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NUMERICAL SOLUTION OF ABEL'S GENERAL FUZZY LINEAR INTEGRAL EQUATIONS BY FRACTIONAL CALCULUS METHOD

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Abstract. The aim of this article is to give a numerical method for solving Abel's general fuzzy linear integral equations with arbitrary kernel. The method is based on approximations of fractional integrals and Caputo derivatives. The convergence analysis for the proposed method is also given and the applicability of the proposed method is illustrated by solving some numerical examples. The results show the utility and the greater potential of the fractional calculus method to solve fuzzy integral equations.

1. Introduction

Let $f(x)$ be a predetermined data function. Then Abel's general integral equation [1, 16] is given by

$$f(x) = \int_0^x \frac{k(x,t)g(t)}{(x-t)^\alpha} dt \; ; \; 0 \leq x \leq b, \tag{1}$$

where α is a known constant such that $0 < \alpha < 1$, $g(t)$ is an unknown function which will be determined and $k(x,t)$ is called kernel of Abel's integral equation. If $k(x,t) = \frac{1}{\Gamma(1-\alpha)}$, then (1) is a fractional integral equation of order $1-\alpha$. Equation (1) gives crisp solution if $f(x)$ is a crisp function, otherwise, the equation may only poses fuzzy solutions whenever $f(x)$ is a fuzzy function. Fuzzy integral equations are important for studying and solving a large number of problems in many areas of applied mathematics, particularly in relation to fuzzy control theory. In many applications of applied mathematics, some of the parameters are represented by fuzzy number rather than crisp number, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat and solve general fuzzy integral equations.

The concept of fuzzy functions was first introduced by Chang and Zadeh [2] and the concept of integration of fuzzy functions was introduced by Dubois and Prade [3]. Later, different approaches to integrate fuzzy functions were also suggested by different authors such as Goetschel and Voxman [4], Kaleva [5], Nanda [6], Ralescu

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and Adams [7], Wang [8], Bede and Gal [9] and many other authors. The numerical solution of fuzzy Riemann integral equations was investigated by Wu in [10].

Let $\widetilde{f}(x)$ and $\widetilde{g}(x)$ be fuzzy functions. Consider the solutions of Abel's general linear fuzzy integral equations

$$\widetilde{f}(x) = \int_0^x \frac{k(x,t)\widetilde{g}(t)}{(x-t)^\alpha}dt \ ; \ 0 \le x \le b, \ \textit{and}\ 0 < \alpha < 1. \tag{2}$$

Let $\big(\underline{f}(x,r), \overline{f}(x,r)\big)$ and $\big(\underline{g}(x,r), \overline{g}(x,r)\big)$ be the parametric representation of fuzzy functions $\widetilde{f}(x)$ and $\widetilde{g}(x)$ respectively. Goetschel and Voxman [4] proved that if $\widetilde{f}(x)$ is continuous in metric D, then its definite integral exists and

$$\underline{\int_a^b f(x,r)dx} = \int_a^b \underline{f}(x,r)dx, \ \text{and} \ \overline{\int_a^b f(x,r)dx} = \int_a^b \overline{f}(x,r)dx. \tag{3}$$

By using above parametrization Eq. (2) can be written as follows:

$$\Big(\underline{f}(x,r), \overline{f}(x,r)\Big) = \left(\int_0^x \frac{k(x,t)\underline{g}(t,r)}{(x-t)^\alpha}dt, \int_0^x \frac{k(x,t)\overline{g}(t,r)}{(x-t)^\alpha}dt\right), \ 0 \le r \le 1. \tag{4}$$

In recent years, there has been a growing interest in the field of fractional derivative and Riemann-Liouville fractional integral [11–14]. The main reason consists in the fact that the theory of derivatives of fractional (non-integer) order stimulates considerable interest in the areas of mathematics, physics, engineering and other sciences. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. In this paper we will discuss the solutions of (2) by using fractional calculus.

The structure of this paper is as follows. We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In section 3, some numerical approximations of fuzzy fractional derivatives are given. The approximate solution and the error bounds of the fuzzy Abel equation by using fractional calculus is given in section 4. In section 5, the proposed method is implemented by solving two illustrative examples and finally, section 6 forms the conclusions.

2. Some Definitions, relations and properties of fuzzy fractional operators

In this section, some definitions about fuzzy functions by using fractional calculus are introduced which will be required in later part of this paper.

Let $C^F[a,b]$ be the space of all continuous fuzzy-valued functions on $[a,b]$. Also, we denote the space of all Lebesgue integrable fuzzy-valued functions on the bounded interval $[a,b] \subset \mathbb{R}$ by $L^F[a,b]$. Now, we define the fuzzy Riemann-Liouville integral of fuzzy-valued functions as follows:-

Definition 2.1. Assume $\widetilde{f} \in C^F[a,b] \bigcap L^F[a,b]$, $a < x < b$ and Γ is Euler's gamma function. Then the fuzzy left and right Riemann-Liouville fractional integral of order α, where $n-1 < \alpha \le n$, $n \in \mathbb{N}$ of $\widetilde{f}$ is defined as,

$$\big({}_aJ_x^\alpha \widetilde{f}\big)(x) = \big[\big({}_aJ_x^\alpha \underline{f}\big)(x,r), \big({}_aJ_x^\alpha \overline{f}\big)(x,r)\big], 0 \le r \le 1,$$

where,

$$\begin{aligned}\left({}_aJ_x^{\alpha}\underline{f}\right)(x,r) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\underline{f}(t,r)dt\\ \left({}_aJ_x^{\alpha}\overline{f}\right)(x,r) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\overline{f}(t,r)dt\end{aligned}$$

and

$$\left({}_xJ_b^{\alpha}\widetilde{f}\right)(x) = \left[\left({}_xJ_b^{\alpha}\underline{f}\right)(x,r), \left({}_xJ_b^{\alpha}\overline{f}\right)(x,r)\right], 0 \le r \le 1,$$

where,

$$\begin{aligned}\left({}_xJ_b^{\alpha}\underline{f}\right)(x,r) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\underline{f}(t,r)dt\\ \left({}_xJ_b^{\alpha}\overline{f}\right)(x,r) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\overline{f}(t,r)dt.\end{aligned}$$

Definition 2.2. Let $\widetilde{f} \in C^F[a,b] \bigcap L^F[a,b]$, $a < x < b$ and Γ is Euler's gamma function. Then the fuzzy left and right Riemann-Liouville fractional derivatives of order α, where $n-1 < \alpha < n$, $n \in \mathbb{N}$ of $\widetilde{f}$ is defined as,

$$\left({}_aD_x^{\alpha}\widetilde{f}\right)(x) = \left[\left({}_aD_x^{\alpha}\underline{f}\right)(x,r), \left({}_aD_x^{\alpha}\overline{f}\right)(x,r)\right], 0 \le r \le 1,$$

where,

$$\begin{aligned}\left({}_aD_x^{\alpha}\underline{f}\right)(x,r) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}\underline{f}(t,r)dt\\ \left({}_aD_x^{\alpha}\overline{f}\right)(x,r) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}\overline{f}(t,r)dt\end{aligned}$$

and

$$\left({}_xD_b^{\alpha}\widetilde{f}\right)(x) = \left[\left({}_xD_b^{\alpha}\underline{f}\right)(x,r), \left({}_xD_b^{\alpha}\overline{f}\right)(x,r)\right], 0 \le r \le 1,$$

where,

$$\begin{aligned}\left({}_xD_b^{\alpha}\underline{f}\right)(x,r) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}\underline{f}(t,r)dt\\ \left({}_xD_b^{\alpha}\overline{f}\right)(x,r) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}\overline{f}(t,r)dt.\end{aligned}$$

