locales closed sublocales coincide with compact sublocales. We try to answer when does a closed sublocale equivalent to a compact sublocale.

Definition 3.1. A frame A is said to be *maximal compact* if,

1. A is compact,
2. if A is a proper subframe of the frame L, then L is not compact.

Theorem 3.2. *Let A be any frame that is not maximal compact. Then there exists a compact sublocale which is not closed in A.*

Proof. Let us assume that $A \subset B$ where the frame B is compact. We assume, without loss of generality, that these are subframes of a boolean frame L according to *Corollary 2.6* of II [10]. Let $b \in B - A$. Consider the singly generated extension $A[b]$. Then $A[b]$ is a compact subframe of B. Then by *Lemma* 2.2, b^c is compact relative to $A[b]$ and hence $\downarrow b^c$ is compact.

Case 1: Suppose $b^c \in A$

Claim: $\mathbf{o}(b^c) = \{x \in A : b^c \to x\} = \{x \in A : b^c \to x = x\}$ is a compact sublocale of A that is not closed.

For, it is the image of $\downarrow b^c$ regarded as a locale under the localic map obtained as the adjoint of the frame homomorphism $j : A \to \downarrow b^c$ defined by $x \to b^c \wedge x$ and since $\downarrow b^c$ is compact as a locale, $\mathbf{o}(b^c)$ is compact in A, by *Theorem* 2.3. If we assume that $\mathbf{o}(b^c)$ is closed in A, then there exists $y \in A$ such that $\mathbf{o}(b^c) = \uparrow_A y$. Since $0 \in \mathbf{o}(b^c)$, $y = 0$ which implies that $\mathbf{o}(b^c) = A$. Hence $b = 0$, which is a contradiction as $b \in B - A$. Hence $\mathbf{o}(b^c)$ is not closed in A.

Case 2: Suppose $b^c \notin A$

Let $p = \bigwedge\{x \in A : b^c \leq x\}$. We claim that $p \neq 1$. For, if $p = 1$, then $F = \uparrow_A b^c$ is a filter. Consider the ideal $I = \{x \in A : x \leq b^c\}$ in L disjoint from F. Now, by *Lemma 2.3* of I [10], there exists a maximal ideal $M \subseteq A$ containing I and disjoint from F. Then, by *Theorem 2.4* of I [10], M is a prime ideal. Now $b^c \wedge b = 0 \in M$ and M is a prime ideal, $b \in M \subseteq A$, which is not true as $b \notin A$. Hence $p \neq 1$.

We prove that $\downarrow_A p$ is compact but not closed in A. For, it needs to prove that p is compact relative to A. Let $S \subseteq A$ with $b^c \leq p = \bigvee S$. Since b^c is compact relative to A, there exists a finite set $F \subseteq S$ with $b^c \leq \bigvee F$. Then $\bigvee F \in \{x \in A : b^c \leq x\}$ and hence $p \leq \bigvee F$. Also $F \subseteq S$ and hence $\bigvee F \leq p$. Combining we get $\bigvee F = p$ where $F \subseteq S$ is finite. Hence p is compact relative to A.

Now $\mathbf{o}(p)$ can be proved to be a compact sublocale but not closed, by repeating the proof in case 1 with b^c replaced by p.

□

Theorem 3.3. *Let A be a compact subframe of a noncompact frame L. Let $a \in A$ and $\uparrow_L a$ be compact. Then $A[b]$ is compact for any $b \in \uparrow_L a$.*

Proof. If $b \in A$, then $A[b] = A$ and is compact. Let $b \notin A$ and $S \subseteq A[b]$ with $\bigvee S = 1$. Let $S = \left\{a_i \vee (a_i' \wedge b) : a_i, a_i' \in A, a_i \leq a_i', i \in I\right\}$. Now $1 = \bigvee S = (\bigvee a_i) \vee [(\bigvee a_i') \wedge b] = (\bigvee a_i') \wedge [(\bigvee a_i) \vee b] = (\bigvee a_i') \wedge [\bigvee (a_i \vee b)]$. Thus $\bigvee_{i \in I} a_i' = 1$ and $\bigvee_{i \in I}(a_i \vee b) = 1$. Since A is compact, there exists a finite subset $J_1 \subseteq I$ with

