INTUITIONISTIC FUZZY PMS-SUBALGEBRA OF A PMS-ALGEBRA

BEZA LAMESGIN DERSEH*, BERHANU ASSAYE ALABA, AND YOHANNES GEDAMU WONDIFRAW

ABSTRACT. In this paper, we introduce the notion of intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra is examined, and some interesting results are obtained.

1. Introduction


In this paper, we introduced the notion of intuitionistic fuzzy PMS-subalgebras of PMS-algebras and investigate some of their properties. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra is examined, and some interesting results are obtained.
2. Preliminaries

In this section, we recall some basic definitions and results that are used in the study of this paper.

**Definition 2.1.** [12] A nonempty set $X$ with a constant $0$ and a binary operation '$*$' is called PMS-algebra if it satisfies the following axioms.

1. $0 * x = x$
2. $(y * x) * (z * x) = z * y$, for all $x, y, z \in X$.

In $X$, we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

**Definition 2.2.** [12] Let $S$ be a nonempty subset of a PMS-algebra $X$, then $S$ is called a PMS-subalgebra of $X$ if $x * y \in S$, for all $x, y \in S$.

**Example 2.3.** [12] Let $Z$ be the set of all integers, and let $*$ be a binary relation on $Z$ defined by $x * y = y - x$, for all $x, y \in Z$, where '$-$' the usual subtraction of integers. Then $(Z, *, 0)$ is a PMS-algebra since

1. $0 * x = x - 0 = x$
2. $(y * x) * (z * x) = (z * x) - (y * x) = (x - z) - (x - y) = y - z = z * y$.

Clearly, the set $E$ of all even integers is a PMS-subalgebra of a PMS-algebra $Z$, since $x * y = y - x \in E$ for all $x, y \in E$.

**Proposition 2.4.** [12] In any PMS-algebra $(X, *, 0)$ the following properties hold for all $x, y, z \in X$.

1. $x * x = 0$
2. $(y * x) * x = y$
3. $x * (y * x) = y * 0$
4. $(y * x) * z = (z * x) * y$
5. $(x * y) * 0 = y * x = (0 * y) * (0 * x)$

**Definition 2.5.** [15] Let $X$ be a nonempty set. A fuzzy subset $A$ of the set $X$ is defined as $A = \{(x, \mu_A(x)) | x \in X\}$ where the mapping $\mu_A : X \rightarrow [0, 1]$ defines the degree of membership.

**Definition 2.6.** [11] A fuzzy set $A$ in a PMS-algebra $X$ is called fuzzy PMS-subalgebra of $X$ if $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

**Definition 2.7.** [2, 4] An intuitionistic fuzzy set (IFS) $A$ in a nonempty set $X$ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$, where the functions $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and the degree of non membership, respectively, satisfying the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

**Remark 2.8.** Ordinary fuzzy sets over $X$ may be viewed as special intuitionistic fuzzy sets with the non membership function $\nu_A(x) = 1 - \mu_A(x)$. So each Ordinary fuzzy set may be written as $\{(x, \mu_A(x), 1 - \mu_A(x)) | x \in X\}$ to define an intuitionistic fuzzy set. For the sake of simplicity we write $A = (\mu_A, \nu_A)$ for an intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$.

**Definition 2.9.** [2–4] Let $A$ and $B$ be two intuitionistic fuzzy subsets of the set $X$, where $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$, then
Let \( A \times B = \{ (x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))) \mid x \in X \} \)

2. \( A \cup B = \{ (x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))) \mid x \in X \} \)

3. \( \bar{A} = \{ (x, \nu_A(x), \mu_A(x)) \mid x \in X \} \)

4. \( \Box A = \{ (x, \mu_A(x), 1 - \mu_A(x)) \mid x \in X \} \)

5. \( \Diamond A = \{ (x, 1 - \nu_A(x), \nu_A(x)) \mid x \in X \} \)

3. Intuitionistic Fuzzy PMS-subalgebra

In this section we introduce the notion of intuitionistic fuzzy PMS-subalgebra and investigated some of its properties. Throughout this and the next section \( X \) denotes a PMS-algebra, unless otherwise specified.

**DEFINITION 3.1.** An intuitionistic fuzzy subset \( A = (\mu_A, \nu_A) \) of a PMS-algebra \( X \) is called an intuitionistic fuzzy PMS-subalgebra of \( X \) if

1. \( \mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\} \) and
2. \( \nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\} \) for all \( x, y \in X \)

**EXAMPLE 3.2.** Let \( X = \{01, 1, 2, 3\} \) be a set with the following table.

\[
\begin{array}{c|cccc}
  * & 0 & 1 & 2 & 3 \\
  \\
 0 & 0 & 1 & 2 & 3 \\
1 & 2 & 0 & 1 & 3 \\
2 & 1 & 2 & 0 & 1 \\
3 & 3 & 1 & 2 & 0 \\
\end{array}
\]

Then \((X, \ast, 0)\) is a PMS-algebra and \( S = \{0, 1, 2\} \) is a PMS-subalgebra \( X \).

Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy set in \( X \) defined by

\[
\mu_A(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0.5 & \text{if } x = 1, 2 \\
0 & \text{if } x = 3 
\end{cases}
\]

and

\[
\nu_A(x) = \begin{cases} 
0 & \text{if } x = 0 \\
0.4 & \text{if } x = 1, 2 \\
1 & \text{if } x = 3 
\end{cases}
\]

For intuitionistic fuzzy set \( A \) in a PMS-algebra \( X \) with membership values \( \mu_A(x) \) and non membership values \( \nu_A(x) \) as defined above, definition 3.1 is satisfied. Therefore \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of the PMS-algebra \( X \).

**LEMMA 3.3.** If \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \), then \( \mu_A(0) \geq \mu_A(x) \) and \( \nu_A(0) \leq \nu_A(x) \) for all \( x \in X \)

**Proof.** Suppose \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \). Since \( x \ast x = 0 \) for every \( x \in X \) by proposition 2.1(1), we have

\[
\mu_A(0) = \mu_A(x \ast x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)
\]

and

\[
\nu_A(0) = \nu_A(x \ast x) \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)
\]

Hence \( \mu_A(0) \geq \mu_A(x) \) and \( \nu_A(0) \leq \nu_A(x) \) for all \( x \in X \).

