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INTUITIONISTIC FUZZY PMS-SUBALGEBRA OF A PMS-ALGEBRA

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Abstract. In this paper, we introduce the notion of intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra is examined, and some interesting results are obtained.

1. Introduction

In 1966, Y. Imai and K. Iseki [7] and in 1980, Iseki [8] introduced the two classes of abstract algebras, BCK-algebra and BCI-algebra respectively. In 2016, Sithar Selvam and Nagalakshmi [12] introduced a new algebraic structure called PMS-algebra. Zadeh [15] first introduced the concept of a fuzzy set in 1965. After the invention of fuzzy sets, Rosenfeld [10] pioneered the study of fuzzy algebraic structures. In 2016, Sithar Selvam and Nagalakshmi [11] fuzzified PMS-subalgebra and PMS-ideal. K.T. Atanassov [2, 4] developed the concept of intuitionistic fuzzy set as a generalization of Zadeh's fuzzy set. Since then, many researches have been done by mathematicians to extend fuzzy mathematical concepts to intuitionistic fuzzy concepts.
A. Zarandi and A. Borumand Saied [16] studied the intuitionistic fuzzy ideal of BG-algebras in 2005. Mohamed Akram [1] discussed the Bifuzzy structure in K-algebras. Senapati et al. [13,14] investigated intuitionistic fuzzification of subalgebras and ideals of BG-algebras. In 2010, M. Chandramouleeswaran and P. Muralikrishna discussed intuitionistic L-Fuzzy subalgebras of BG and BF algebras. Intuitionistic fuzzy structures of B-algebras were studied by Y. H. Kim and T. E. Jeong [9].
In this paper, we introduced the notion of intuitionistic fuzzy PMS-subalgebras of PMS-algebras and investigate some of their properties. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra is examined, and some interesting results are obtained.

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2. Preliminaries

In this section, we recall some basic definitions and results that are used in the study of this paper

Definition 2.1. [12] A nonempty set X with a constant 0 and a binary operation $'*'$ is called PMS-algebra if it satisfies the following axioms.

1. $0 * x = x$
2. $(y * x) * (z * x) = z * y$, for all $x, y, z \in X$.

In X, we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

Definition 2.2. [12] Let S be a nonempty subset of a PMS-algebra X, then S is called a PMS-sub algebra of X if $x * y \in S$, for all $x, y \in S$.

Example 2.3. [12] Let Z be the set of all integers, and let $*$ be a binary relation on Z defined by $x * y = y - x$, for all $x, y \in Z$, where '-' the usual subtraction of integers. Then $(Z, *, 0)$ is a PMS-algebra since

1. $0 * x = x - 0 = x$
2. $(y * x) * (z * x) = (z * x) - (y * x) = (x - z) - (x - y) = y - z = z * y$.

Clearly, the set E of all even integers is a PMS-subalgebra of a PMS-algebra Z, since $x * y = y - x \in E$ for all $x, y \in E$.

Proposition 2.4. *[12] In any PMS-algebra* $(X, *, 0)$ *the following properties hold for all* $x, y, z \in X$.

1. $x * x = 0$
2. $(y * x) * x = y$
3. $x * (y * x) = y * 0$
4. $(y * x) * z = (z * x) * y$
5. $(x * y) * 0 = y * x = (0 * y) * (0 * x)$

Definition 2.5. [15] Let X be a nonempty set. A fuzzy subset A of the set X is defined as $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$ where the mapping $\mu_A : X \to [0, 1]$ defines the degree of membership

Definition 2.6. [11] A fuzzy set A in a PMS-algebra X is called fuzzy PMS-subalgebra of X if $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$

Definition 2.7. [2, 4] An intuitionistic fuzzy set (IFS) A in a nonempty set X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and the degree of non membership, respectively, satisfying the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

Remark 2.8. Ordinary fuzzy sets over X may be viewed as special intuitionistic fuzzy sets with the non membership function $\nu_A(x) = 1 - \mu_A(x)$. So each Ordinary fuzzy set may be written as $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$ to define an intuitionistic fuzzy set. For the sake of simplicity we write $A = (\mu_A, \nu_A)$ for an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$.

Definition 2.9. [2–4] Let A and B be two intuitionistic fuzzy subsets of the set X, where $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$, then

1. $A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle \, | x \in X\}$
2. $A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x)),\ min(\nu_A(x), \nu_B(x))\rangle \, | x \in X\}$
3. $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \, | x \in X\}$
4. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \, | x \in X\}$
5. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \, | x \in X\}$

3. Intuitionistic Fuzzy PMS-subalgebra

In this section we introduce the notion of intuitionistic fuzzy PMS-subalgebra and investigated some of its properties. Throughout this and the next section X denotes a PMS-algebra, unless otherwise specified.

Definition 3.1. An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is called an intuitionistic fuzzy PMS-subalgebra of X if

1. $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ and
2. $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$

Example 3.2. Let $X = \{01, 1, 2, 3\}$ be a set with the following table.

*	0	1	2	3
0	0	1	2	3
1	2	0	1	2
2	1	2	0	1
3	3	1	2	0

Then $(X, *, 0)$ is a PMS-algebra and $S = \{0, 1, 2\}$ is a PMS-subalgebra X.
Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0.5 & \text{if} \quad x = 1, 2 \\ 0 & \text{if} \quad x = 3 \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 0.4 & \text{if} \quad x = 1, 2 \\ 1 & \text{if} \quad x = 3 \end{cases}$$

For intuitionistic fuzzy set A in a PMS-algebra X with membership values $\mu_A(x)$ and non membership values $\nu_A(x)$ as defined above, definition 3.1 is satisfied. Therefore $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS- subalgebra of the PMS-algebra X.