$\bigvee_{j_1 \in J_1} a_{j_1}{}' = 1$. Also $a_i \vee b \geq b \geq a$ and hence $a_i \vee b \in \uparrow_L a$. Since $\uparrow_L a$ is compact, there exists a finite subset $J_2 \subseteq I$ with $\bigvee_{j_2 \in J_2}(a_{j_2} \vee b) = 1$. Set $J = J_1 \bigcup J_2$ and $F = \{a_j \vee (a_j{}' \wedge b) : j \in J\}$. Clearly $F \subseteq S$ and F is finite. Then $\bigvee F = (\bigvee a_j) \vee [(\bigvee a_j{}') \wedge b] = (\bigvee a_j{}') \wedge (\bigvee (a_j \vee b)) = 1 \wedge 1 = 1$, because $J_1 \subseteq J, J_2 \subseteq J$ and $\bigvee_{j_1 \in J_1} a_{j_1}{}' = 1, \bigvee_{j_2 \in J_2}(a_{j_2} \vee b) = 1$. Hence $A[b]$ is compact. □

THEOREM 3.4. *Let L be any non-compact frame. Let $A \subseteq L$ be maximal compact and let $a \in A$. Then $\uparrow_L a$ is compact if and only if $\uparrow_L a = \uparrow_A a$.*

Proof. Assume that $\uparrow_L a$ is compact. Let $b \in \uparrow_L a$ and $b \notin A$. Then $A[b]$ is compact by *Theorem* 3.3, contradicts the maximality of A. Hence $b \in A$ and $\uparrow_L a \subseteq \uparrow_A a$. Also $\uparrow_A a \subseteq \uparrow_L a$, since $A \subseteq L$. Hence $\uparrow_L a = \uparrow_A a$. Conversely, if $\uparrow_L a = \uparrow_A a$, $\uparrow_A a$ is a closed sublocale of A and hence compact. □

We state the following definition due to J.Paseka and B.$\breve{S}$marda [6] for proving the next result.

DEFINITION 3.5. Define $F_C = \{a \in L : \uparrow a$ is compact in L $\}$. Then the locale generated by the set $\{(l, 0_L) : l \in L\} \bigcup \{(a, 1) : a \in F_C\}$ is defined as L_{F_C}. L_{F_C} is a compact locale called the *one point compactification* [6] of L.

THEOREM 3.6. *Let A be a maximal compact subframe of the frame L. If K is a compact sublocale of A, then K must be closed in L.*

Proof. Assume that K is not closed in L.
Consider $\overline{K}$ the closure of K in L. Then there exists $\beta \in L$ such that $\overline{K} = \uparrow_L \beta$. If $\beta \in A$, then by *Theorem* 3.4, $\uparrow_A \beta = \uparrow_L \beta$. Thus K is closed in A and hence in L. So we assume that $\beta \in L - A$. Consider the singly generated extension $A[\beta]$ of the frame A by adding the element β.

Claim: $A[\beta]$ is compact.
Let $S \subseteq A[\beta]$ with $\bigvee S = 1$. Then we can express
$S = \{a_i \vee (a_i{}' \wedge \beta) : a_i, a_i{}' \in A, a_i \leq a_i{}', i \in I\}$. Now $\bigvee S = (\bigvee a_i) \vee [(\bigvee a_i{}') \wedge \beta] = (\bigvee a_i{}') \wedge [(\bigvee a_i) \vee \beta] = (\bigvee a_i{}') \wedge [\bigvee (a_i \vee \beta)] = 1$. Thus $\bigvee_{i \in I} a_i{}' = 1$ and $\bigvee_{i \in I}(a_i \vee \beta) = 1$. Since A is compact, there exists a finite set $J_1 \subseteq I$ with $\bigvee_{j_1 \in J_1} a_{j_1}{}' = 1$. Consider the one point compactification L_{F_C} of L. By *Theorem* 3.4, $\uparrow_A a = \uparrow_L a$ for any $a \in A$. But $\uparrow_A a$ being a closed sublocale of A is compact in A and hence in L. Hence $A \subseteq F_C$. Now $a_i \vee \beta \in L$ and $a_i \in F_C$. Hence by definition of L_{F_C}, we have $(a_i \vee \beta, 0) \vee (a_i, 1) = (a_i \vee \beta, 1) \in L_{F_C}$. Now

$$\begin{aligned} \bigvee_{i \in I}(a_i \vee \beta, 1) &= (\bigvee_{i \in I}(a_i \vee \beta), 1) \\ &= (1, 1) \end{aligned}$$

Since L_{F_C} is compact, there exists a finite subset $J_2 \subseteq I$ with $\bigvee_{j_2 \in J_2}(a_{j_2} \vee \beta, 1) = (1, 1)$ and hence $\bigvee_{j_2 \in J_2}(a_{j_2} \vee \beta) = 1$. Set $J = J_1 \bigcup J_2$ and $F = \{a_j \vee (a_j{}' \wedge \beta) : j \in J\}$. Clearly $F \subseteq S$ and F is finite. As seen before, $\bigvee F = 1$. Thus $A[\beta]$ is compact. But $A \subset A[\beta]$ and this contradicts the maximality of A. Hence K must be closed in L. □

Now we state and prove the main theorem characterizing maximal compact frames.