For $\alpha = n$,

$$\left({}_aD_x^{\alpha}\widetilde{f}\right)(x) = \left(\frac{d^n}{dx^n}\underline{f}(x,r), \frac{d^n}{dx^n}\overline{f}(x,r)\right)$$

and

$$\left({}_xD_b^{\alpha}\widetilde{f}\right)(x) = \left(\frac{d^n}{dx^n}\underline{f}(x,r), \frac{d^n}{dx^n}\overline{f}(x,r)\right).$$

Definition 2.3. Let $\widetilde{f} \in C^F[a,b] \bigcap L^F[a,b]$, $a < x < b$ and Γ is Euler's gamma function. Then the fuzzy left and right Caputo fractional derivatives of order α, where $n-1 < \alpha < n$, $n \in \mathbb{N}$ of $\widetilde{f}$ is defined as,

$$\left({}_a^cD_x^{\alpha}\widetilde{f}\right)(x) = \left[\left({}_a^cD_x^{\alpha}\underline{f}\right)(x,r), \left({}_a^cD_x^{\alpha}\overline{f}\right)(x,r)\right],\ 0 \le r \le 1,$$

where,

$$\left({}_{a}^{c}D_{x}^{\alpha}\underline{f}\right)(x,r)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}\underline{f}^{(n)}(t,r)dt$$
$$\left({}_{a}^{c}D_{x}^{\alpha}\overline{f}\right)(x,r)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}\overline{f}^{(n)}(t,r)dt$$

and

$$\left({}_{x}^{c}D_{b}^{\alpha}\widetilde{f}\right)(x)=\left[\left({}_{x}^{c}D_{b}^{\alpha}\underline{f}\right)(x,r),\left({}_{x}^{c}D_{b}^{\alpha}\overline{f}\right)(x,r)\right],\ 0\leq r\leq 1,$$

where,

$$\left({}_{x}^{c}D_{b}^{\alpha}\underline{f}\right)(x,r)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}(t-x)^{n-\alpha-1}\underline{f}^{(n)}(t,r)dt$$
$$\left({}_{x}^{c}D_{b}^{\alpha}\overline{f}\right)(x,r)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}(t-x)^{n-\alpha-1}\overline{f}^{n}(t,r)dt.$$

For $\alpha=n$,

$$\left({}_{a}^{c}D_{x}^{\alpha}\widetilde{f}\right)(x)=\left(\frac{d^{n}}{dx^{n}}\underline{f}(x,r),\frac{d^{n}}{dx^{n}}\overline{f}(x,r)\right)$$

and

$$\left({}_{x}^{c}D_{b}^{\alpha}\widetilde{f}\right)(x)=\left(\frac{d^{n}}{dx^{n}}\underline{f}(x,r),\frac{d^{n}}{dx^{n}}\overline{f}(x,r)\right).$$

DEFINITION 2.4. Let $\widetilde{f}\in C^{F}[a,b]\bigcap L^{F}[a,b]$ and $h>0$. The fuzzy left and right Grunwald-Letnikov fractional derivatives of order α (>0) of $\widetilde{f}$ are defined as,

$$\left(D^{\alpha}\widetilde{f}\right)(x)=\left[\left(D^{\alpha}\underline{f}\right)(x,r),\left(D^{\alpha}\overline{f}\right)(x,r)\right],\ 0\leq r\leq 1,$$

where,

$$\left(D^{\alpha}\underline{f}\right)(x,r)=\lim_{h\to 0}h^{-\alpha}\sum_{s=0}^{\infty}(-1)^{s}\binom{\alpha}{s}\underline{f}(x-sh,r)$$
$$\left(D^{\alpha}\overline{f}\right)(x,r)=\lim_{h\to 0}h^{-\alpha}\sum_{s=0}^{\infty}(-1)^{s}\binom{\alpha}{s}\overline{f}(x-sh,r)$$

and

$$\left(D^{-\alpha}\widetilde{f}\right)(x)=\left[\left(D^{-\alpha}\underline{f}\right)(x,r),\left(D^{-\alpha}\overline{f}\right)(x,r)\right],\ 0\leq r\leq 1,$$

where,

$$\left(D^{-\alpha}\underline{f}\right)(x,r)=\lim_{h\to 0}h^{\alpha}\sum_{s=0}^{\infty}\begin{bmatrix}\alpha\\ s\end{bmatrix}\underline{f}(x-sh,r)$$
$$\left(D^{-\alpha}\overline{f}\right)(x,r)=\lim_{h\to 0}h^{\alpha}\sum_{s=0}^{\infty}\begin{bmatrix}\alpha\\ s\end{bmatrix}\overline{f}(x-sh,r)$$

and,

$$\begin{bmatrix}\alpha\\ s\end{bmatrix}=\frac{\alpha(\alpha+1)(\alpha+2)......(\alpha+s-1)}{s!}.$$

The following theorem establishes relations between fuzzy Caputo and Riemann-Liouville fractional derivatives of $\widetilde{f}$.

THEOREM 2.1. *Let $t > 0$ and $n \in \mathbb{N}$. Then for $n-1 < \alpha < n$, the following relations between fuzzy Riemann-Liouville and fuzzy Caputo operators hold:-*

$$\left({}_a^c D_x^\alpha \widetilde{f}\right)(x) = \left({}_a D_x^\alpha \widetilde{f}\right)(x) - \sum_{k=0}^{n-1} \frac{\widetilde{f}^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}$$

and

$$\left({}_x^c D_b^\alpha \widetilde{f}\right)(x) = \left({}_x D_b^\alpha \widetilde{f}\right)(x) - \sum_{k=0}^{n-1} \frac{\widetilde{f}^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha},$$

where,

$$\left({}_a^c D_x^\alpha \widetilde{f}\right)(x) = \left[\left({}_a^c D_x^\alpha \underline{f}\right)(x,r), \left({}_a^c D_x^\alpha \overline{f}\right)(x,r)\right], \ \left({}_x^c D_b^\alpha \widetilde{f}\right)(x) = \left[\left({}_x^c D_b^\alpha \underline{f}\right)(x,r), \left({}_x^c D_b^\alpha \overline{f}\right)(x,r)\right]$$
$$\left({}_a D_x^\alpha \widetilde{f}\right)(x) = \left[\left({}_a D_x^\alpha \underline{f}\right)(x,r), \left({}_a D_x^\alpha \overline{f}\right)(x,r)\right], \ \left({}_x D_b^\alpha \widetilde{f}\right)(x) = \left[\left({}_x D_b^\alpha \underline{f}\right)(x,r), \left({}_x D_b^\alpha \overline{f}\right)(x,r)\right]$$

and

$$\widetilde{f}^{(k)}(a) = \left(\underline{f}^{(k)}(a,r), \overline{f}^{(k)}(a,r)\right), \ \widetilde{f}^{(k)}(b) = \left(\underline{f}^{(k)}(b,r), \overline{f}^{(k)}(b,r)\right).$$

Proof. Here we define Taylor series expansion of $\widetilde{f}$ about a as,

$$\begin{aligned} \underline{f}(x,r) &= \underline{f}(a,r) + (x-a)\underline{f}'(a,r) + \frac{(x-a)^2}{2!}\underline{f}''(a,r) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}\underline{f}^{(n-1)}(a,r) + \underline{R}_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{(x-a)^k}{\Gamma(k+1)}\underline{f}^{(k)}(a,r) + \underline{R}_{n-1}, \end{aligned}$$

$$\begin{aligned} \overline{f}(x,r) &= \overline{f}(a,r) + (x-a)\overline{f}'(a,r) + \frac{(x-a)^2}{2!}\overline{f}''(a,r) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}\overline{f}^{(n-1)}(a,r) + \overline{R}_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{(x-a)^k}{\Gamma(k+1)}\overline{f}^{(k)}(a,r) + \overline{R}_{n-1}, \end{aligned}$$

where,

$$\begin{aligned} \underline{R}_{n-1} &= \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}\underline{f}^{(n)}(t,r)dt = {}_aJ_x^n \underline{f}^{(n)}(x,r) \\ \overline{R}_{n-1} &= \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}\overline{f}^{(n)}(t,r)dt = {}_aJ_x^n \overline{f}^{(n)}(x,r). \end{aligned}$$