Lemma 3.3. *If $A = (\mu_A, \nu_A)$is an intuitionistic fuzzy PMS-subalgebra of X, then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$*

Proof. Suppose $A = (\mu_A, \nu_A)$is an11 intuitionistic fuzzy PMS-subalgebra of X. Since $x * x = 0$ for every $x \in X$ by proposition 2.1(1), we have
$\mu_A(0) = \mu_A(x * x) \geq min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$ and
$\nu_A(0) = \nu_A(x * x) \leq max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)$
Hence $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$ □

Lemma 3.4. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X, if $x * y \leq z$, then $\mu_A(x) \geq min\{1\mu_A(y), \mu_A(z)\}$ and $\nu_A(x) \leq max\{\nu_A(y), \nu_A(z)\}$.*

Proof. Suppose $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X. Let $x, y, z \in X$ such that $x * y \leq z$. Then by the binary relation $\leq$ defined in X, we have

$(x*y)*z=0$. Thus by definition 2.1 and proposition 2.4 (4), we have

$$\begin{aligned}\mu_A(x)=\mu_A(0*x)&=\mu_A(((x*y)*z)*x)\\&=\mu_A(((z*y)*x)*x)\\&=\mu_A((x*x)*(z*y))\\&=\mu_A(0*(z*y))\\&=\mu_A(z*y)\geq min\{\mu_A(z),\mu_A(y)\}\end{aligned}$$

Hence $\mu_A(x)\geq min\{\mu_A(z),\mu_A(y)\}$

Similarly, $\nu_A(x)\leq max\{\nu_A(z),\nu_A(y)\}$ □

THEOREM 3.5. *Let $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X and let $x\in X$, then $\mu_A(x*y)=\mu_A(y)$ and $\nu_A(x*y)=\nu_A(y)$ for each $y\in X$ if and only if $\mu_A(x)=\mu_A(0)$ and $\nu_A(x)=\nu_A(0)$, where 0 is a constant in X.*

Proof. Suppose $\mu_A(x*y)=\mu_A(y)$ and $\nu_A(x*y)=\nu_A(y)$ for each $y\in X$. Then we need to show that $\mu_A(x)=\mu_A(0)$ and $\nu_A(x)=\nu_A(0)$, where 0 is a constant in X. By lemma 3.3, $\mu_A(0)\geq\mu_A(x)$ and $\nu_A(0)\leq\nu_A(x)$ for each $x\in X$. By proposition 2.4 (2) $(x*0)*0=x$. Then $\mu_A(x)=\mu_A((x*0)*0)\geq min\{\mu_A(x*0),\mu_A(0)\}=\mu_A(0)$.
Also, $\nu_A(x)=\nu_A((x*0)*0)\leq max\{\nu_A(x*0),\nu_A(0)\}=\nu_A(0)$.
Hence $\mu_A(x)\geq\mu_A(0)$ and $\nu_A(x)\leq\nu_A(0)$.
Therefore $\mu_A(x)=\mu_A(0)$ and $\nu_A(x)=\nu_A(0)$
Conversely, Suppose $\mu_A(x)=\mu_A(0)$ and $\nu_A(x)=\nu_A(0)$. Then we need to prove that $\mu_A(x*y)=\mu_A(y)$ and $\nu_A(x*y)=\nu_A(y)$, for each $y\in X$.
By lemma 3.3 $\mu_A(x)\geq\mu_A(y)$ and $\nu_A(x)\leq\nu_A(y)$ for each $y\in X$. Since A is an intuitionistic fuzzy PMS-subalgebra of X, Then $\mu_A(x*y)\geq min\{\mu_A(x),\mu_A(y)\}=\mu_A(y)$ and $\nu_A(x*y)\leq max\{\nu_A(x),\nu_A(y)\}=\nu_A(y)$. Thus $\mu_A(x*y)\geq\mu_A(y)$ and $\nu_A(x*y)\leq\nu_A(y)$ for each $y\in X$.
But, using Proposition 2.4 (2) and 2.4 (5) it follws that

$$\begin{aligned}\mu_A(y)=\mu_A((y*x)*x)&\geq min\{\mu_A(y*x),\mu_A(x)\}\\&=min\{\mu_A((x*y)*0),\mu_A(x)\}\\&\geq min\{min\{\mu_A(x*y),\mu_A(0)\},\mu_A(x)\}\\&=min\{\mu_A(x*y),\mu_A(x)\}=\mu_A(x*y)\end{aligned}$$

and

$$\begin{aligned}\nu_A(y)=\nu_A((y*x)*x)&\leq max\{\nu_A(y*x),\nu_A(x)\}\\&=max\{\nu_A((x*y)*0),\nu_A(x)\}\\&\leq max\{max\{\nu_A(x*y),\nu_A(0)\},\nu_A(x)\}\\&=max\{\nu_A(x*y),\nu_A(x)\}=\nu_A(x*y)\end{aligned}$$

Hence $\mu_A(x*y)=\mu_A(y)$ and $\nu_A(x*y)=\nu_A(y)$ for each $y\in X$. □

THEOREM 3.6. *Let $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X. If $\mu_A(x*y)=\mu_A(0)$ and $\nu_A(x*y)=\nu_A(0)$ for all $x,y\in X$, then $\mu_A(x)=\mu_A(y)$ and $\nu_A(x)=\nu_A(y)$*

Proof. Let $x, y \in X$ such that $\mu_A(x * y) = \mu_A(0)$ and $\nu_A(x * y) = \nu_A(0)$.
Claim $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$

$$\begin{aligned}
\text{Now,} \quad \mu_A(x) &= \mu_A((y * y) * x) \\
&= \mu_A((x * y) * y) \\
&\geq min\{\mu_A(x * y), \mu_A(y)\} \\
&= min\{\mu_A(0), \mu_A(y)\} = \mu_A(y) \\
\text{Conversely,} \quad \mu_A(y) &= \mu_A((x * x) * y) \\
&= \mu_A((y * x) * x) \\
&\geq min\{\mu_A(y * x), \mu_A(y)\} \\
&= min\{\mu_A((x * y) * 0), \mu_A(y)\} \\
&\geq min\{min\{\mu_A(x * y), \mu_A(0)\}, \mu_A(x)\} \\
&= min\{\mu_A(0), \mu_A(x)\} = \mu_A(x)
\end{aligned}$$