**LEMMA 3.4.** Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy PMS-subalgebra of \( X \), if \( x \ast y \leq z \), then \( \mu_A(x) \geq \min\{1, \mu_A(y), \mu_A(z)\} \) and \( \nu_A(x) \leq \max\{\nu_A(y), \nu_A(z)\} \).

**Proof.** Suppose \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy PMS-subalgebra of \( X \). Let \( x, y, z \in X \) such that \( x \ast y \leq z \). Then by the binary relation \( \leq \) defined in \( X \), we have

\[
\mu_A(0) = \mu_A(x \ast x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)
\]

and

\[
\nu_A(0) = \nu_A(x \ast x) \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)
\]

Hence \( \mu_A(0) \geq \mu_A(x) \) and \( \nu_A(0) \leq \nu_A(x) \) for all \( x \in X \).
\((x \ast y) \ast z = 0\). Thus by definition 2.1 and proposition 2.4 (4), we have
\[
\mu_A(x) = \mu_A(0 \ast x) = \mu_A(((x \ast y) \ast x) \ast x) \\
= \mu_A(((x \ast y) \ast x) \ast x) \\
= \mu_A((x \ast y) \ast (z \ast y)) \\
= \mu_A(0 \ast (z \ast y)) \\
= \mu_A(z \ast y) \geq \min\{\mu_A(z), \mu_A(y)\}
\]
Hence \(\mu_A(x) \geq \min\{\mu_A(z), \mu_A(y)\}\)

Similarly, \(\nu_A(x) \leq \max\{\nu_A(z), \nu_A(y)\}\)

**Theorem 3.5.** Let \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \(X\). If \(x \in X\) and let \(x, y \in X\) such that \(\mu_A(x \ast y) = \mu_A(y)\) and \(\nu_A(x \ast y) = \nu_A(y)\) for each \(y \in X\), then \(\mu_A(x) = \mu_A(0)\) and \(\nu_A(x) = \nu_A(0)\), where 0 is a constant in \(X\).

**Proof.** Suppose \(\mu_A(x \ast y) = \mu_A(y)\) and \(\nu_A(x \ast y) = \nu_A(y)\) for each \(y \in X\). Then we need to show that \(\mu_A(x) = \mu_A(0)\) and \(\nu_A(x) = \nu_A(0)\), where 0 is a constant in \(X\). By lemma 3.3, \(\mu_A(0) \geq \mu_A(x)\) and \(\nu_A(0) \leq \nu_A(x)\) for each \(x \in X\). By proposition 2.4 (2) \((x \ast 0) \ast 0 = x\). Then \(\mu_A(x) = \mu_A((x \ast 0) \ast 0) \geq \min\{\mu_A(x \ast 0), \mu_A(0)\}\) = \(\mu_A(0)\).

Also, \(\nu_A(x) = \nu_A((x \ast 0) \ast 0) \leq \max\{\nu_A(x \ast 0), \nu_A(0)\}\) = \(\nu_A(0)\).

Hence \(\mu_A(x) \geq \mu_A(0)\) and \(\nu_A(x) \leq \nu_A(0)\).

Therefore \(\mu_A(x) = \mu_A(0)\) and \(\nu_A(x) = \nu_A(0)\).

Conversely, Suppose \(\mu_A(x) = \mu_A(0)\) and \(\nu_A(x) = \nu_A(0)\). Then we need to prove that \(\mu_A(x \ast y) = \mu_A(y)\) and \(\nu_A(x \ast y) = \nu_A(y)\), for each \(y \in X\).

By lemma 3.3 \(\mu_A(x) \geq \mu_A(y)\) and \(\nu_A(x) \leq \nu_A(y)\) for each \(y \in X\). Since \(A\) is an intuitionistic fuzzy PMS-subalgebra of \(X\), then \(\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}\) = \(\mu_A(y)\) and \(\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}\) = \(\nu_A(y)\). Thus \(\mu_A(x \ast y) \geq \mu_A(y)\) and \(\nu_A(x \ast y) \leq \nu_A(y)\) for each \(y \in X\).

But, using Proposition 2.4 (2) and 2.4 (5) it follows that
\[
\mu_A(y) = \mu_A((x \ast y) \ast x) \geq \min\{\mu_A(x \ast y), \mu_A(x)\}
\]
\[
= \min\{\mu_A((x \ast y) \ast 0), \mu_A(x)\}
\]
\[
\geq \min\{\min\{\mu_A(x \ast y), \mu_A(0)\}, \mu_A(x)\}
\]
\[
= \min\{\mu_A(x \ast y), \mu_A(x)\} = \mu_A(x \ast y)
\]
and
\[
\nu_A(y) = \nu_A((x \ast y) \ast x) \leq \max\{\nu_A(x \ast y), \nu_A(x)\}
\]
\[
= \max\{\nu_A((x \ast y) \ast 0), \nu_A(x)\}
\]
\[
\leq \max\{\max\{\nu_A(x \ast y), \nu_A(0)\}, \nu_A(x)\}
\]
\[
= \max\{\nu_A(x \ast y), \nu_A(x)\} = \nu_A(x \ast y)
\]

Hence \(\mu_A(x \ast y) = \mu_A(y)\) and \(\nu_A(x \ast y) = \nu_A(y)\) for each \(y \in X\). □

**Theorem 3.6.** Let \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \(X\). If \(\mu_A(x \ast y) = \mu_A(0)\) and \(\nu_A(x \ast y) = \nu_A(0)\) for all \(x, y \in X\), then \(\mu_A(x) = \mu_A(y)\) and \(\nu_A(x) = \nu_A(y)\).
Proof. Let \( x, y \in X \) such that \( \mu_A(x * y) = \mu_A(0) \) and \( \nu_A(x * y) = \nu_A(0) \).