Using linearity property of fuzzy Riemann-Liouville fractional derivative, the fuzzy Riemann-Liouville fractional derivative of $\widetilde{f}$ can be given as,

$$
\begin{aligned}
& ({}_aD_x^\alpha \widetilde{f})(x) \\
= & \big({}_aD_x^\alpha \underline{f}(x,r),\ {}_aD_x^\alpha \overline{f}(x,r)\big) \\
= & \left({}_aD_x^\alpha\Big(\sum_{k=0}^{n-1}\frac{(x-a)^k}{\Gamma(k+1)}\underline{f}^{(k)}(a,r)+\underline{R}_{n-1}\Big),\ {}_aD_x^\alpha\Big(\sum_{k=0}^{n-1}\frac{(x-a)^k}{\Gamma(k+1)}\overline{f}^{(k)}(a,r)+\overline{R}_{n-1}\Big)\right) \\
= & \left(\sum_{k=0}^{n-1}\frac{\frac{d^\alpha}{dx^\alpha}(x-a)^k}{\Gamma(k+1)}\underline{f}^{(k)}(a,r)+\ {}_aD_x^\alpha\underline{R}_{n-1},\sum_{k=0}^{n-1}\frac{\frac{d^\alpha}{dx^\alpha}(x-a)^k}{\Gamma(k+1)}\overline{f}^{(k)}(a,r)+\ {}_aD_x^\alpha\overline{R}_{n-1}\right) \\
= & \left(\sum_{k=0}^{n-1}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\frac{(x-a)^{(k-\alpha)}}{\Gamma(k+1)}\underline{f}^{(k)}(a,r)+{}_aD_x^\alpha\,{}_aJ_x^n\underline{f}^{(n)}(x,r),\right. \\
& \left.\sum_{k=0}^{n-1}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\frac{(x-a)^{(k-\alpha)}}{\Gamma(k+1)}\overline{f}^{(k)}(a,r)+{}_aD_x^\alpha\,{}_aJ_x^n\overline{f}^{(n)}(x,r)\right) \\
= & \left(\sum_{k=0}^{n-1}\frac{(x-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)}\underline{f}^{(k)}(a,r)+\ {}_aJ_x^{n-\alpha}\underline{f}^{(n)}(x,r),\sum_{k=0}^{n-1}\frac{(x-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)}\overline{f}^{(k)}(a,r)+\right. \\
& \left. {}_aJ_x^{n-\alpha}\overline{f}^{(n)}(x,r)\right) \\
= & {}_a^cD_x^\alpha\widetilde{f}(x)+\sum_{k=0}^{n-1}\frac{(x-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)}\widetilde{f}^{(k)}(a).
\end{aligned}
$$

Therefore,

$$
({}_a^cD_x^\alpha\widetilde{f})(x) \;=\; ({}_aD_x^\alpha\widetilde{f})(x)-\sum_{k=0}^{n-1}\frac{(x-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)}\widetilde{f}^{(k)}(a).
$$

Similarly, in above, if we replace b in place of x and x in place of a, then we can get the second result. □

Remark 2.1. *If $\widetilde{f}\in C^F[a,b]\bigcap L^F[a,b]$ and $\widetilde{f}^{(k)}(a)=0,\ k=0,1,....,n-1$, then,*

$$ {}_a^cD_x^\alpha\widetilde{f}=\ {}_aD_x^\alpha\widetilde{f}. $$

Also, if $\widetilde{f}^{(k)}(b)=0,\ k=0,1,....,n-1$, then,

$$ {}_x^cD_b^\alpha\widetilde{f}=\ {}_xD_b^\alpha\widetilde{f}. $$

If L is an arbitrary fractional operator, then

$$ L(t\widetilde{f}+s\widetilde{g})=tL(\widetilde{f})+sL(\widetilde{g})\ \forall\widetilde{f},\widetilde{g}\in C^F[a,b]\bigcap L^F[a,b],\ t,s\in\mathbb{R}. $$

Diethelm [15,17,18] uses the product trapezoidal rule with respect to the weight function $(t_k-u)^{\alpha-1}$ to approximate the Riemann-Liouville fractional integrals. Similarly, we can define approximate fuzzy Riemann-Liouville fractional integrals as,

$$ \int_{t_0}^{t_k}(t_k-u)^{\alpha-1}\widetilde{f}(u)du\simeq\int_{t_0}^{t_k}(t_k-u)^{\alpha-1}\widetilde{f}_k^*(u)du, $$

where, $\widetilde{f}(u) = (\underline{f}(u,r), \overline{f}(u,r))$, $\widetilde{f}_k^*(u) = (\underline{f_k^*}(u,r), \overline{f_k^*}(u,r))$ and $\widetilde{f}_k^*$ is the piecewise linear interpolation of $\widetilde{f}$ whose nodes are chosen at $t_j = jh$, $j = 0, 1, 2, \ldots.., n$ and $h = \dfrac{b-a}{n}$.

Theorem 2.2.
(a) Let $\widetilde{f} \in C^1[0,b]$ be a fuzzy valued function. Then,

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \widetilde{f}(t)dt - \sum_{j=0}^{k} b_{j,k+1} \widetilde{f}(t_j) \right| \le \frac{1}{\alpha} ||\widetilde{f}'||_\infty t_{k+1}^\alpha h.$$

(b) For $\widetilde{f}(t) = t^p$, $p \in (0,1)$, then we obtain,

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \widetilde{f}(t)dt - \sum_{j=0}^{k} b_{j,k+1} \widetilde{f}(t_j) \right| \le c_{\alpha,p}^{Re} t_{k+1}^{\alpha+p-1} h,$$

where $c_{\alpha,p}^{Re}$ is a constant that depends only on α and p and

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}[(k+1-j)^\alpha - (k-j)^\alpha], \ t_j = jh.$$

Proof. By construction of the quadrature rule, in both cases we find the quadrature error as

$$\left(\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \underline{f}(t,r)dt - \sum_{j=0}^{k} b_{j,k+1} \underline{f}(t_j,r), \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \overline{f}(t,r)dt - \sum_{j=0}^{k} b_{j,k+1} \overline{f}(t_j,r) \right)$$
$$= \left(\sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1} - t)^{\alpha-1} (\underline{f}(t,r) - \underline{f}(t_j,r)dt, \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1} - t)^{\alpha-1} (\overline{f}(t,r) - \overline{f}(t_j,r)dt \right). \quad (5)$$

Now in order to prove statement (a), we apply Mean Value Theorem to the second factor of the integration part on the right hand side of (5), to get