Thus $\mu_A(x) = \mu_A(y)$

By similar argument we have $\nu_A(x) = \nu_A(y)$ □

THEOREM 3.7. *The intersection of any two intuitionistic fuzzy PMS-sub algebras of X is also an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra X.
Claim: $A \cap B$ is an intuitionistic fuzzy PMS-subalgebra of X.Then for $x, y \in X$, we have

$$\begin{aligned}
\mu_{A \cap B}(x * y) &= min\{\mu_A(x * y), \mu_B(x * y)\} \\
&\geq min\{min\{\mu_A(x), \mu_A(y)\}, min\{\mu_B(x), \mu_B(y)\}\} \\
&= min\{min\{\mu_A(x), \mu_B(x)\}, min\{\mu_A(y), \mu_B(y)\}\} \\
&= min\{\mu_{A \cap B}(x), \mu_{A \cap B}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A \cap B}(x * y) &= max\{\nu_A(x * y), \nu_B(x * y)\} \\
&\leq max\{max\{\nu_A(x), \nu_A(y)\}, max\{\nu_B(x), \nu_B(y)\}\} \\
&= max\{max\{\nu_A(x), \nu_B(x)\}, max\{\nu_A(y), \nu_B(y)\}\} \\
&= max\{\nu_{A \cap B}(x), \nu_{A \cap B}(y)\}
\end{aligned}$$

Hence $A \cap B$ is an intuitionistic fuzzy PMS-subalgebra of X □

The above theorem proves that the intersection of any two intuitionistic fuzzy PMS-subalgebras of X is again an intuitionistic fuzzy subalgebra of X. It can also be generalized to any family of intuitionistic fuzzy PMS-subalgebra of X as follows:

COROLLARY 3.8. *If $\{A_i : i \in I\}$ be a family of intuitionistic fuzzy PMS-subalgebra of X, then $\cap_{i \in I}$ is also an intuitionistic fuzzy PMS-subalgebra of X,where $\cap_{i \in I} \mu_{A_i}(x) = \inf_{i \in I} \mu_{A_i}(x)$ and $\cap_{i \in I} \nu_{A_i}(x) = \sup_{i \in I} \mu_{A_i}(x)$*

REMARK 3.9. The union of any two intutionistic fuzzy PMS-subalgebras of a PMS-algebra X is not necassarily an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X.

EXAMPLE 3.10. Let X={0,1,2,3} be a set with the table as in example 3.2 and $A = (\mu_A, \mu_A)$ is an intuitionistic fuzzy set in X as defind in example 3.2. Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy set in X defined by

$$\mu_B(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0.6 & \text{if} \quad x = 1,3 \\ 0 & \text{if} \quad x = 2 \end{cases} \quad \text{and} \quad \nu_B(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 0.2 & \text{if} \quad x = 1,3 \\ 1 & \text{if} \quad x = 2 \end{cases}$$

Now, $\mu_{A\cup B}(1*0) = \mu_{A\cup B}(2) = max\{\mu_A(2), \mu_B(2)\} = max\{0.5, 0\} = 0.5$ (i)

$$\begin{aligned} min\{\mu_{A\cup B}(1), \mu_{A\cup B}(0)\} &= min\{max\{\mu_A(1), \mu_B(1)\}, max\{\mu_A(0), \mu_B(0)\}\} \\ &= min\{max\{0.5, 0.6\}, max\{1, 1\}\} \\ &= min\{0.6, 1\} = 0.6 \qquad (ii) \end{aligned}$$

and

also, $\nu_{A\cup B}(1*0) = \nu_{A\cup B}(2) = min\{\nu_A(2), \nu_B(2)\} = min\{0.4, 1\} = 0.4$ (iii)

$$\begin{aligned} max\{\nu_{A\cup B}(1), \nu_{A\cup B}(0)\} &= max\{min\{\mu_A(1), \nu_B(1)\}, min\{\nu_A(0), \nu_B(0)\}\} \\ &= max\{min\{0.4, 0.2\}, min\{0, 0\}\} \\ &= max\{0.2, 0\} = 0.2 \qquad (iv) \end{aligned}$$

From (i) and (ii) we see that $\mu_{A\cup B}(1 * 0) = 0.5 < 0.6 = min\{\mu_{A\cup B}(1), \mu_{A\cup B}(0)\}$ and from (iii) and (iv) we see that $\nu_{A\cup B}(1 * 0) = 0.4 > 0.2 = max\{\nu_{A\cup B}(1), \nu_{A\cup B}(0)\}$ which is a contradiction.This shows that the union of any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra X may not be an intutionistic fuzzy PMS-subalgebra.

LEMMA 3.11. *Let $A = (\mu_A, \nu_A)$ be an intutionistic fuzzy set in X. Then the following statements hold for any $x, y \in X$.*

1. $1 - max\{\mu_A(x), \mu_A(y)\} = min\{1 - \mu_A(x), 1 - \mu_A(y)\}$
2. $1 - min\{\mu_A(x), \mu_A(y)\} = max\{1 - \mu_A(x), 1 - \mu_A(y)\}$.
3. $1 - max\{\nu_A(x), \nu_A(y)\} = min\{1 - \nu_A(x), 1 - \nu_A(y)\}$
4. $1 - min\{\nu_A(x), \nu_A(y)\} = max\{1 - \nu_A(x), 1 - \nu_A(y)\}$.