Claim \( \mu_A(x) = \mu_A(y) \) and \( \nu_A(x) = \nu_A(y) \)

Now, \( \mu_A(x) = \mu_A((y * y) * x) \)
\[ = \mu_A((x * y) * y) \]
\[ \geq \min\{\mu_A(x * y), \mu_A(y)\} \]
\[ = \min\{\mu_A(0), \mu_A(y)\} = \mu_A(y) \]

Conversely, \( \mu_A(y) = \mu_A((x * x) * y) \)
\[ = \mu_A((y * x) * x) \]
\[ \geq \min\{\mu_A(y * x), \mu_A(y)\} \]
\[ = \min\{\mu_A((x * y) * 0), \mu_A(y)\} \]
\[ \geq \min\{\min\{\mu_A(x * y), \mu_A(0)\}, \mu_A(x)\} \]
\[ = \min\{\mu_A(0), \mu_A(x)\} = \mu_A(x) \]

Thus \( \mu_A(x) = \mu_A(y) \)

By similar argument we have \( \nu_A(x) = \nu_A(y) \) \( \square \)

**Theorem 3.7.** The intersection of any two intuitionistic fuzzy PMS-subalgebras of \( X \) is also an intuitionistic fuzzy PMS-subalgebra of \( X \).

Proof. Let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) be any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra \( X \).

Claim: \( A \cap B \) is an intuitionistic fuzzy PMS-subalgebra of \( X \). Then for \( x, y \in X \), we have

\[
\mu_{A \cap B}(x * y) = \min\{\mu_A(x * y), \mu_B(x * y)\} \\
\geq \min\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}\} \\
= \min\{\min\{\mu_A(x), \mu_A(x)\}, \min\{\mu_A(y), \mu_B(y)\}\} \\
= \min\{\mu_A(x), \mu_B(x)\}
\]

and

\[
\nu_{A \cap B}(x * y) = \max\{\nu_A(x * y), \nu_B(x * y)\} \\
\leq \max\{\max\{\nu_A(x), \nu_A(y)\}, \max\{\nu_B(x), \nu_B(y)\}\} \\
= \max\{\max\{\nu_A(x), \nu_B(x)\}, \max\{\nu_A(y), \nu_B(y)\}\} \\
= \max\{\nu_A(x), \nu_B(x)\}
\]

Hence \( A \cap B \) is an intuitionistic fuzzy PMS-subalgebra of \( X \) \( \square \)

The above theorem proves that the intersection of any two intuitionistic fuzzy PMS-subalgebras of \( X \) is again an intuitionistic fuzzy subalgebra of \( X \). It can also be generalized to any family of intuitionistic fuzzy PMS-subalgebra of \( X \) as follows:

**Corollary 3.8.** If \( \{A_i : i \in I\} \) be a family of intuitionistic fuzzy PMS-subalgebra of \( X \), then \( \bigcap_{i \in I} A_i \) is also an intuitionistic fuzzy PMS-subalgebra of \( X \), where \( \cap_{i \in I} \mu_{A_i}(x) = \inf_{i \in I} \mu_{A_i}(x) \) and \( \cap_{i \in I} \nu_{A_i}(x) = \sup_{i \in I} \mu_{A_i}(x) \)

**Remark 3.9.** The union of any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra \( X \) is not necessarily an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \).
Example 3.10. Let $X = \{0, 1, 2, 3\}$ be a set with the table as in example 3.2 and $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set in $X$ as defined in example 3.2. Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy set in $X$ defined by

$$
\mu_B(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0.6 & \text{if } x = 1, 3 \\
0 & \text{if } x = 2
\end{cases}
$$

and

$$
\nu_B(x) = \begin{cases} 
0.2 & \text{if } x = 1, 3 \\
1 & \text{if } x = 2
\end{cases}
$$

Now, $\mu_{A*B} = \mu_{A*B}(2) = \max\{\mu_A(2), \mu_B(2)\} = \max\{0.5, 0\} = 0.5$ (i)

$\min\{\mu_{A*B}(1), \mu_{A*B}(0)\} = \min\{\max\{\mu_A(1), \mu_B(1)\}, \max\{\mu_A(0), \mu_B(0)\}\}$

$= \min\{\max\{0.5, 0.6\}, \max\{1, 1\}\}$

$= \min\{0.6, 1\} = 0.6$ (ii)

and

$\nu_{A*B}(1*0) = \nu_{A*B}(2) = \min\{\nu_A(2), \nu_B(2)\} = \min\{0.4, 1\} = 0.4$ (iii)

$\max\{\nu_{A*B}(1), \nu_{A*B}(0)\} = \max\{\min\{\mu_A(1), \nu_B(1)\}, \min\{\nu_A(0), \nu_B(0)\}\}$

$= \max\{\min\{0.4, 0.2\}, \min\{0, 0\}\}$

$= \max\{0.2, 0\} = 0.2$ (iv)

From (i) and (ii) we see that $\mu_{A*B}(1*0) = 0.5 < 0.6 = \min\{\mu_{A*B}(1), \mu_{A*B}(0)\}$ and from (iii) and (iv) we see that $\nu_{A*B}(1*0) = 0.4 > 0.2 = \max\{\nu_{A*B}(1), \nu_{A*B}(0)\}$ which is a contradiction. This shows that the union of any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra $X$ may not be an intuitionistic fuzzy PMS-subalgebra.

Lemma 3.11. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in $X$. Then the following statements hold for any $x, y \in X$.

1. $1 - \max\{\mu_A(x), \mu_A(y)\} = \min\{1 - \mu_A(x), 1 - \mu_A(y)\}$
2. $1 - \min\{\mu_A(x), \mu_A(y)\} = \max\{1 - \mu_A(x), 1 - \mu_A(y)\}$
3. $1 - \max\{\nu_A(x), \nu_A(y)\} = \min\{1 - \nu_A(x), 1 - \nu_A(y)\}$
4. $1 - \min\{\nu_A(x), \nu_A(y)\} = \max\{1 - \nu_A(x), 1 - \nu_A(y)\}$

Now, we can prove the next two theorems using the above Lemma.

Theorem 3.12. An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra $X$ is an intuitionistic fuzzy PMS-subalgebra of $X$ if and only if the fuzzy subsets $\mu_A$ and $\nu_A$ are fuzzy subalgebras of $X$.

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$.

Claim: The fuzzy subsets $\mu_A$ and $\nu_A$ of $X$ are fuzzy subalgebras of $X$. Clearly, $\mu_A$ is a fuzzy PMS-subalgebra of $X$ directly follows from the fact that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. Now for all $x, y \in X$, $\nu_A(x*y) = 1 - \nu_A(x*y) \geq 1 - \max\{\nu_A(x), \nu_A(y)\}$

$$= \min\{1 - \nu_A(x), 1 - \nu_A(y)\} \quad \text{(By Lemma 3.11(3))}$$

Therefore $\nu_A$ is a fuzzy PMS-subalgebra of $X$.