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \underline{f}(t,r)dt - \sum_{j=0}^{k} b_{j,k+1} \underline{f}(t_j,r), \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \overline{f}(t,r)dt - \sum_{j=0}^{k} b_{j,k+1} \overline{f}(t_j,r) \right|$$

$$\begin{aligned}
&= \left(\sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1} - t)^{\alpha-1} \Big| \underline{f}(t,r) - \underline{f}(t_j,r) \Big| dt, \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1} - t)^{\alpha-1} \Big| \overline{f}(t,r) - \overline{f}(t_j,r) \Big| dt \right) \\
&\le ||(\underline{f}', \overline{f}')||_\infty \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1} - t)^{\alpha-1} (t - jh)dt \\
&= ||\widetilde{f}'||_\infty \frac{h^{1+\alpha}}{\alpha} \sum_{j=0}^{k} \left(\frac{1}{1+\alpha} [(k+1-j)^{1+\alpha} - (k-j)^{1+\alpha}] - (k-j)^\alpha \right) \\
&= ||\widetilde{f}'||_\infty \frac{h^{1+\alpha}}{\alpha} \left(\frac{(k+1)^{1+\alpha}}{1+\alpha} - \sum_{j=0}^{k} j^\alpha \right) = ||\widetilde{f}'||_\infty \frac{h^{1+\alpha}}{\alpha} \left(\int_0^{k+1} t^\alpha dt - \sum_{j=0}^{k} j^\alpha \right) \\
&\le ||\widetilde{f}'||_\infty \frac{h^{1+\alpha}}{\alpha} (k+1)^\alpha = ||\widetilde{f}'||_\infty \frac{h}{\alpha} t_{k+1}^\alpha.
\end{aligned}$$

Hence

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\widetilde{f}(t)dt - \sum_{j=0}^{k} b_{j,k+1}\widetilde{f}(t_j)\right| \leq \frac{1}{\alpha}||\widetilde{f}'||_\infty t_{k+1}^{\alpha} h.$$

Since the integrand is monotonic, we applied some standard results from quadrature theory [19](Theorem 97) to find that this term is bounded by the total variation of the integrand, viz. the quantity $(k+1)^\alpha$.
Similarly, to prove statement (b), use monotonicity of $\widetilde{f}$ in (5), to get

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\underline{f}(t,r)dt - \sum_{j=0}^{k} b_{j,k+1}\underline{f}(t_j,r), \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\overline{f}(t,r)dt - \sum_{j=0}^{k} b_{j,k+1}\overline{f}(t_j,r)\right|$$

$$\begin{aligned}
\leq\ & \Bigg|\Bigg(\sum_{j=0}^{k}\int_{jh}^{(j+1)h} (t_{k+1}-t)^{\alpha-1}\big(\underline{f}(t_{j+1},r)-\underline{f}(t_j,r)\big)dt, \\
& \qquad \sum_{j=0}^{k}\int_{jh}^{(j+1)h} (t_{k+1}-t)^{\alpha-1}\big(\overline{f}(t_{j+1},r)-\overline{f}(t_j,r)\big)dt\Bigg)\Bigg| \\
\leq\ & \Bigg(\sum_{j=0}^{k}\big|\underline{f}(t_{j+1},r)-\underline{f}(t_j,r)\big|\frac{h^\alpha}{\alpha}\big((k+1-j)^\alpha-(k-j)^\alpha\big), \\
& \sum_{j=0}^{k}\big|\overline{f}(t_{j+1},r)-\overline{f}(t_j,r)\big|\frac{h^\alpha}{\alpha}\big((k+1-j)^\alpha-(k-j)^\alpha\big)\Bigg) \\
=\ & \frac{h^{\alpha+p}}{\alpha}\sum_{j=0}^{k}\big((j+1)^p-j^p\big)\big((k+1-j)^\alpha-(k-j)^\alpha\big) \\
\leq\ & \frac{h^{\alpha+p}}{\alpha}\left(2(k+1)^\alpha-2k^\alpha+p\alpha\sum_{j=1}^{k-1} j^{p-1}(k-j+q)^{\alpha-1}\right) \\
& \leq \frac{h^{\alpha+p}}{\alpha}\left(2\alpha(k+q)^{\alpha-1}+p\alpha\sum_{j=1}^{k-1} j^{p-1}(k-j+q)^{\alpha-1}\right)
\end{aligned}$$

by applications of the Mean Value Theorem. Here $q=0$, if $\alpha \leq 1$, otherwise $q=1$. The term in parentheses is bounded from above by $C_{\alpha,p}(k+1)^{p+\alpha-1}$, where $C_{\alpha,p}$ is a constant depending on α and p but not on k by using the Euler-MacLaurin formula [20](Theorem 3.7). □

Next we give the result for product trapezoidal formula. The proof of this theorem is very similar to the proof of above theorem so we omit the proof.

Theorem 2.3. *Let $\widetilde{f} \in C^2[0,b]$ be a fuzzy valued function. Then,*

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\widetilde{f}(t)dt - \sum_{j=0}^{k+1} a_{j,k+1}\widetilde{f}(t_j)\right| \leq c_\alpha^{Tr}||\widetilde{f}''||_\infty t_{k+1}^{\alpha} h^2,$$

where c_α^{Tr} is a constant depends only on α and

$$a_{j,k} = \frac{h^\alpha}{\alpha(\alpha+1)}\begin{cases} \big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big),\ if\ j=0, \\ \big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big),\ if\ 1\leq j\leq k-1, \\ 1,\ if\ j=k. \end{cases}$$

3. Numerical approximations to fuzzy fractional operators

THEOREM 3.1. *Suppose that the interval $[0, b]$ is subdivided into k subintervals $[x_j, x_{j+1}]$ of equal lengths $h = \dfrac{b}{k}$ by using the nodes $x_j = jh$, for $j = 0, 1, 2, ..., k$. The modified fuzzy trapezoidal rule*

$$\begin{aligned}\Big(T(\underline{f}, h, \alpha), T(\overline{f}, h, \alpha)\Big) &= \frac{h^\alpha}{\Gamma(\alpha+2)}\Big(\big((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha\big)\underline{f}(0, r) + \underline{f}(b, r)\\ &+ \sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}\big)\underline{f}(x_j, r),\\ &\big((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha\big)\overline{f}(0, r) + \overline{f}(b, r)\\ &+ \sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}\big)\overline{f}(x_j, r)\Big)\end{aligned}$$

is an approximation to the fuzzy fractional integral ${}_0J_x^\alpha \widetilde{f}(x)|_{x=b}$, where

$$\begin{aligned}{}_0J_x^\alpha \widetilde{f}(x)|_{x=b} &= {}_0J_x^\alpha(\underline{f}(x, r), \overline{f}(x, r))|_{x=b} = T(\underline{f}, \overline{f}, h, \alpha) - E_T(\underline{f}, \overline{f}, h, \alpha)\\ &= (T(\underline{f}, h, \alpha) - E_T(\underline{f}, h, \alpha), T(\overline{f}, h, \alpha) - E_T(\overline{f}, h, \alpha)).\end{aligned}$$

Furthermore, if $\widetilde{f} \in C^2[0, b]$, then

$$|E_T(\underline{f}, \overline{f}, h, \alpha)| \leq c_\alpha^{'}||\widetilde{f}''||_\infty b^\alpha h^2 = O(h^2),$$

where $c_\alpha^{'}$ is a constant depending only on α.