Now, we can prove the next two theorems using the above Lemma.

THEOREM 3.12. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is an intuitionistic fuzzy PMS-subalgebra of X if and only if the fuzzy subsets μ_A and $\bar{\nu_A}$ are fuzzy subalgebras of X.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. Claim: The fuzzy subsets μ_A and $\bar{\nu_A}$ of X are fuzzy subalgebras of X. Clearly, μ_A is a fuzzy PMS-subalgebra of X directly follows from the fact that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. Now for all $x, y \in X$,

$$\begin{aligned} \bar{\nu_A}(x * y) = 1 - \nu_A(x * y) &\geq 1 - max\{\nu_A(x), \nu_A(y)\} \\ &= min\{1 - \nu_A(x), 1 - \nu_A(y)\} \quad \text{(By Lemma 3.11(3))} \\ &= min\{\bar{\nu_A}(x), \bar{\nu_A}(y)\} \end{aligned}$$

Therefore $\bar{\nu_A}$ is a fuzzyPMS-subalgebra of X

Conversely, Suppose μ_A and $\bar{\nu_A}$ are fuzzy PMS-subalgebras of X. So, we need to show that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. Since μ_A and $\bar{\nu_A}$ are fuzzy PMS-subalgebras of X, we have that $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\bar{\nu_A}(x * y) \geq min\{\bar{\nu_A}(x), \bar{\nu_A}(y)\}$, for all $x, y \in X$. Now it suffices to show that

$\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.

$$1 - \nu_A(x * y) = \bar{\nu_A}(x * y) \geq min\{\bar{\nu_A}(x), \bar{\nu_A}(y)\}$$
$$= min\{1 - \nu_A(x), 1 - \nu_A(y)\}$$
$$= 1 - max\{\nu_A(x), \nu_A(y)\} \quad \text{(By Lemma 3.11(3))}$$
$$\Rightarrow \nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}, \text{ for all } x, y \in X.$$

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X. □

COROLLARY 3.13. *If μ_A is a fuzzy PMS-subalgebra of X, then $A = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Suppose μ_A is a fuzzy PMS-subalgebra of X. Then we want to show that $A = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy PMS-subalgebra of X. Since μ_A is a fuzzy PMS-subalgebra of X, it follows that $\mu_A(x*y) \geq min\{\mu_A(x), \mu_A(y)\}$ .Then it suffices to show that $\bar{\mu_A}(x * y) \leq max\{\bar{\mu_A}(x), \bar{\mu_A}(y)\}$.

$$\bar{\mu_A}(x * y) = 1 - \mu_A(x * y) \leq 1 - min\{\mu_A(x), \mu_A(x)\}$$
$$= max\{1 - \mu_A(x), 1 - \mu_A(x)\}$$
$$= max\{\bar{\mu_A}(x), \bar{\mu_A}(x)\}$$
$$\Rightarrow \bar{\mu_A}(x * y) \leq max\{\bar{\mu_A}(x), \bar{\mu_A}(y)\}$$

Hence $A = (\mu_A, \bar{\mu_A})$ is an intuitionistic fuzzy PMS-subalgebra of X. □

COROLLARY 3.14. *If $\bar{\nu_A}$ is a fuzzy PMS-subalgebra of X, then $A = (\bar{\nu_A}, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Similar to corollary 3.13 □

THEOREM 3.15. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X is an intuitionistic fuzzy PMS-subalgebra of X if and only if $\Box A = (\mu_A, \bar{\mu_A})$ and $\Diamond A = (\bar{\nu_A}, \nu_A)$ are intuitionistic fuzzy PMS-subalgebra of X.*

Proof. Assume that an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X is an intuitionistic fuzzy PMS-subalgebra of X, then

$$\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\} \text{ and } \nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}.$$

Claim: $\Box A = (\mu_A, \bar{\mu_A})$ and $\Diamond A = (\bar{\nu_A}, \nu_A)$ are intuitionistic fuzzy PMS-subalgebras of X.

(i) To show that $\Box A$ is an intuitionistic fuzzy PMS-subalgebra of X, it suffices to show that $\bar{\mu_A}(x * y) \leq max\{\bar{\mu_A}(x), \bar{\mu_A}(y)\}$,for all $x, y \in X$. Let $x, y \in X$, then

$$\bar{\mu_A}(x * y) = 1 - \mu_A(x * y) \leq 1 - min\{\mu_A(x), \mu_A(y)\}$$
$$= max\{1 - \mu_A(x), 1 - \mu_A(y)\}$$
$$= max\{\bar{\mu_A}(x), \bar{\mu_A}(y)\}$$
$$\Rightarrow \bar{\mu_A}(x * y) \leq max\{\bar{\mu_A}(x), \bar{\mu_A}(y)\}, \forall x, y \in X.$$

Hence $\Box A$ is an intuitionistic fuzzy PMS-subalgebra of X

(ii) To show that $\Diamond A$ is an intuitionistic fuzzy PMS-subalgebra of X, it suffices to show that $\bar{\nu_A}(x * y) \geq min\{\bar{\nu_A}(x), \bar{\nu_A}(y)\}$, for all $x, y \in X$. Let $x, y \in X$, then

$$\bar{\nu_A}(x * y) = 1 - \nu_A(x * y) \geq 1 - max\{\bar{\nu_A}(x), \bar{\nu_A}(y)\}$$
$$= min\{1 - \nu_A(x), 1 - \nu_A(y)\}$$
$$= min\{\bar{\nu_A}(x), \bar{\mu_A}(y)\}$$
$$\Rightarrow \bar{\nu_A}(x * y) \geq min\{\bar{\nu_A}(x), \bar{\nu_A}(y)\}, \forall x, y \in X.$$

Hence $\Diamond A$ is an intuitionistic fuzzy PMS-subalgebra of X.
The proof of the converse of this theorem is trivial. □

4. Level Subsets of Intuitionistic Fuzzy PMS-subalgebras

In this section, the idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. Characterizations of level subsets of a fuzzy PMS-subalgebra of a PMS-algebra are given.