Conversely, Suppose $\mu_A$ and $\nu_A$ are fuzzy PMS-subalgebras of $X$. So, we need to show that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. Since $\mu_A$ and $\nu_A$ are fuzzy PMS-subalgebras of $X$, we have that $\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x*y) \geq \min\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in X$. Now it suffices to show that
\[\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}\] for all \(x, y \in X\).

\[
1 - \nu_A(x \ast y) = \nu_A(x \ast y) \geq \min\{\nu_A(x), \nu_A(y)\} \\
= \min\{1 - \nu_A(x), 1 - \nu_A(y)\} \\
= 1 - \max\{\nu_A(x), \nu_A(y)\} \quad \text{(By Lemma 3.11(3))}
\]

\[\Rightarrow \nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}, \text{ for all } x, y \in X.
\]

Hence \(A = (\mu_A, \nu_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\). \(\Box\)

**Corollary 3.13.** If \(\mu_A\) is a fuzzy PMS-subalgebra of \(X\), then \(A = (\mu_A, \bar{\mu}_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\).

**Proof.** Suppose \(\mu_A\) is a fuzzy PMS-subalgebra of \(X\). Then we want to show that \(A = (\mu_A, \bar{\mu}_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\). Since \(\mu_A\) is a fuzzy PMS-subalgebra of \(X\), it follows that \(\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}\). Then it suffices to show that \(\bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}\).

\[\bar{\mu}_A(x \ast y) = 1 - \mu_A(x \ast y) \leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\
= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\
= \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}
\]

\[\Rightarrow \bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}
\]

Hence \(A = (\mu_A, \bar{\mu}_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\). \(\Box\)

**Corollary 3.14.** If \(\bar{\nu}_A\) is a fuzzy PMS-subalgebra of \(X\), then \(A = (\bar{\nu}_A, \nu_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\).

**Proof.** Similar to corollary 3.13 \(\Box\)

**Theorem 3.15.** An intuitionistic fuzzy subset \(A = (\mu_A, \nu_A)\) of \(X\) is an intuitionistic fuzzy PMS-subalgebra of \(X\) if and only if \(\Box A = (\mu_A, \bar{\mu}_A)\) and \(\Diamond A = (\bar{\nu}_A, \nu_A)\) are intuitionistic fuzzy PMS-subalgebras of \(X\).

**Proof.** Assume that an intuitionistic fuzzy subset \(A = (\mu_A, \nu_A)\) of \(X\) is an intuitionistic fuzzy PMS-subalgebra of \(X\), then \(\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}\) and \(\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}\).

Claim: \(\Box A = (\mu_A, \bar{\mu}_A)\) and \(\Diamond A = (\bar{\nu}_A, \nu_A)\) are intuitionistic fuzzy PMS-subalgebras of \(X\).

(i) To show that \(\Box A\) is an intuitionistic fuzzy PMS-subalgebra of \(X\), it suffices to show that \(\bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}\), for all \(x, y \in X\). Let \(x, y \in X\), then

\[
\bar{\mu}_A(x \ast y) = 1 - \mu_A(x \ast y) \leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\
= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\
= \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}
\]

\[\Rightarrow \bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}, \forall x, y \in X.
\]

Hence \(\Box A\) is an intuitionistic fuzzy PMS-subalgebra of \(X\).

(ii) To show that \(\Diamond A\) is an intuitionistic fuzzy PMS-subalgebra of \(X\), it suffices to show that \(\bar{\nu}_A(x \ast y) \geq \min\{\bar{\nu}_A(x), \bar{\nu}_A(y)\}\), for all \(x, y \in X\). Let \(x, y \in X\), then

\[
\bar{\nu}_A(x \ast y) = 1 - \nu_A(x \ast y) \geq 1 - \max\{\bar{\nu}_A(x), \bar{\nu}_A(y)\} \\
= \min\{1 - \nu_A(x), 1 - \nu_A(y)\} \\
= \min\{\bar{\nu}_A(x), \bar{\nu}_A(y)\}
\]

\[\Rightarrow \bar{\nu}_A(x \ast y) \geq \min\{\bar{\nu}_A(x), \bar{\nu}_A(y)\}, \forall x, y \in X.
\]
Hence \(\Diamond A\) is an intuitionistic fuzzy PMS-subalgebra of \(X\).

The proof of the converse of this theorem is trivial. \(\Box\)

4. Level Subsets of Intuitionistic Fuzzy PMS-subalgebras

In this section, the idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. Characterizations of level subsets of a fuzzy PMS-subalgebra of a PMS-algebra are given.

**Theorem 4.1.** If \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy PMS-subalgebra of \(X\), then the sets \(X_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(0)\}\) and \(X_{\nu_A} = \{x \in X | \nu_A(x) = \nu_A(0)\}\) are PMS-subalgebra of \(X\).

**Proof.** Suppose \(A = (\mu_A, \nu_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\) and let \(x, y \in X_{\mu_A}\). Then \(\mu_A(x) = \mu_A(0) = \mu_A(y)\). So, \(\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}\) \(= \min\{\mu_A(0), \mu_A(0)\} = \mu_A(0)\). \(\Rightarrow \mu_A(x * y) \geq \mu_A(0)\). By Lemma 3.3, we get that \(\mu_A(x * y) = \mu_A(0)\) which imply that \(x * y \in X_{\mu_A}\). Also, \(\nu_A(x) = \nu_A(0) = \nu_A(y)\) and so \(\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} = \max\{\nu_A(0), \nu_A(0)\} = \nu_A(0)\). \(\Rightarrow \nu_A(x * y) \leq \nu_A(0)\). By Lemma 3.3, we get that \(\nu_A(x * y) = \nu_A(0)\) which imply that \(x * y \in X_{\nu_A}\).