Proof. By using above theorem it can noted that

$$\int_0^b (b-u)^{\alpha-1}\widetilde{f}(u)du \simeq \int_0^b (b-u)^{\alpha-1}\widetilde{f}_k^*(u)du = \sum_{j=0}^{k} a_{j,k}\widetilde{f}(x_j),$$

where,

$$a_{j,k} = \frac{h^\alpha}{\alpha(\alpha+1)}\begin{cases} ((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha), \ if\ j = 0,\\ ((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}), \ if\ 1 \leq j \leq k-1,\\ 1, \ if\ j = k.\end{cases}$$

By definition,

$$\big({}_0J_x^\alpha \widetilde{f}\big)(x) = \big({}_0J_x^\alpha \underline{f}(x, r), {}_0J_x^\alpha \overline{f}(x, r)\big), \ 0 \leq r \leq 1.$$

where,

$$\big({}_0J_x^\alpha \underline{f}\big)(x, r) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\underline{f}(t, r)dt$$

and

$$\big({}_0J_x^\alpha \overline{f}\big)(x, r) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\overline{f}(t, r)dt.$$

Also

$$\int_0^b (b-u)^{\alpha-1}\widetilde{f}(u)du \simeq \sum_{j=0}^{k} a_{j,k}\widetilde{f}(x_j)$$

or

$$\Big(\int_0^b (b-u)^{\alpha-1}\underline{f}(u,r)du, \int_0^b (b-u)^{\alpha-1}\overline{f}(u,r)du\Big) \simeq \Big(\sum_{j=0}^{k} a_{j,k}\underline{f}(x_j,r), \sum_{j=0}^{k} a_{j,k}\overline{f}(x_j,r)\Big).$$

Therefore,

$$\begin{aligned}
\big({}_0J_b^\alpha \widetilde{f}\big)(b) &= \big({}_0J_b^\alpha \underline{f}(b,r),\ {}_0J_b^\alpha \overline{f}(b,r)\big) \\
&\simeq \frac{1}{\Gamma(\alpha)}\Big(\sum_{j=0}^{k} a_{j,k}\underline{f}(x_j,r), \sum_{j=0}^{k} a_{j,k}\overline{f}(x_j,r)\Big) \\
&\simeq \frac{1}{\Gamma(\alpha)}\frac{h^\alpha}{\alpha(\alpha+1)}\Big(\big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\underline{f}(0,r)+\underline{f}(b,r) \\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\underline{f}(x_j,r), \\
&\quad \big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\overline{f}(0,r)+\overline{f}(b,r) \\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\overline{f}(x_j,r)\Big).
\end{aligned}$$

Since,

$${}_0J_x^\alpha \widetilde{f}(x)|_{x=b} = \big(T(\underline{f},h,\alpha)-E_T(\underline{f},h,\alpha), T(\overline{f},h,\alpha)-E_T(\overline{f},h,\alpha)\big),$$

hence

$$\begin{aligned}
\Big(T(\underline{f},h,\alpha),T(\overline{f},h,\alpha)\Big) &= \frac{h^\alpha}{\Gamma(\alpha+2)}\Big(\big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\underline{f}(0,r)+\underline{f}(b,r) \\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\underline{f}(x_j,r), \\
&\quad \big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\overline{f}(0,r)+\overline{f}(b,r) \\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\overline{f}(x_j,r)\Big).
\end{aligned}$$

Thus, by using Theorem 2.3,

$$\begin{aligned}
&\Big|\int_0^b (b-u)^{\alpha-1}\widetilde{f}(u)du - \int_0^b (b-u)^{\alpha-1}\widetilde{f}_k'(u)du\Big| \le c_\alpha^{'}||\widetilde{f}''||_\infty b^\alpha h^2 \\
&\Big|E_T(\widetilde{f},h,\alpha)\Big| \le c_\alpha'||\widetilde{f}''||_\infty b^\alpha h^2 = O(h^2).
\end{aligned}$$

□

THEOREM 3.2. *Suppose that the interval $[0,b]$ is subdivided into k subintervals $[x_j,x_{j+1}]$ of equal distances $h=\frac{b}{k}$ by using the nodes $x_j = jh,\ j=0,1,2,......,k$.*

Then the modified fuzzy trapezoidal rule for $n-1<\alpha\leq n, n\in\mathbb{N}$ is given by;

$$\begin{aligned}
&\Big(C(\underline{f},h,\alpha),C(\overline{f},h,\alpha)\Big)\\
=\ &\frac{h^{n-\alpha}}{\Gamma(n-\alpha+2)}\Big(\big((k-1)^{n-\alpha+1}-(k-n+\alpha-1)k^{n-\alpha}\big)\underline{f}^{(n)}(0,r)+\underline{f}^{(n)}(b,r)\\
&+\sum_{j=1}^{k-1}\big((k-j+1)^{n-\alpha+1}-2(k-j)^{n-\alpha+1}+(k-j-1)^{n-\alpha+1}\big)\underline{f}^{(n)}(x_j,r),\\
&\big((k-1)^{n-\alpha+1}-(k-n+\alpha-1)k^{n-\alpha}\big)\overline{f}^{(n)}(0,r)+\overline{f}^{(n)}(b,r)\\
&+\sum_{j=1}^{k-1}\big((k-j+1)^{n-\alpha+1}-2(k-j)^{n-\alpha+1}+(k-j-1)^{n-\alpha+1}\big)\overline{f}^{(n)}(x_j,r)\Big)
\end{aligned}$$

is an approximation to the Caputo fuzzy fractional derivative

$$\big({}_0^cD_x^\alpha\widetilde{f}\big)(x)|_{x=b}=\big[\big({}_0^cD_x^\alpha\underline{f}\big)(x,r),\ \big({}_0^cD_x^\alpha\overline{f}\big)(x,r)\big]_{x=b}$$

where,

$$\big({}_0^cD_x^\alpha\underline{f}\big)(x,r)|_{x=b}=\frac{1}{\Gamma(n-\alpha)}\int_0^b(b-u)^{n-\alpha-1}\underline{f}^{(n)}(u,r)du$$

$$\big({}_0^cD_x^\alpha\overline{f}\big)(x,r)|_{x=b}=\frac{1}{\Gamma(n-\alpha)}\int_0^b(b-u)^{n-\alpha-1}\overline{f}^{(n)}(u,r)du.$$

Thus

$$\big({}_0^cD_x^\alpha\widetilde{f}\big)(x)\Big|_{x=b}=\big(C(\underline{f},h,\alpha)-E_c(\underline{f},h,\alpha),C(\overline{f},h,\alpha)-E_c(\overline{f},h,\alpha)\big).$$

Furthermore, if $\widetilde{f}\in C^{n+2}[0,b]$, then

$$|E_c(\underline{f},\overline{f},h,\alpha)|\leq c'_{n-\alpha}||\widetilde{f}^{(n+2)}||_\infty b^{n-\alpha}h^2=O(h^2),$$

where $c'_{n-\alpha}$ is a constant depending only on α.

Proof.

$$\big({}_0^cD_x^\alpha\widetilde{f}\big)(x)|_{x=b}=\big[\big({}_0^cD_x^\alpha\underline{f}\big)(x,r),\ \big({}_0^cD_x^\alpha\overline{f}\big)(x,r)\big]_{x=b}$$

where,

$$\begin{aligned}
\big({}_0^cD_x^\alpha\underline{f}\big)(x,r)|_{x=b}&=\frac{1}{\Gamma(n-\alpha)}\int_0^b(b-u)^{n-\alpha-1}\underline{f}^{(n)}(u,r)du\\
\big({}_0^cD_x^\alpha\overline{f}\big)(x,r)|_{x=b}&=\frac{1}{\Gamma(n-\alpha)}\int_0^b(b-u)^{n-\alpha-1}\overline{f}^{(n)}(u,r)du,
\end{aligned}$$

replacing α by $n-\alpha$ and $(\underline{f}(u,r),\overline{f}(u,r))$ by $(\underline{f}^n(u,r),\overline{f}^n(u,r))$ in previous theorem, we can obtain,

$$\begin{aligned}
&\left(\int_0^b(b-u)^{n-\alpha-1}\underline{f}(u,r)du,\int_0^b(b-u)^{n-\alpha-1}\overline{f}(u,r)du\right)\\
\simeq\ &\left(\sum_{j=0}^k a_{j,k}\underline{f}(u_j,r),\sum_{j=0}^k a_{j,k}\overline{f}(u_j,r)\right).
\end{aligned}$$