THEOREM 4.1. *If $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X, then the sets $X_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(0)\}$ and $X_{\nu_A} = \{x \in X | \nu_A(x) = \nu_A(0)\}$ are PMS -subalgebra of X*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X and let $x, y \in X_{\mu_A}$. Then $\mu_A(x) = \mu_A(0) = \mu_A(y)$. So, $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ $= min\{\mu_A(0), \mu_A(0)\} = \mu_A(0)$. $\Rightarrow \mu_A(x * y) \geq \mu_A(0)$. By Lemma 3.3, we get that $\mu_A(x * y) = \mu_A(0)$ which imply that $x * y \in X_{\mu_A}$. Also, Let $x, y \in X_{\nu_A}$. Then $\nu_A(x) = \nu_A(0) = \nu_A(y)$ and so $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\} = max\{\nu_A(0), \nu_A(0)\} = \nu_A(0)$. $\Rightarrow \nu_A(x * y) \leq \nu_A(0)$. By Lemma 3.3, we get that $\nu_A(x * y) = \nu_A(0)$ which imply that $x * y \in X_{\nu_A}$.
Hence, the sets X_{μ_A} and X_{ν_A} are PMS-subalgebras of X. □

THEOREM 4.2. *Let S be a nonempty subset of a PMS-algebra X and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by*

$$\mu_A(x) = \begin{cases} p & \text{if } x \in S \\ q & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} r & \text{if } x \in S \\ s & \text{if } x \notin S \end{cases}$$

for all $p, q, r, s \in [0, 1]$ with $p \geq q, r \leq s$ and $0 \leq p + r \leq 1, 0 \leq q + s \leq 1$. Then A is an intuitionistic fuzzy PMS-subalgebra of X if and only if S is a PMS-subalgebra of X. Furthermore, in this situation, $X_{\mu_A} = S = X_{\nu_A}$.

Proof. Let A be an intuitionistic fuzzy PMS-subalgebra of X.Then we want to show that S is a PMS-subalgebra of X. Let $x, y \in X$ such that $x, y \in S$.
Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X, we have
$\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\} = min\{p, p\} = p$ and
$\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\} = max\{r, r\} = r$.
Hence $x * y \in S$. So, S is a PMS-subalgebra of X.
Conversely, suppose that S is a PMS-subalgebra of X. We claim to show that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X.
Let $x, y \in X$. Now consider the following cases

case (i). 1If $x, y \in S$, then $x * y \in S$, since S is a PMS-subalgebra of X. Thus, $\mu_A(x * y) = p = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) = r = max\{\nu_A(x), \nu_A(y)\}$

case (ii). If $x \in S, y \notin S$, then $\mu_A(x) = p, \mu_A(y) = q$ and $\nu_A(x) = r, \nu_A(y) = s$. Thus, $\mu_A(x * y) \geq q1 = min\{p, q\} = min\{\mu_A(x), \mu_A(y)\}$ implies $\mu_A(x * y) \geq min\{\mu_A(x)\mu_A(y)\}$ and $\nu_A(x * y) \leq s = max\{r, s\} = max\{\nu_A(x), \nu_A(y)\}$ implies $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$

case (iii). If $x \notin S, y \in S$, then interchanging the roles of x and y in Case (ii), yields similar results $\mu_A(x * y) \geq min\{\mu_A(x), \mu(y)\}$ and $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$

case (iv). If $x, y \notin S$, then $\mu_A(x) = q = \mu_A(y)$ and $\nu_A(x) = s = \nu_A$, this implies that $\mu_A(x * y) \geq q = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq s = max\{\nu_A(x), \nu_A(y)\}$
Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X.

Furthermore, we have
$X_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(0)\} = \{x \in X | \mu_A(x) = p\} = S$ and
$X_{\nu_A} = \{x \in X | \nu_A(x) = \nu_A(0)\} = \{x \in X | \nu_A(x) = r\} = S$.
Hence $X_{\mu_A} = S = X_{\nu_A}$. □

Definition 4.3. Let $A = (\mu_A, \nu_A)$ be any intuitionistic fuzzy subset of a PMS-algebra X such that $t, s \in [0, 1]$, then the set $U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\}$ is called an upper t-level set of an intuitionistic fuzzy subset A of X and the set $L(\mu_A, s) = \{x \in X : \nu_A(x) \leq s\}$ is called a lower s-level set of an intuitionistic fuzzy subset A of X

Theorem 4.4. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is an intuitionistic fuzzy PMS-subalgebra of X if and only if the the nonempty level subsets $U(\mu_A, t)$ and $L(\nu_A, s)$ of A are PMS-subalgebras of X for all $t, s \in [0, 1]$ with $0 \leq t + s \leq 1$.*