Hence, the sets \(X_{\mu_A}\) and \(X_{\nu_A}\) are PMS-subalgebras of \(X\). \(\Box\)

**Theorem 4.2.** Let \(S\) be a nonempty subset of a PMS-algebra \(X\) and \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy set in \(X\) defined by

\[
\mu_A(x) = \begin{cases} p & \text{if } x \in S \\ q & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} r & \text{if } x \in S \\ s & \text{if } x \notin S \end{cases}
\]

for all \(p, q, r, s \in [0, 1]\) with \(p \geq q, r \leq s\) and \(0 \leq p + r \leq 1, 0 \leq q + s \leq 1\). Then \(A\) is an intuitionistic fuzzy PMS-subalgebra of \(X\) if and only if \(S\) is a PMS-subalgebra of \(X\). Furthermore, in this situation, \(X_{\mu_A} = S = X_{\nu_A}\).

**Proof.** Let \(A\) be an intuitionistic fuzzy PMS-subalgebra of \(X\). Then we want to show that \(S\) is a PMS-subalgebra of \(X\). Let \(x, y \in X\) such that \(x, y \in S\).

Since \(A = (\mu_A, \nu_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\), we have

\[
\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = \min\{p, q\} \quad \text{and} \quad \nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} = \max\{r, s\} = r.
\]

Hence \(x * y \in S\). So, \(S\) is a PMS-subalgebra of \(X\).

Conversely, suppose that \(S\) is a PMS-subalgebra of \(X\). We claim to show that \(A = (\mu_A, \nu_A)\) is an intuitionistic fuzzy PMS-subalgebra of \(X\).

Let \(x, y \in X\). Now consider the following cases:

- **Case (i).** If \(x, y \in S\), then \(x * y \in S\), since \(S\) is a PMS-subalgebra of \(X\). Thus, \(\mu_A(x * y) = p = \min\{\mu_A(x), \mu_A(y)\}\) and \(\nu_A(x * y) = r = \max\{\nu_A(x), \nu_A(y)\}\).

- **Case (ii).** If \(x \in S, y \notin S\), then \(\mu_A(x) = p, \mu_A(y) = q\) and \(\nu_A(x) = r, \nu_A(y) = s\). Thus, \(\mu_A(x * y) \geq q\) \(= \min\{p, q\} = \min\{\mu_A(x), \mu_A(y)\}\) implies \(\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}\) and \(\nu_A(x * y) \leq s = \max\{r, s\} = \max\{\nu_A(x), \nu_A(y)\}\) implies \(\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}\).

- **Case (iii).** If \(x \notin S, y \in S\), then interchanging the roles of \(x\) and \(y\) in Case (ii), yields similar results \(\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}\) and \(\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}\).
case (iv). If \( x, y \notin S \), then \( \mu_A(x) = q = \mu_A(y) \) and \( \nu_A(x) = s = \nu_A \), this implies that
\[
\mu_A(x \ast y) \geq q = \min \{ \mu_A(x), \mu_A(y) \} \quad \text{and} \quad \nu_A(x \ast y) \leq s = \max \{ \nu_A(x), \nu_A(y) \}
\]
Hence \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \).

Furthermore, we have
\[
X_{\mu_A} = \{ x \in X | \mu_A(x) = \mu_A(0) \} = \{ x \in X | \mu_A(x) = p \} = S \quad \text{and} \quad X_{\nu_A} = \{ x \in X | \nu_A(x) = \nu_A(0) \} = \{ x \in X | \nu_A(x) = r \} = S.
\]
Hence \( X_{\mu_A} = S = X_{\nu_A} \).

**Definition 4.3.** Let \( A = (\mu_A, \nu_A) \) be any intuitionistic fuzzy subset of a PMS-algebra \( X \) such that \( t, s \in [0, 1] \), then the set \( U(\mu_A, t) = \{ x \in X: \mu_A(x) \geq t \} \) is called an upper \( t \)-level set of an intuitionistic fuzzy subset \( A \) of \( X \) and the set \( L(\mu_A, s) = \{ x \in X: \nu_A(x) \leq s \} \) is called a lower \( s \)-level set of an intuitionistic fuzzy subset \( A \) of \( X \).

**Theorem 4.4.** An intuitionistic fuzzy subset \( A = (\mu_A, \nu_A) \) of a PMS-algebra \( X \) is an intuitionistic fuzzy PMS-subalgebra of \( X \) if and only if the nonempty level subsets \( U(\mu_A, t) \) and \( L(\nu_A, s) \) of \( A \) are PMS-subalgebras of \( X \) for all \( t, s \in [0, 1] \) with \( 0 \leq t + s \leq 1 \).

**Proof.** Assume that \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \) such that \( U(\mu_A, t) \neq \emptyset \) and \( L(\nu_A, s) \neq \emptyset \). Now we claim that \( U(\mu_A, t) \) and \( L(\nu_A, s) \) are PMS-subalgebras of \( X \) for all \( t, s \in [0, 1] \) with \( 0 \leq t + s \leq 1 \). Let \( x, y \in U(\mu_A, t) \), then we have \( \mu_A(x) \geq t \) and \( \mu_A(y) \geq t \). Thus \( \mu_A(x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \} \geq \min \{ t, t \} = t \)
\[
\Rightarrow \ x \ast y \in U(\mu_A, t)
\]
Hence \( U(\mu_A, t) \) is a PMS-subalgebra of \( X \).

Also, let \( x, y \in L(\nu_A, s) \), then \( \nu_A(x) \leq s \) and \( \nu_A(y) \leq s \)
\[
\therefore \nu_A(x \ast y) \leq \max \{ \nu_A(x), \nu_A(y) \} \leq \max \{ t, t \} = t \Rightarrow x \ast y \in L(\nu_A, s)
\]
Hence \( L(\nu_A, s) \) is a PMS-subalgebra of \( X \).

Conversely, Suppose that \( U(\mu_A, t) \) and \( L(\nu_A, s) \) are PMS-subalgebras of \( X \) for all \( t, s \in [0, 1] \) with \( 0 \leq t + s \leq 1 \).

Claim: \( A \) is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \).

Let \( x, y \in X \) such that \( \mu_A(x) = t_1 \) and \( \mu_A(y) = t_2 \) for \( t_1, t_2 \in [0, 1] \). Then \( x \in U(\mu_A, t_1) \) and \( y \in U(\mu_A, t_2) \).