and thus

$$\begin{aligned}&\Big(C(\underline{f},h,\alpha),C(\overline{f},h,\alpha)\Big)\\
=\;&\frac{h^{n-\alpha}}{\Gamma(n-\alpha+2)}\Big(\big((k-1)^{n-\alpha+1}-(k-n+\alpha-1)k^{n-\alpha}\big)\underline{f}^{(n)}(0,r)+\underline{f}^{(n)}(b,r)\\
&+\sum_{j=1}^{k-1}\big((k-j+1)^{n-\alpha+1}-2(k-j)^{n-\alpha+1}+(k-j-1)^{n-\alpha+1}\big)\underline{f}^{(n)}(x_j,r),\\
&\big((k-1)^{n-\alpha+1}-(k-n+\alpha-1)k^{n-\alpha}\big)\overline{f}^{(n)}(0,r)+\overline{f}^{(n)}(b,r)\\
&+\sum_{j=1}^{k-1}\big((k-j+1)^{n-\alpha+1}-2(k-j)^{n-\alpha+1}+(k-j-1)^{n-\alpha+1}\big)\overline{f}^{(n)}(x_j,r)\Big).\end{aligned}$$

Also, by using Theorem 2.3, we can get

$$\Big|\int_0^b (b-u)^{\alpha-1}\widetilde{f}(u)du-\int_0^b (b-u)^{\alpha-1}\widetilde{f}_k'(u)du\Big|\leq c_{n-\alpha}^{'}||\widetilde{f}^{(n+2)}||_\infty b^{n-\alpha}h^2$$
$$\Big|E_c(\widetilde{f},h,\alpha)\Big|\leq c_{n-\alpha}^{'}||\widetilde{f}^{(n+2)}||_\infty b^{n-\alpha}h^2=O(h^2)$$

where $c_{n-\alpha}^{'}$ is a constant depending only on α. □

4. Solution of Abel's fuzzy integral equation of first kind

Consider the following Abel's fuzzy integral equation of first kind:-

$$\widetilde{f}(x)=\Big(\underline{f}(x,r),\overline{f}(x,r)\Big)=\Big(\int_0^x\frac{\underline{g}(t,r)}{(x-t)^\alpha}dt,\int_0^x\frac{\overline{g}(t,r)}{(x-t)^\alpha}dt\Big) \tag{6}$$

$0<\alpha<1, 0\leq x\leq b, 0\leq r\leq 1$, where $\widetilde{f}\in C^1[a,b]$ is given function satisfying $\widetilde{f}(0)=0$ and $\widetilde{g}$ is the unknown function $\widetilde{g}(t)=\big(\underline{g}(t,r),\overline{g}(t,r)\big)$. The following theorem will give solution of (6) by using fractional calculus.

THEOREM 4.1. *The solution of above problem* (6) *by using fractional calculus is*

$$\widetilde{g}(x)=\frac{\sin(\alpha\pi)}{\pi}\Big(\int_0^x\frac{\underline{f}'(t,r)}{(x-t)^{1-\alpha}}dt,\int_0^x\frac{\overline{f}'(t,r)}{(x-t)^{1-\alpha}}dt\Big)$$

Proof. Since,

$$\big({}_0J_x^\alpha\widetilde{f}\big)(x)=\big[\big({}_0J_x^\alpha\underline{f}\big)(x,r),\big({}_0J_x^\alpha\overline{f}\big)(x,r)\big],0\leq r\leq 1,$$

where,

$$\big({}_0J_x^\alpha\underline{f}\big)(x,r)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\underline{f}(t,r)dt,$$
$$\big({}_0J_x^\alpha\overline{f}\big)(x,r)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\overline{f}(t,r)dt,$$

so, we can write (6) in the equivalent form as

$$\widetilde{f}(x)=\big(\underline{f}(x,r),\overline{f}(x,r)\big)=\Gamma(1-\alpha)\big({}_0J_x^{1-\alpha}\underline{g}(x,r),\ {}_0J_x^{1-\alpha}\overline{g}(x,r)\big).$$

Also, we know that ${}_a^cD_x^\alpha {}_aJ_x^\alpha \widetilde{f}(x) = \widetilde{f}(x)$. So,

$$
\begin{aligned}
{}_0^cD_x^{1-\alpha}\widetilde{f}(x) &= \Gamma(1-\alpha)\Big({}_0^cD_x^{1-\alpha}{}_0J_x^{1-\alpha}\underline{g}(x,r), {}_0^cD_x^{1-\alpha}{}_0J_x^{1-\alpha}\overline{g}(x,r)\Big) \\
&= \Gamma(1-\alpha)\big(\underline{g}(x,r), \overline{g}(x,r)\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widetilde{g}(x) &= \big(\underline{g}(x,r), \overline{g}(x,r)\big) \\
&= \frac{1}{\Gamma(1-\alpha)}\big({}_0^cD_x^{1-\alpha}\underline{f}(x,r),\ {}_0^cD_x^{1-\alpha}\overline{f}(x,r)\big) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\Big(\int_0^x \frac{\underline{f}'(t,r)}{(x-t)^{1-\alpha}}dt, \int_0^x \frac{\overline{f}'(t,r)}{(x-t)^{1-\alpha}}dt\Big) \\
&= \frac{\sin(\alpha\pi)}{\pi}\Big(\int_0^x \frac{\underline{f}'(t,r)}{(x-t)^{1-\alpha}}dt, \int_0^x \frac{\overline{f}'(t,r)}{(x-t)^{1-\alpha}}dt\Big)
\end{aligned}
$$

□

THEOREM 4.2. *Suppose that the interval $[0,b]$ is subdivided into k subintervals $[t_j, t_{j+1}]$ of equal distances $h = \frac{b}{k}$ by using the nodes $t_j = jh,\ j = 0, 1, 2, ..., k$ and $0 < x < b$. An approximate solution $\widetilde{\hat{g}}$ to the solution $\widetilde{g}$ of the Abel's general fuzzy integral equation (6) is given by,*

$$
\begin{aligned}
\widetilde{\hat{g}}(x) &= \big(\underline{\hat{g}}(x,r), \overline{\hat{g}}(x,r)\big) \\
&= \frac{h^\alpha}{\Gamma(1-\alpha)\Gamma(\alpha+2)}\Big(\big((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha\big)\underline{f}'(0,r) + \underline{f}'(x,r) \\
&\quad + \sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}\big)\underline{f}'(t_j,r), \\
&\quad \big((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha\big)\overline{f}'(0,r) + \overline{f}'(x,r) \\
&\quad + \sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}\big)\overline{f}'(t_j,r)\Big)
\end{aligned}
$$

Moreover, if $\widetilde{f} \in C^3[0,b]$, then

$$
\begin{aligned}
\widetilde{g}(x) &= \big(\underline{g}(x,r), \overline{g}(x,r)\big) \\
&= \Big(\underline{\hat{g}}(x,r) - \frac{1}{\Gamma(1-\alpha)}\underline{E}(x,r), \overline{\hat{g}}(x,r) - \frac{1}{\Gamma(1-\alpha)}\overline{E}(x,r)\Big)
\end{aligned}
$$

with

$$
|\widetilde{E}(x)| = |\big(\underline{E}(x,r), \overline{E}(x,r)\big)| \le c_\alpha^{'}||\widetilde{f}'''||_\infty x^\alpha h^2 = O(h^2)
$$

where $c_\alpha^{'}$ is the constant depending only on α and $||\widetilde{f}'''||_\infty = \max_{t\in[0,x]}|\widetilde{f}'''(t)|$.