Proof. Assume that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X such that $U(\mu_A, t) \neq \phi$ and $L(\nu_A, s) \neq \phi$. Now we claim that $U(\mu_A, t)$ and $L(\nu_A, t)$ are PMS-subalgebras of X for all $t, s \in [0, 1]$ with $0 \leq t + s \leq 1$. Let $x, y \in U(\mu_A, t)$, then we have $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$. Thus $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\} \geq min\{t, t\} = t$
$\Rightarrow x * y \in U(\mu_A, t)$
Hence $U(\mu_A, t)$ is a PMS-subalgebra of X.
Also, let $x, y \in L(\nu_A, s)$, then $\nu_A(x) \leq s$ and $\nu_A(y) \leq s$
So, $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\} \leq max\{t, t\} = t \Rightarrow x * y \in L(\nu_A, s)$
Hence $L(\nu_A, s)$ is a PMS-subalgebra of X.
Conversely, Suppose that $U(\mu_A, t)$ and $L(\nu_A, s)$ are PMS-subalgebra of X for all $t, s \in [0, 1]$ with $0 \leq t + s \leq 1$
Claim: A is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X.
Let $x, y \in X$ such that $\mu_A(x) = t_1$ and $\mu_A(y) = t_2$ for $t_1, t_2 \in [0, 1]$. Then $x \in U(\mu_A, t_1)$ and $y \in U(\mu_A, t_2)$.
Choose $t = min\{t_1, t_2\}$, then $t \leq t_1$ and $t \leq t_2$
$\Rightarrow U(\mu_A, t_1) \subseteq U(\mu_A, t)$ and $U(\mu_A, t_2) \subseteq U(\mu_A, t)$.
$\Rightarrow x, y \in U(\mu_A, t)$,
Since $U(\mu_A, t)$ is a PMS-Subalgebra of X, it follows that $x * y \in U(\mu_A, t)$.
Thus $\mu_A(x * y) \geq t = min\{t_1, t_2\} = min\{\mu_A(x), \mu_A(y)\}$.
Hence $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.
And also, let $x, y \in X$ such that $\nu_A(x) = s_1$ and $\nu_A(y) = s_2$ for $s_1, s_2 \in [0, 1]$.
Then $x \in L(\nu_A, s_1)$ and $y \in L(\nu_A, s_2)$.
Choose $s = max\{s_1, s_2\}$, then $s_1 \leq s$ and $s_2 \leq s$
$\Rightarrow L(\nu_A, s_1) \subseteq L(\nu_A, s)$ and $L(\nu_A, s_2) \subseteq L(\nu_A, s)$.
$\Rightarrow x, y \in U(\nu_A, s)$,
Since $L(\nu_A, s)$ is a PMS-subalgebra of X, it follows that $x * y \in L(\nu_A, s)$.
Thus $\nu_A(x * y) \leq s = max\{s_1, s_2\} = max\{\nu_A(x), \nu_A(y)\}$.
Hence $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.
Hence A is an intuitionistic fuzzy PMS-subalgebra of a PMS -algebra X. □

Remark 4.5. The PMS-subalgebras $U(\mu_A, t)$ and $L(\nu_A, s)$ of X for all $t, s \in [0,1]$ obtained in the above theorem are called level PMS-subalgebras of X.

Corollary 4.6. *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra X is an intuitionistic fuzzy PMS-subalgebra of X if and only if the level subsets $U(\mu_A, t)$ and $L(\nu_A, s)$ of A are PMS-subalgebras of X for all $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $0 \le t + s \le 1$*

Theorem 4.7. *Let S be a subset of X and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by*

$$\mu_A(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s & \text{if } x \in S \\ 1 & \text{if } x \notin S \end{cases}$$

for all $t, s \in [0,1]$ such that $0 \le t + s \le 1$. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X, then S is a level PMS-subalgebra of X.

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X. Then we need to show that S is a level PMS-subalgebra of X. Let $x, y \in S$, then $\mu_A(x) = t = \mu_A(y)$ and $\nu_A(x) = s = \nu_A(y)$. So, $\mu_A(x * y) \ge min\{\mu_A(x), \mu_A(y)\} = min\{t, t\} = t$ and $\nu_A(x * y) \le max\{\mu_A(x), \nu_A(y)\} = max\{s, s\} = s$ which implies that $x * y \in S$. Hence S is a PMS-subalgebra of X. Also, by theorem 4.4, $U(\mu_A, t)$ is a level subalgebra of X, and
$U(\mu_A, t) = \{x \in X : \mu_A(x) \ge t\} = S = \{x \in X : \nu_A(x) \le s\}$.
Thus, S is a level PMS-Subalgebra of X corresponding to the intuitionistic fuzzy PMS-subalgebra $A = (\mu_A, \nu_A)$ of X. □

Theorem 4.8. *If S is any PMS-subalgebra of X, then there exists an intuitionistic fuzzy PMS-subalgebra A of X, in which S satisfies both the upper level and lower level PMS-subalgebra of A in X.*

Proof. Let S be a PMS-subalgebra of a PMS-algebra X and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s & \text{if } x \in S \\ 1 & \text{if } x \notin S \end{cases}$$

for all $t, s \in [0,1]$ such that $0 \le t + s \le 1$.
Clearly, $U(\mu_A, t) = \{x \in X : \mu_A(x) \ge t\} = S$. Let $x, y \in X$. To prove that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra X, we consider the following cases:

case(i). If $x, y \in S$, then $x * y \in S$. Since S is a PMS-subalgebra of a PMS-algebra X. $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = t$ and $\nu_A(x) = \nu_A(y) = \nu_A(x * y) = s$.
Therefore $\mu_A(x * y) = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) = max\{\nu_A(x), \nu_A(y)\}$

case(ii). If $x \in S, y \notin S$, then we have $\mu_A(x) = t, \mu_A(y) = 0$ and $\nu_A(x) = s, \nu_A(y) = 1$. Thus, $\mu_A(x*y) \ge 0 = min\{t, 0\} = min\{\mu_A(x), \mu_A(y)\}$ which implies that $\mu_A(x * y) \ge min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \le 1 = max\{s, 1\} = max\{\nu_A(x), \nu_A, (y)\}$ implies $\nu_A(x * y) \le \max\{\nu_A(x), \nu_A(y)\}$

case(iii). If $x \notin S, y \in S$, then interchanging the roles of x and y in Case (ii), yields similar results $\mu_A(x * y) \ge min\{\mu_A(x), \mu(y)\}$ and $\nu_A(x * y) \le max\{\nu_A(x), \nu_A(y)\}$

case(iv). If $x, y \notin S$ then $\mu_A(x) = 0 = \mu_A(y)$ and $\nu_A(x) = 1 = \nu_A(y)$.Then
$\mu_A(x * y) \ge 0 = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \le 1 = max\{\nu_A(x), \nu_A(y)\}$.