Choose \( t = \min \{ t_1, t_2 \} \), then \( t \leq t_1 \) and \( t \leq t_2 \)
\[
\Rightarrow U(\mu_A, t_1) \subseteq U(\mu_A, t) \quad \text{and} \quad U(\mu_A, t_2) \subseteq U(\mu_A, t).
\]
\[
\Rightarrow x, y \in U(\mu_A, t),
\]
Since \( U(\mu_A, t) \) is a PMS-Subalgebra of \( X \), it follows that \( x \ast y \in U(\mu_A, t) \).
Thus \( \mu_A(x \ast y) \geq t = \min \{ t_1, t_2 \} = \min \{ \mu_A(x), \mu_A(y) \} \).
Hence \( \mu_A(x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \} \) for all \( x, y \in X \).

And also, let \( x, y \in X \) such that \( \nu_A(x) = s_1 \) and \( \nu_A(y) = s_2 \) for \( s_1, s_2 \in [0, 1] \).

Then \( x \in L(\nu_A, s_1) \) and \( y \in L(\nu_A, s_2) \).

Choose \( s = \max \{ s_1, s_2 \} \), then \( s_1 \leq s \) and \( s_2 \leq s \),
\[
\Rightarrow L(\nu_A, s_1) \subseteq L(\nu_A, s) \quad \text{and} \quad L(\nu_A, s_2) \subseteq L(\nu_A, s).
\]
\[
\Rightarrow x, y \in L(\nu_A, s),
\]
Since \( L(\nu_A, s) \) is a PMS-subalgebra of \( X \), it follows that \( x \ast y \in L(\nu_A, s) \).
Thus \( \nu_A(x \ast y) \leq s = \max \{ s_1, s_2 \} = \max \{ \nu_A(x), \nu_A(y) \} \).
Hence \( \nu_A(x \ast y) \leq \max \{ \nu_A(x), \nu_A(y) \} \) for all \( x, y \in X \).

Hence \( A \) is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \).
Remarque 4.5. Les sub-algebres PMS des $U(\mu_A, t)$ et $L(\nu_A, s)$ de $X$ pour tout $t, s \in [0,1]$ obtenues dans le théorème précédent sont appelées des sous-algebres PMS de $X$.

Corollaire 4.6. Un ensemble flou intuitionniste $A = (\mu_A, \nu_A)$ d’une PMS-algèbre $X$ est un ensemble flou PMS-sous-algèbre de $X$ si et seulement si les sous-ensembles flous $U(\mu_A, t)$ et $L(\nu_A, s)$ de $A$ sont des sous-algebres PMS de $X$ pour tout $t \in Im(\mu_A)$ et $s \in Im(\nu_A)$ avec $0 \leq t + s \leq 1$.

Theorem 4.7. Si $S$ est un ensemble de $X$ et $A = (\mu_A, \nu_A)$ est un ensemble flou intuitionniste dans $X$ défini par

$$\mu_A(x) = \begin{cases} t & \text{si } x \in S \\ 0 & \text{si } x \notin S \end{cases} \quad \text{et} \quad \nu_A(x) = \begin{cases} s & \text{si } x \in S \\ 1 & \text{si } x \notin S \end{cases}$$

pour tout $t, s \in [0,1]$ tel que $0 \leq t + s \leq 1$. Si $A = (\mu_A, \nu_A)$ est un ensemble flou PMS-sous-algèbre de $X$, alors $S$ est un ensemble flou PMS-sous-algèbre de $X$.


démonstration. Soit $A = (\mu_A, \nu_A)$ être un ensemble flou PMS-sous-algèbre de $X$. On montrera que $S$ est un ensemble flou PMS-sous-algèbre de $X$. Soit $x, y \in S$, alors $\mu_A(x) = t = \mu_A(y)$ et $\nu_A(x) = s = \nu_A(y)$. Donc, $\mu_A(x * y) = \min\{\mu_A(x), \mu_A(y)\} = \min\{t, t\} = t$ et $\nu_A(x * y) = \max\{\mu_A(x), \nu_A(y)\} = \max\{s, s\} = s$ qui implique que $x * y \in S$. Par conséquent, $S$ est un ensemble flou PMS-sous-algèbre de $X$. En outre, par le théorème 4.4, $U(\mu_A, t)$ est un ensemble flou PMS-sous-algèbre de $X$, et

$$U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\} = S = \{x \in X : \nu_A(x) \leq s\}.$$

Ainsi, $S$ est un ensemble flou PMS-sous-algèbre de $X$ correspondant à l’ensemble flou PMS-sous-algèbre $A = (\mu_A, \nu_A)$ de $X$.


démonstration. Soit $S$ un ensemble flou PMS-sous-algèbre de $X$ et $A = (\mu_A, \nu_A)$ soit un ensemble flou intuitionniste dans $X$ défini par

$$\mu_A(x) = \begin{cases} t & \text{si } x \in S \\ 0 & \text{si } x \notin S \end{cases} \quad \text{et} \quad \nu_A(x) = \begin{cases} s & \text{si } x \in S \\ 1 & \text{si } x \notin S \end{cases}$$

pour tout $t, s \in [0,1]$ tels que $0 \leq t + s \leq 1$.

Clairement, $U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\} = S$. Soit $x, y \in X$. Pour montrer que $A = (\mu_A, \nu_A)$ est un ensemble flou PMS-sous-algèbre de $X$, nous considérons les cas suivants:

- **cas(i)**. Si $x, y \in S$, alors $x * y \in S$. Puisque $S$ est un ensemble flou PMS-sous-algèbre de $X$, $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = t$ et $\nu_A(x) = \nu_A(y) = \nu_A(x * y) = s$. Par conséquent, $\mu_A(x * y) = \min\{\mu_A(x), \mu_A(y)\}$ et $\nu_A(x * y) = \max\{\nu_A(x), \nu_A(y)\}$

- **cas(ii)**. Si $x \in S, y \notin S$, alors nous avons $\mu_A(x) = t, \mu_A(y) = 0$ et $\nu_A(x) = s, \nu_A(y) = 1$. Par conséquent, $\mu_A(x * y) \geq 0 = \min\{t, 0\} = \min\{\mu_A(x), \mu_A(y)\}$ et $\nu_A(x * y) \leq 1 = \max\{s, 1\} = \max\{\nu_A(x), \nu_A(y)\}$ implique $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

- **cas(iii)**. Si $x \notin S, y \in S$, alors interchanger le rôle de $x$ et $y$ dans le cas (ii), donne des résultats similaires $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ et $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

- **cas(iv)**. Si $x, y \notin S$ alors $\mu_A(x) = 0 = \mu_A(y)$ et $\nu_A(x) = 1 = \nu_A(y)$. Par conséquent, $\mu_A(x * y) \geq 0 = \min\{\mu_A(x), \mu_A(y)\}$ et $\nu_A(x * y) \leq 1 = \max\{\nu_A(x), \nu_A(y)\}$.
So, in all cases we get $\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x*y) \leq \max\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in X$.