Proof. We want to approximate the Caputo derivatives i.e; to approximate

$$\begin{aligned}
\widetilde{g}(x) &= \big(\underline{g}(x,r), \overline{g}(x,r)\big) \\
&= \frac{1}{\Gamma(1-\alpha)}\big({}_0^cD_x^{1-\alpha}\underline{f}(x,r),\ {}_0^cD_x^{1-\alpha}\overline{f}(x,r)\big) \\
&= \frac{\sin(\alpha\pi)}{\pi}\Gamma(\alpha)\big({}_0^cD_x^{1-\alpha}\underline{f}(x,r),\ {}_0^cD_x^{1-\alpha}\overline{f}(x,r)\big)
\end{aligned}$$

If the Caputo fractional derivative of order $1-\alpha$ for $\widetilde{f},\ 0<\alpha<1$ is calculated at collocation nodes $t_j,\ j=0,1,....,k$, then for $a=0, n=1$

$$\begin{aligned}
{}_0^cD_x^{1-\alpha}\widetilde{f}(x) &= \big({}_0^cD_x^{1-\alpha}\underline{f}(x,r),\ {}_0^cD_x^{1-\alpha}\overline{f}(x,r)\big) \\
&= \frac{1}{\Gamma(\alpha)}\Big(\int_0^x (x-t)^{\alpha-1}\underline{f}'(t,r)dt, \int_0^x (x-t)^{\alpha-1}\overline{f}'(t,r)dt\Big)
\end{aligned}$$

Using Theorem (3.2), for $n=1$

$$\begin{aligned}
&\Big(C(\underline{f},h,1-\alpha), C(\overline{f},h,1-\alpha)\Big) \\
= &\frac{h^\alpha}{\Gamma(\alpha+2)}\Big(\big((k-1)^{1+\alpha}-(k-1-\alpha)k^\alpha\big)\underline{f}'(0,r)+\underline{f}'(x,r) \\
&+\sum_{j=1}^{k-1}\big((k-j+1)^{1+\alpha}-2(k-j)^{1+\alpha}+(k-j-1)^{1+\alpha}\big)\underline{f}'(t_j,r), \\
&\big((k-1)^{1+\alpha}-(k-1-\alpha)k^\alpha\big)\overline{f}'(0,r)+\overline{f}'(x,r) \\
&+\sum_{j=1}^{k-1}\big((k-j+1)^{1+\alpha}-2(k-j)^{1+\alpha}+(k-j-1)^{1+\alpha}\big)\overline{f}'(t_j,r)\Big),
\end{aligned}$$

where,

$$\begin{aligned}
{}_0^cD_x^{\alpha}\widetilde{f}(x)|_{x=b} &= C(\underline{f},\overline{f},h,\alpha)-E_c(\underline{f},\overline{f},h,\alpha) \\
&= \Big(C(\underline{f},h,\alpha)-E_c(\underline{f},h,\alpha), C(\overline{f},h,\alpha)-E_c(\overline{f},h,\alpha)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{g}(x) &= \frac{1}{\Gamma(1-\alpha)}\big({}_0^cD_x^{1-\alpha}\underline{f}(x,r), {}_0^cD_x^{1-\alpha}\overline{f}(x,r)\big)\\
&= \frac{1}{\Gamma(1-\alpha)}\Big(C(\underline{f},h,1-\alpha)-E_c(\underline{f},h,1-\alpha), C(\overline{f},h,1-\alpha)-E_c(\overline{f},h,1-\alpha)\Big)\\
&= \frac{h^\alpha}{\Gamma(1-\alpha)\Gamma(\alpha+2)}\Big(\big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\underline{f}'(0,r)+\underline{f}'(x,r)\\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\underline{f}'(t_j,r),\\
&\quad \big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\overline{f}'(0,r)+\overline{f}'(x,r)\\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\overline{f}'(t_j,r)\Big)\\
&\quad -\frac{1}{\Gamma(1-\alpha)}\big(E_c(\underline{f},h,1-\alpha),E_c(\overline{f},h,1-\alpha)\big)\\
&= \widetilde{\hat{g}}(x)-\frac{1}{\Gamma(1-\alpha)}\widetilde{E}(x),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\hat{g}}(x) &= \big(\underline{\hat{g}}(x,r),\overline{\hat{g}}(x,r)\big)\\
&= \frac{h^\alpha}{\Gamma(1-\alpha)\Gamma(\alpha+2)}\Big(\big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\underline{f}'(0,r)+\underline{f}'(x,r)\\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\underline{f}'(t_j,r),\\
&\quad \big((k-1)^{\alpha+1}-(k-\alpha-1)k^\alpha\big)\overline{f}'(0,r)+\overline{f}'(x,r)\\
&\quad +\sum_{j=1}^{k-1}\big((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1}\big)\overline{f}'(t_j,r)\Big)
\end{aligned}$$

Hence, by using Theorem 2.3, we can get

$$\begin{aligned}
\widetilde{E}(x) &= \Big(E_c(\underline{f},h,1-\alpha),E_c(\overline{f},h,1-\alpha)\Big)\\
&\quad |\widetilde{E}(x)|\le c_\alpha^{'}||\widetilde{f}^{'''}||_\infty x^\alpha h^2=O(h^2)
\end{aligned}$$

where $c_\alpha^{'}$ is the constant depending only on α and $||\widetilde{f}^{'''}||_\infty = \max_{t\in[0,x]}|\widetilde{f}^{'''}(t)|$. □

5. Numerical examples

In this section we will show usefulness of our results by solving two Abel's integral equations.

Example 5.1. Consider the Abel's general fuzzy integral equation,

$$\Big(\frac{4}{3}rx^{\frac{3}{2}},\frac{4}{3}(2-r)x^{\frac{3}{2}}\Big)=\Big(\int_0^x\frac{\underline{g}(t,r)}{(x-t)^{1/2}}dt,\int_0^x\frac{\overline{g}(t,r)}{(x-t)^{1/2}}dt\Big) \tag{7}$$

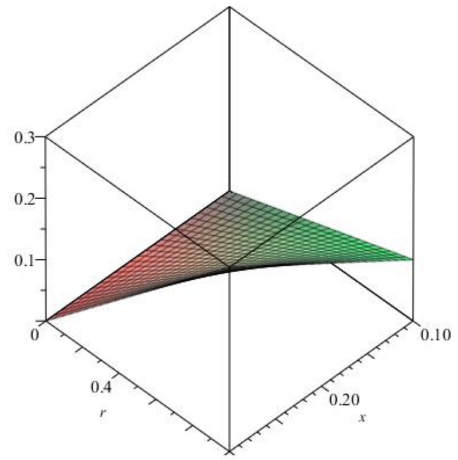

Figure 1. Lower Approximation

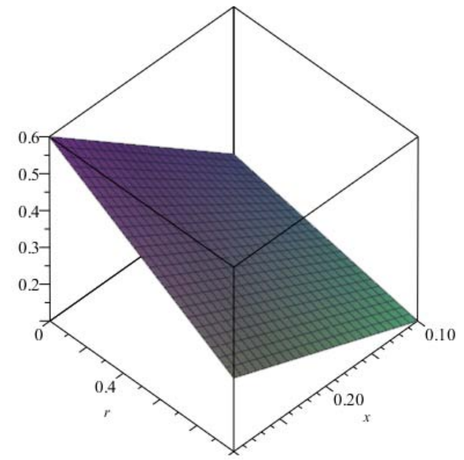

Figure 2. Upper Approximation

Here,

$$\underline{f}(x,r)=\frac{4}{3}rx^{\frac{3}{2}},\overline{f}(x,r)=\frac{4}{3}(2-r)x^{\frac{3}{2}},\ \underline{f'}(x,r)=2rx^{\frac{1}{2}},\overline{f}'(x,r)=2(2-r)x^{\frac{1}{2}}.$$