So, in all cases we get $\mu_A(x*y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x*y) \leq max\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in X$.

Thus, A is an intuitionistic fuzzy PMS-subalgebra of X. □

We can also prove the following theorem as a generalization of theorem 4.8.

THEOREM 4.9. *Let $\{S_i\}$ be any family of a PMS-subalgebra of a PMS-algebra X such that $S_0 \subset S_1 \subset S_2 \subset ... \subset S_n = X$, then there exists an intuitionistic fuzzy PMS-subalgebra $A = (\mu_A, \nu_A)$ of X whose level PMS-subalgebras are exactly the PMS-subalgebras $\{S_i\}$.*

Proof. Suppose $t_0 > t_1 > t_2 > ... > t_n$ and $s_0 < s_1 < s_2 ... < s_n$ where each $t_i, s_i \in [0,1]$ with $0 \leq t_i + s_i \leq 1$. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set defined by

$$\mu_A(x) = \begin{cases} t_0 & \text{if } x \in S_0 \\ t_i & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n. \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s_0 & \text{if } x \in S_0 \\ s_i & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n. \end{cases}$$

Now, We claim that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X and $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 \leq i \leq n$.
Let $x, y \in X$ Then, we consider the following two cases
Case (i): Let $x, y \in S_i - S_{i-1}$. Therefore by the definition of $A = (\mu_A, \nu_A)$, we have $\mu_A(x) = t_i = \mu_A(y)$ and $\nu_A(x) = s_i = \nu_A(y)$. Since S_i is a PMS-subalgebra of X, it follows that $x*y \in S_i$, and so either $x*y \in S_i - S_{i-1}$ or $x*y \in S_{i-1}$ or $x*y \in S_{i-1} - S_{i-2}$.
$\Rightarrow \mu_A(x) = t_i$ or $\mu_A(x) = t_{i-1} > t_i$ and $\nu_A(x) = s_i$ or $\nu_A(x) = s_{i-1} > s_i$.
In any case we conclude that

$$\mu_A(x*y) \geq t_i = min\{\mu_A(x), \mu_A(y)\} \text{ and } \nu_A(x*y) \leq s_i = max\{\nu_A(x), \nu_A(y)\}.$$

Case (ii): For $i > j$, $t_j > t_i$, $s_j < s_i$ and $S_j \subset S_i$. Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$. Then, $\mu_A(x) = t_i, \mu_A(y) = t_j > t_i, \nu_A(x) = s_i$ and $\nu_A(y) = s_j < s_i$. Then $x * y \in S_i$ since S_i is a PMS-subalgebra of X and $S_j \subset S_i$.
Hence $\mu_A(x * y) \geq t_i = min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq s_i = max\{\nu_A(x), \nu_A(y)\}$ by case (i). Thus $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of X.
Also,from the definition of $A = (\mu_A, \nu_A)$, it follows that $Im(\mu_A) = \{t_0, t_1, ..., t_n\}$ and $Im(\nu_A) = \{s_0, s_1, ..., s_n\}$. So, $U(\mu_A, t_i)$ and $L(\nu_A, s_i)$ are the level subalgebras of A for $0 \leq i \leq n$, and form the chains,
$U(\mu_A, t_0) \subset ... \subset U(\mu_A, t_n) = X$ and $L(\nu_A, s_0) \subset ... \subset L(\nu_A, s_n) = X$.
Now, $U(\mu_A, t_0) = \{x \in X : \mu_A(x) \geq t_0\} = S_0 = \{x \in X : \nu_A(x) \leq s_0\} = L(\nu_A, s_0)$.
Finally, we prove that $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$.
Now let $x \in S_i$, then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$. This implies $x \in (\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$. Hence $S_i \subseteq (\mu_A, t_i)$ and $S_i \subseteq L(\nu_A, s_i)$. If $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$, then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$ which implies that $x \notin S_j$ for $j > i$. For otherwise, if $x \in S_j$, then $\mu_A(x) \geq t_j$ and $\nu_A(x) \leq s_j$, which implies $t_i > \mu_A(x) \geq t_j$ and $s_i < \nu_A(x) \leq s_j$. This contradicts the assumption that $x \in U(\mu_A, t_i)$ and $x \in L(\nu_A, s_i)$. Hence $\mu_A(x) \in \{t_0, t_1, ..., t_n\}$ and $\nu_A(x) \in \{s_0, s_1, ..., s_n\}$. So $x \in S_k$ for some $k \leq i$. As $S_k \subseteq S_i$, it follows that $x \in S_i$. Hence $U(\mu_A, t_i) \subseteq S_i$ and $L(\nu_A, s_i) \subseteq S_i$. Therefore $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$. □

Note that the number of PMS-subalgebras of a finite PMS–algebra X is finite whereas the number of level PMS-subalgebras of an intuitionistic fuzzy PMS-subalgebra A appears to be infinite. However, every level PMS-subalgebra of X is a PMS-subalgebra of X, not all of these PMS-subalgebras are unique. The next theorem illustrates this situation.