Thus, $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$. \hfill \Box

We can also prove the following theorem as a generalization of theorem 4.8.

**Theorem 4.9.** Let $\{S_i\}$ be any family of a PMS-subalgebra of a PMS-algebra $X$ such that $S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_n = X$, then there exists an intuitionistic fuzzy PMS-subalgebra $A = (\mu_A, \nu_A)$ of $X$ whose level PMS-subalgebras are exactly the PMS-subalgebras $\{S_i\}$.

**Proof.** Suppose $t_0 > t_1 > t_2 > \ldots > t_n$ and $s_0 < s_1 < s_2 \ldots < s_n$ where each $t_i, s_i \in [0, 1]$ with $0 \leq t_i + s_i \leq 1$. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set defined by

$$
\mu_A(x) = \begin{cases} 
t_0 & \text{if } x \in S_0 \\
t_i & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n.
\end{cases}
\quad \text{and} \quad
\nu_A(x) = \begin{cases} 
s_0 & \text{if } x \in S_0 \\
s_i & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n.
\end{cases}
$$

Now, We claim that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$ and $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 \leq i \leq n$.

Let $x, y \in X$ Then, we consider the following two cases

**Case (i):** Let $x, y \in S_i - S_{i-1}$. Therefore by the definition of $A = (\mu_A, \nu_A)$, we have $\mu_A(x) = t_i = \mu_A(y)$ and $\nu_A(x) = s_i = \nu_A(y)$. Since $S_i$ is a PMS-subalgebra of $X$, it follows that $x*y \in S_i$, and so either $x*y \in S_i - S_{i-1}$ or $x*y \in S_{i-1}$ or $x*y \in S_{i-1} - S_{i-2}$.

$\Rightarrow \mu_A(x) = t_i$ or $\mu_A(x) = t_{i-1} > t_i$ and $\nu_A(x) = s_i$ or $\nu_A(x) = s_{i-1} > s_i$.

In any case we conclude that $\mu_A(x*y) \geq t_i = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x*y) \leq s_i = \max\{\nu_A(x), \nu_A(y)\}$.

**Case (ii):** For $i > j$, $t_j > t_i$, $s_j < s_i$ and $S_j \subseteq S_i$. Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$. Then, $\mu_A(x) = t_i, \mu_A(y) = t_j > t_i, \nu_A(x) = s_i$ and $\nu_A(y) = s_j < s_i$. Then $x*y \in S_i$ since $S_i$ is a PMS-subalgebra of $X$ and $S_j \subseteq S_i$.

Hence $\mu_A(x*y) \geq t_i = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x*y) \leq s_i = \max\{\nu_A(x), \nu_A(y)\}$ by case (i). Thus $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$.

Also, from the definition of $A = (\mu_A, \nu_A)$, it follows that $Im(\mu_A) = \{t_0, t_1, \ldots, t_n\}$ and $Im(\nu_A) = \{s_0, s_1, \ldots, s_n\}$. So, $U(\mu_A, t_i)$ and $L(\nu_A, s_i)$ are the level subalgebras of $A$ for $0 \leq i \leq n$, and form the chains,

$U(\mu_A, t_0) \subset \ldots \subset U(\mu_A, t_n) = X$ and $L(\nu_A, s_0) \subset \ldots \subset L(\nu_A, s_n) = X$.

Now, $U(\mu_A, t_0) = \{x \in X : \mu_A(x) \geq t_0\} = S_0 = \{x \in X : \nu_A(x) \leq s_0\} = L(\nu_A, s_0)$. Finally, we prove that $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$.

Now let $x \in S_i$, then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$. This implies $x \in (\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$. Hence $S_i \subseteq (\mu_A, t_i)$ and $S_i \subseteq (\nu_A, s_i)$. If $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$, then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$ which implies that $x \notin S_j$ for $j > i$. For otherwise, if $x \in S_j$, then $\mu_A(x) \geq t_j$ and $\nu_A(x) \leq s_j$, which implies $t_i > \mu_A(x) \geq t_j$ and $s_i < \nu_A(x) \leq s_j$. This contradicts the assumption that $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$. Hence $\mu_A(x) \notin \{t_0, t_1, \ldots, t_n\}$ and $\nu_A(x) \notin \{s_0, s_1, \ldots, s_n\}$. So $x \in S_k$ for some $k \leq i$. As $S_k \subseteq S_i$, it follows that $x \in S_i$. Hence $U(\mu_A, t_i) \subseteq S_i$ and $L(\nu_A, s_i) \subseteq S_i$. Therefore $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$. \hfill \Box

Note that the number of PMS-subalgebras of a finite PMS-algebra $X$ is finite whereas the number of level PMS-subalgebras of an intuitionistic fuzzy PMS-subalgebra $A$ appears to be infinite. However, every level PMS-subalgebra of $X$ is a PMS-subalgebra.
of $X$, not all of these PMS-subalgebras are unique. The next theorem illustrates this situation.

**Theorem 4.10.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$, then

(i). The upper level PMS-subalgebras $U(\mu_A, t_1)$ and $U(\mu_A, t_2)$, (with $t_1 < t_2$) of an intuitionistic fuzzy PMS-subalgebra $A$ are equal if and only if there is no $x \in X$ such that $t_1 \leq \mu_A(x) < t_2$.

(ii). The lower level PMS-subalgebras $L(\nu_A, s_1)$ and $L(\nu_A, s_2)$, (with $s_1 > s_2$) of an intuitionistic fuzzy PMS-subalgebra $A$ are equal if and only if there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

**Proof.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$. Since the proofs for both (i) and (ii) are similar, here we prove for only (ii).