The Exact solution is,

$$\begin{aligned}(\underline{g}(x,r),\overline{g}(x,r)) &= \frac{\sin\frac{\pi}{2}}{\pi}\Big(\int_0^x \frac{2rt^{1/2}}{(x-t)^{1/2}}dt, \int_0^x \frac{2(2-r)t^{1/2}}{(x-t)^{1/2}}dt\Big)\\ &= \frac{1}{\pi}\Big(2r\int_0^x \frac{t^{1/2}}{(x-t)^{1/2}}dt, 2(2-r)\int_0^x \frac{t^{1/2}}{(x-t)^{1/2}}dt\Big).\end{aligned}$$

Since,

$$\int_0^x \frac{t^{1/2}}{(x-t)^{1/2}}dt=\frac{1}{x^{1/2}}\int_0^x \frac{t^{1/2}}{(1-\frac{t}{x})^{1/2}}dt=x\frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(2)}=\frac{\pi x}{2},$$

then the solution is

$$\big(\underline{g}(x,r),\overline{g}(x,r)\big)=\frac{1}{\pi}\Big(2r\frac{\pi x}{2},2(2-r)\frac{\pi x}{2}\Big)=(rx,(2-r)x).$$

For $r\in[0,1]$, Table 1 contains approximate values of lower case, where as Table 2 contains approximate values of upper case.

Table 1. Approximated values lower solution

x_i	$k=10$	$k=100$	$k=1000$	*Exact solution*	Error $\Delta_{k=1000}$
0.1	0.04979017284	0.04999337466	0.04999967243	0.05000000000	0.00000032757
0.2	0.09958034560	0.09998674925	0.09999934535	0.1000000000	0.00000065465
0.3	0.1493705186	0.1499801239	0.1499990180	0.1500000000	0.000000982

TABLE 2. Approximated values upper solution

x_i	$k = 10$	$k = 100$	$k = 1000$	*Exact solution*	Error $\Delta_{k=1000}$
0.1	0.1493705185	0.1499801241	0.1499990174	0.1499999999	0.0000009825
0.2	0.2987410368	0.2999602477	0.2999980364	0.2999999998	0.0000019634
0.3	0.4481115555	0.4499403726	0.4499970526	0.4499999998	0.0000029472

EXAMPLE 5.2. Consider the Abel's general fuzzy integral equation,

$$\Big(\frac{2}{3}rx^{\frac{3}{2}}, \frac{2}{3}(2-r)x^{\frac{3}{2}}\Big) = \Big(\int_0^x \frac{\underline{g}(t,r)}{(x-t)^{1/3}}dt, \int_0^x \frac{\overline{g}(t,r)}{(x-t)^{1/3}}dt\Big) \tag{8}$$

Here,

$$\underline{f}(x,r) = \frac{2}{3}rx^{\frac{3}{2}}, \overline{f}(x,r) = \frac{2}{3}(2-r)x^{\frac{3}{2}},\ \underline{f}'(x,r) = rx^{\frac{1}{2}}, \overline{f}'(x,r) = (2-r)x^{\frac{1}{2}}.$$

The Exact solution of the equation is,

$$\begin{aligned}\big(\underline{g}(x,r), \overline{g}(x,r)\big) &= \frac{\sin\frac{\pi}{3}}{\pi}\Big(\int_0^x \frac{rt^{1/2}}{(x-t)^{1/3}}dt, \int_0^x \frac{(2-r)t^{1/2}}{(x-t)^{1/3}}dt\Big)\\ &= \frac{\sqrt{3}}{2\pi}\Big(r\int_0^x \frac{t^{1/2}}{(x-t)^{1/3}}dt, (2-r)\int_0^x \frac{t^{1/2}}{(x-t)^{1/3}}dt\Big).\end{aligned}$$

Since

$$\int_0^x \frac{t^{1/2}}{(x-t)^{1/3}}dt = \frac{1}{x^{1/3}}\int_0^x \frac{t^{1/2}}{(1-\frac{t}{x})^{1/3}}dt = x^{7/6}\frac{\Gamma(3/2)\Gamma(2/3)}{\Gamma(13/6)} = x^{7/6}\frac{\frac{1}{2}\sqrt{\pi}\Gamma(2/3)}{\Gamma(13/6)},$$

the solution is

$$(\underline{g}(x,r), \overline{g}(x,r)) = \frac{\sqrt{3}}{2\pi}\Big(\frac{\frac{1}{2}\sqrt{\pi}\Gamma(2/3)}{\Gamma(13/6)}rx^{7/6}, \frac{\frac{1}{2}\sqrt{\pi}\Gamma(2/3)}{\Gamma(13/6)}(2-r)x^{7/6}\Big).$$

For $r \in [0,1]$, Table 3 contains approximate values of lower case, where as Table 4 contains approximate values of upper case.

TABLE 3. Approximated values lower solution

x_i	$k = 10$	$k = 100$	$k = 1000$	*Exact solution*	Error $\Delta_{k=1000}$
0.6	0.2266616908	0.2272571240	0.2272762232	0.2272760804	0.0000001428
0.7	0.2577312708	0.2584083229	0.2584300402	0.2584298778	0.0000001624
0.8	0.2880671512	0.2888238944	0.2888481682	0.2888479866	0.0000001816

TABLE 4. Approximated values upper solution

x_i	$k = 10$	$k = 100$	$k = 1000$	*Exact solution*	Error $\Delta_{k=1000}$
0.6	0.6799850715	0.6817713709	0.6818286695	0.6818282411	0.0000004284
0.7	0.7731938131	0.7752249684	0.7752901188	0.7752896332	0.0000004856
0.8	0.8642014538	0.8664716826	0.8665445048	0.8665439592	0.0000005456

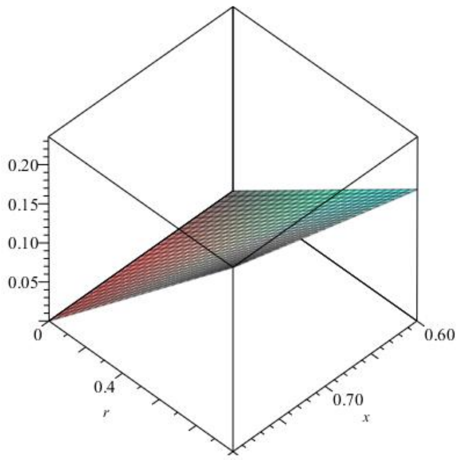

FIGURE 3. Lower Approximation

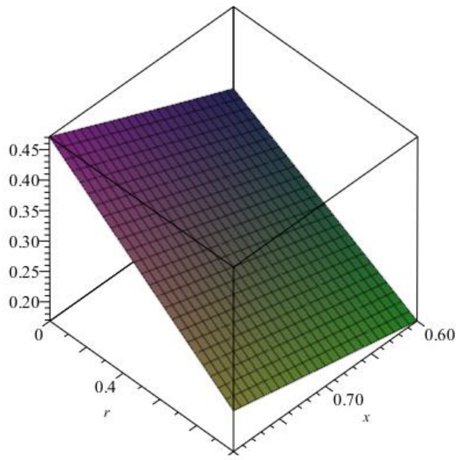

FIGURE 4. Upper Approximation

6. Conclusion

In this paper, we derived Riemann fractional integrals and Caputo derivatives for fuzzy functions. Then we solved linear Abel's general fuzzy integral equations by using above mentioned methods. Particularly, we converted linear Abel's fuzzy integral equation into two crisp linear Abel's integral equations based on the embedding method and then solved it. In the end we illustrated the results by solving two Abel's general fuzzy integral equations.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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