THEOREM 4.10. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X, then*

(i). *The upper level PMS-subalgebras $U(\mu_A, t_1)$ and $U(\mu_A, t_2)$,(with $t_1 < t_2$) of an intuitionistic fuzzy PMS-subalgebra A are equal if and only if there is no $x \in X$ such that $t_1 \leq \mu_A(x) < t_2$.*

(ii). *The lower level PMS-sub algebras $L(\nu_A, s_1)$ and $L(\nu_A, s_2)$,(with $s_1 > s_2$) of an intuitionistic fuzzy PMS-subalgebra A are equal if and only if there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.*

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X. Since the proofs for both (i) and (ii) are similar, here we prove for only (ii).
Suppose that $L(\nu_A, s_1) = L(\nu_A, s_2)$, for $s_1 > s_2$. Then we claim that there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$. Assume that there exists $x \in X$ such that $s_1 \geq \mu_A(x) < s_2.$

$$\Rightarrow x \in L(\nu_A, s_1) \text{ but } x \notin L(\nu_A, s_2)$$
$$\Rightarrow L(\mu_A, s_2) \text{ is a proper subset of } L(\nu_A, s_1).$$

This contradicts to the assumption that $L(\nu_A, s_1) = U(\nu_A, s_2)$.
Hence there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2.$
Conversely, suppose that there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2.$. Then we prove that $L(\nu_A, s_1) = L(\nu_A, s_2)$.
Since $s_1 > s_2$, we get $L(\nu_A, s_2) \subseteq L(\nu_A, s_1)$ (1)
Now, $x \in L(\nu_A, s_1) \Rightarrow \nu_A(x) \leq s_1$.

$$\Rightarrow \nu_A(x) \leq s_2, \quad \text{(Since } \nu_A(x) \text{ does not lie between } s_1 \text{ and } s_2\text{)}.$$
$$\Rightarrow x \in L(\nu_A, s_2).$$

Hence $L(\nu_A, s_1) \subseteq L(\nu_A, t_2)$ (2)
From (1) and (2) we get $L(\nu_A, s_1) = L(\nu_A, s_2)$. □

REMARK 4.11. As the consequence of Theorem 4.10, the level subalgebras of an intuitionistic fuzzy PMS-algebra $A = (\mu_A, \nu_A)$ of a finite PMS-algebra X form a chain,
$U(\mu_A, t_0) \subset U(\mu_A, t_1) \subset ... \subset U(\mu_A, t_n) = X$ and $L(\nu_A, s_0) \subset L(\nu_A, s_1) \subset ... \subset L(\nu_A, s_n) = X$, where $t_0 > t_1 > ... > t_n$ and $s_0 < s_1 < ... < s_n$.

COROLLARY 4.12. *Let X be a finite PMS-algebra and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X.*

(i). *If $Im(\mu_A) = \{t_1, \ldots, t_n\}$, then the family of PMS-subalgebras $\{U(\mu_A, t_i)|1 \leq i \leq n\}$, constitutes all the upper level PMS-subalgebras of A in X.*

(ii). *If $Im(\nu_A) = \{s_1, \ldots, s_n\}$, then the family of PMS-subalgebras $\{L(\nu_A, s_i)|1 \leq i \leq n\}$, constitutes all the lower level PMS-subalgebras of A in X.*

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X such that $Im(\mu_A) = \{t_1, t_2, \ldots, t_n\}$ with $t_1 < t_2 < \ldots < t_n$ and $Im(\nu_A) = \{s_1, s_2, \ldots, s_n\}$ with $s_1 > s_2 > \ldots < s_n$.

(i). Let $t \in [0, 1]$ and $t \notin Im(\mu_A)$. Now, we can consider the following cases.
case (1). If $t \leq t_1$, then $U(\mu_A, t_1) = X = U(\mu_A, t)$.
case (2). If $t > t_n$, then $U(\mu_A, t) = \{x \in X|\mu_A(x) \geq t\} = \{x \in X|\mu_A(x) > t_n\} = \emptyset$
case (3). If $t_{i-1} < t < t_i$, then $U(\mu_A, t) = U(\mu_A, t_i)$ by theorem 4.10(i), since

there is no $x \in X$ such that $t \leq \mu_A(x) < t_i$. Thus for any $t \in [0,1]$, the level PMS-subalgebra is one of $\{U(\mu_A, t_i)|i = 1, 2, \ldots, n\}$.

(ii). proof of (ii) is similar to (i) □

Corollary 4.13. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS–subalgebra of X with finite images.*

(i). *If $U(\mu_A, t_i) = U(\mu_A, t_j)$ for any $t_i, t_j \in Im(\mu_A)$, then $t_i = t_j$.*

(ii). *If $L(\nu_A, s_i) = L(\nu_A, s_j)$ for any $s_i, s_j \in Im(\nu_A)$, then $s_i = s_j$.*

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of X with finite images. Here we only prove (ii).the prove of (i) can be done similarly. Assume $L(\nu_A, s_i) = L(\nu_A, t_j)$ for $s_i, s_j \in Im(\nu_A)$. So to show that $s_i = s_j$ assume on contrary, that is, $s_i \neq s_j$. Without loss of generality assume $s_i > s_j$.

Let $x \in L(\nu_A, s_j)$, then $\nu_A(x) \leq s_j < s_i$.

$\Rightarrow \nu_A(x) < s_i$

$\Rightarrow x \in L(\nu_A, s_i)$

Let $x \in X$ such that $s_i > \nu_A(x) > s_j$. Then $x \in L(\nu_A, s_i)$ but $x \notin L(\nu_A, s_j)$

$\Rightarrow L(\nu_A, s_j) \subset L(\nu_A, s_i)$

$\Rightarrow L(\nu_A, t_i) \neq L(\nu_A, t_j)$ which contradics the hypothesis that $L(\nu_A, s_i) = L(\nu_A, s_j)$. Therefore, $s_i = s_j$. □

5. Conclusion

In this paper, we introduced the notion of intuitionistic fuzzy PMS-subalgebras of PMS-algebras and some results are obtained. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between an intuitionistic fuzzy sets in a PMS-algebra and their level sets is discussed and some interesting results are obtained. The concepts can further be extended to intuitionistic fuzzy ideals of a PMS-algebra for new results in our future work.

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