Theorem 3.7. *Let L be any non-compact frame. A subframe A of L is maximal compact if and only if the closed sublocales of A are exactly the compact sublocales of A.*

Proof. Assume that A is maximal compact. Since every closed sublocale of a compact frame is compact, it needs to prove that compact sublocales are closed. Let K be a compact sublocale of A. Assume that K is not closed in A. Since A is maximal compact, by *Theorem*, K must be closed in L. Then there exists $\beta \in L - A$ such that $K = \uparrow_L \beta$ as K is not closed in A. Since K is compact $\uparrow_L \beta$ is compact. Now by *Theorem*, $A[\beta]$ is compact. This contradicts the maximality of A and hence K must be closed in A. Conversely assume that the closed sublocales of A are exactly the compact sublocales. If A is not maximal compact, then by *Theorem* there exists a compact sublocale which is not closed in A, a contradiction. Hence A is maximal compact. □

Corollary 3.8. *Every compact regular frame is maximal compact.*

Proof. Closed sublocales of compact frames are compact and compact sublocales of regular frames are closed. Hence the result follows. □

Corollary 3.9. *A compact Hausdorff frame is maximal compact.*

Proof. A compact Hausdorff frame is regular. The result follows from *Corollary* 3.8 □

Corollary 3.10. *Let A be any compact frame. Then no subframe of A is regular.*

Proof. If a subframe of a compact frame is regular, then it is maximal compact because of being regular and compact, a contradiction. □

Corollary 3.11. *The topological space $(X, \Omega X)$ is a C-C space if and only if ΩX is a maximal compact frame.*

Proof. Assume that $(X, \Omega X)$ is a C-C space. Then it is M.R.C by *Theorem* 2.4. Then ΩX is maximal compact. Conversely, if ΩX is a maximal compact frame, then it is M.R.C. by *Theorem* 3.7. Hence $(X, \Omega X)$ is M.R.C and thus a C-C space by *Theorem* 2.4. □

Corollary 3.12. *Let A be a spatial maximal compact frame. Then it is compact and subfit.*

Proof. Since A is a maximal compact frame, by *Corollary* 3.11, the topological space which corresponds to A will be a C-C space which compact and T_1 by *Theorem* 2.6. The frame of opens of a T_1 topological space being subfit, the result follows. □

Example 3.13. Let (X, τ) be a cofinite topological space. It is compact and T_1 but not a C-C space. Then the frame τ is subfit and compact. But τ is not a maximal compact frame, by *Corollary* 3.11.

The following is an example of a maximal compact frame which is compact but not Hausdorff.

Example 3.14. Let $(R, \Omega R)$ be the space of rationals with the relative topology and let $(R, \Omega R^*)$ be the one point compactification of $(R, \Omega R)$. Then it is proved in [9] that $(R, \Omega R^*)$ is not Hausdorff but it is a C-C space. Since $(R, \Omega R^*)$ is not Hausdorff, the frame ΩR^* is not a Hausdorff frame, as the topological space representing a Hausdorff spatial frame is Hausdorff. Again by *Corollary* 3.11, the frame ΩR^* is a maximal compact frame as $(R, \Omega R^*)$ is a C-C space.

THEOREM 3.15. *Let A be a non spatial maximal compact frame. Then it cannot be subfit.*

Proof. Since A is compact it is subfit and by *Theorem 2.11* of [8] a compact subfit frame is spatial, a contradiction. □

4. Application

A frame is said to be *reversible* [4], if every order preserving self bijection is a frame isomorphism. A characterization for reversible frames is given in [4]. It is also proved that a frame that is maximal or minimal with respect to some frame isomorphic property is reversible. Thus compact Hausdorff frames are reversible. Also a compact regular frame is reversible. Hence the characterization for maximal compact frames can be used as a method to identify reversible frames.

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Jayaprasad P N
Department of Mathematics, Government College Kottayam,
Kottayam, Kerala, India
E-mail: jayaprasadpn@gmail.com

Madhavan Namboothiri N M
Department of Mathematics, Government College Kottayam,
Kottayam, Kerala, India
E-mail: madhavangck@gmail.com

Santhosh P K
Dept. of Applied Science, Government Engineering College,
Kozhicode, Kerala, India
E-mail: santhoshgpm2@gmail.com

Varghese Jacob
Department of Mathematics, Government College Kottayam
Kottayam, Kerala, India
E-mail: drvarghesejacob@gmail.com