Suppose that $L(\nu_A, s_1) = L(\nu_A, s_2)$, for $s_1 > s_2$. Then we claim that there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

Assume that there exists $x \in X$ such that $s_1 \geq \nu_A(x) < s_2$.

$$\Rightarrow x \in L(\nu_A, s_1) \text{ but } x \notin L(\nu_A, s_2)$$

$$\Rightarrow L(\mu_A, s_2) \text{ is a proper subset of } L(\nu_A, s_1).$$

This contradicts to the assumption that $L(\nu_A, s_1) = U(\nu_A, s_2)$.

Hence there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

Conversely, suppose that there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

Then we prove that $L(\nu_A, s_1) = L(\nu_A, s_2)$.

Since $s_1 > s_2$, we get $L(\nu_A, s_2) \subseteq L(\nu_A, s_1)$ (1)

Now, $x \in L(\nu_A, s_1) \Rightarrow \nu_A(x) \leq s_1$.

$$\Rightarrow \nu_A(x) \leq s_2. \quad (\text{Since } \nu_A(x) \text{ does not lie between } s_1 \text{ and } s_2).$$

$$\Rightarrow x \in L(\nu_A, s_2).$$

Hence $L(\nu_A, s_1) \subseteq L(\nu_A, s_2)$ (2)

From (1) and (2) we get $L(\nu_A, s_1) = L(\nu_A, s_2)$. \qed

**Remark 4.11.** As the consequence of Theorem 4.10, the level subalgebras of an intuitionistic fuzzy PMS-algebra $A = (\mu_A, \nu_A)$ of a finite PMS-algebra $X$ form a chain,

$$U(\mu_A, t_0) \subset U(\mu_A, t_1) \subset ... \subset U(\mu_A, t_n) = X \text{ and } L(\nu_A, s_0) \subset L(\nu_A, s_1) \subset ... \subset L(\nu_A, s_n) = X,$$

where $t_0 > t_1 > ... > t_n$ and $s_0 < s_1 < ... < s_n$.

**Corollary 4.12.** Let $X$ be a finite PMS-algebra and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$.

(i). If $\text{Im}(\mu_A) = \{t_1, \ldots, t_n\}$, then the family of PMS-subalgebras $\{U(\mu_A, t_i)|1 \leq i \leq n\}$, constitutes all the upper level PMS-subalgebras of $A$ in $X$.

(ii). If $\text{Im}(\nu_A) = \{s_1, \ldots, s_n\}$, then the family of PMS-subalgebras $\{L(\nu_A, s_i)|1 \leq i \leq n\}$, constitutes all the lower level PMS-subalgebras of $A$ in $X$.

**Proof.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$ such that $\text{Im}(\mu_A) = \{t_1, t_2, \ldots, t_n\}$ with $t_1 < t_2 < \ldots < t_n$ and $\text{Im}(\nu_A) = \{s_1, s_2, \ldots, s_n\}$ with $s_1 > s_2 > \ldots > s_n$.

(i). Let $t \in [0, 1]$ and $t \notin \text{Im}(\mu_A)$. Now, we can consider the following cases.

case (1). If $t \leq t_1$, then $U(\mu_A, t_1) = X = U(\mu_A, t)$.

case (2). If $t > t_n$, then $U(\mu_A, t) = \{x \in X|\mu_A(x) \geq t\} = \{x \in X|\mu_A(x) > t\} = \emptyset$

case (3). If $t_{i-1} < t < t_i$, then $U(\mu_A, t) = U(\mu_A, t_i)$ by theorem 4.10(i), since
there is no $x \in X$ such that $t \leq \mu_A(x) < t_i$. Thus for any $t \in [0,1]$, the level PMS-subalgebra is one of $\{U(\mu_A, t_i) | i = 1, 2, \ldots, n\}$.

(ii). proof of (ii) is similar to (i) \hfill \Box

**Corollary 4.13.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$ with finite images.

(i). If $U(\mu_A, t_i) = U(\mu_A, t_j)$ for any $t_i, t_j \in Im(\mu_A)$, then $t_i = t_j$.

(ii). If $L(\nu_A, s_i) = L(\nu_A, s_j)$ for any $s_i, s_j \in Im(\nu_A)$, then $s_i = s_j$.

**Proof.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$ with finite images. Here we only prove (ii). the prove of (i) can be done similarly. Assume $L(\nu_A, s_i) = L(\nu_A, t_j)$ for $s_i, s_j \in Im(\nu_A)$. So to show that $s_i = s_j$ assume on contrary, that is, $s_i \neq s_j$. Without loss of generality assume $s_i > s_j$.

Let $x \in L(\nu_A, s_j)$, then

\[ \nu_A(x) \leq s_j < s_i. \]

\[ \Rightarrow \nu_A(x) < s_i \]

\[ \Rightarrow x \in L(\nu_A, s_i) \]

Let $x \in X$ such that $s_i > \nu_A(x) > s_j$. Then $x \in L(\nu_A, s_i)$ but $x \notin L(\nu_A, s_j)$

\[ \Rightarrow L(\nu_A, s_j) \subseteq L(\nu_A, s_i) \]

\[ \Rightarrow L(\nu_A, t_i) \neq L(\nu_A, t_j) \] which contradics the hypothesis that

\[ L(\nu_A, s_i) = L(\nu_A, s_j). \]

Therefore, $s_i = s_j$. \hfill \Box

5. Conclusion

In this paper, we introduced the notion of intuitionistic fuzzy PMS-subalgebras of PMS-algebras and some results are obtained. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between an intuitionistic fuzzy sets in a PMS-algebra and their level sets is discussed and some interesting results are obtained. The concepts can further be extended to intuitionistic fuzzy ideals of a PMS-algebra for new results in our future work.

**References**


Beza Lamesgin Derseh  
Department of Mathematics, Debre Markos University, Debre Markos, Ethiopia  
E-mail: dbezalem@gmail.com

Berhanu Assaye Alaba  
Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia  
E-mail: birhanu.assaye290113@gmail.com

Yohannes Gedamu Wondifraw  
Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia  
E-mail: yohannesg27@gmail.com