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RIGHT-ANGLED ARTIN GROUPS ON PATH GRAPHS, CYCLE GRAPHS AND COMPLETE BIPARTITE GRAPHS

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Abstract. For a finite simplicial graph Γ, let $G(\Gamma)$ denote the right-angled Artin group on the complement graph of Γ. For path graphs P_k, cycle graphs C_ℓ and complete bipartite graphs $K_{n,m}$, this article characterizes the embeddability of $G(K_{n,m})$ in $G(P_k)$ and in $G(C_\ell)$.

1. Introduction

Throughout this article all graphs are simple. For a graph Γ, let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of Γ, respectively. For a finite graph Γ, the *right-angled Artin group* (RAAG) on Γ is the group presented by

$$A(\Gamma) = \langle\, a \in V(\Gamma) \mid [a,b] = 1 \text{ if } \{a,b\} \in E(\Gamma)\,\rangle.$$

It is well-known that two RAAGs $A(\Gamma_1)$ and $A(\Gamma_2)$ are isomorphic as groups if and only if Γ_1 and Γ_2 are isomorphic as graphs [4].

The following opposite convention is often used as well.

$$G(\Gamma) = \langle\, a \in V(\Gamma) \mid [a,b] = 1 \text{ if } \{a,b\} \notin E(\Gamma)\,\rangle$$

In other words, $G(\Gamma) = A(\bar{\Gamma})$, where $\bar{\Gamma}$ denotes the complement graph of Γ. The present article uses this convention. For example, if Γ is the path graph P_n on $n \geqslant 2$ vertices $a_1, \ldots, a_n$ as in Figure 2(a), then

$$G(P_n) = \langle\, a_1, \ldots, a_n \mid [a_i, a_j] = 1 \text{ if } |i-j| \geqslant 2\,\rangle.$$

If Γ is the complete bipartite graph $K_{n,m}$ with vertex set $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\}$ as in Figure 1(a), then

$$G(K_{n,m}) = \left\langle \begin{array}{l} a_1, \ldots, a_n, \\ b_1, \ldots, b_m \end{array} \,\middle|\, \begin{array}{ll} [a_i, a_j] = 1 & \text{for } i,j \in \{1, \ldots, n\}, \\ [b_k, b_\ell] = 1 & \text{for } k,\ell \in \{1, \ldots, m\} \end{array} \right\rangle \;\cong\; \mathbb{Z}^n * \mathbb{Z}^m.$$

Definition 1.1. (i) For a subset $A \subset V(\Gamma)$, the subgraph Λ of Γ with $V(\Lambda) = A$ and $E(\Lambda) = \{\{a,b\} \in E(\Gamma) : a, b \in A\}$ is called the *subgraph of* Γ *induced by* A.

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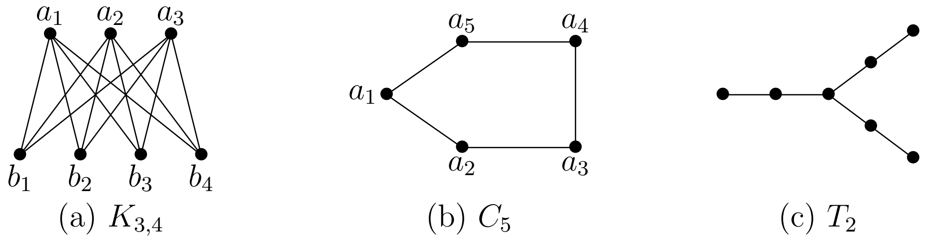

FIGURE 1. Complete bipartite graph, cycle graph, tripod

(ii) If a graph Λ embeds into Γ as an induced subgraph, we write $\Lambda \leqslant \Gamma$.
(iii) If a group H embeds into a group G, i.e. if there exists a monomorphism from H to G, then we write $H \leqslant G$.
(iv) For elements g, h of a group, g^h and $[g, h]$ denote the conjugate $h^{-1}gh$ and the commutator $g^{-1}h^{-1}gh$, respectively.

It is easy to see that $\Gamma_1 \leqslant \Gamma_2$ implies $G(\Gamma_1) \leqslant G(\Gamma_2)$, however, the converse does not hold. The following is a fundamental question for RAAGs.

> [Embeddability Problem] Is there an algorithm to decide whether or not there exists an embedding between two given RAAGs?

The embeddability problem has been studied in various papers, e.g. [1, 3, 5–11]. In particular, the following are known for path graphs and cycle graphs. Let C_n denote the cycle graph on $n \geqslant 3$ vertices as in Figure 1(b).

(i) For $m, n \geqslant 4$, $A(C_m) \leqslant A(C_n)$ if and only if $m = n + k(n-4)$ for some $k \geqslant 0$ [8];
(ii) $G(P_m) \leqslant G(P_n)$ (resp. $G(C_m) \leqslant G(C_n)$) if and only if $n \geqslant m$ [5, 7];
(iii) $G(P_m) \leqslant G(C_n)$ if and only if $n \geqslant m+1$ [5];
(iv) $G(C_m) \leqslant G(P_n)$ if and only if $n \geqslant 2m-2$ [10];
(v) $G(T_2) \leqslant G(P_{22})$, where T_2 denotes the tripod in Figure 1(c) [11].

This article shows the following embeddability between RAAGs on path graphs, cycle graphs and complete bipartite graphs.

THEOREM 1.2. *For $n \geqslant 2$ and $n \geqslant m$, the following hold.*
(i) $G(K_{n,m}) \cong \mathbb{Z}^n * \mathbb{Z}^m \leqslant G(P_k)$ *if and only if* $k \geqslant 2n-1$.
(ii) $G(K_{n,m}) \cong \mathbb{Z}^n * \mathbb{Z}^m \leqslant G(C_\ell)$ *if and only if* $\ell \geqslant 2n$.

As a tool to solve the embeddability problem, Sang-hyun Kim and Thomas Koberda [8] introduced the notion of extension graphs. The *extension graph*, denoted by Γ^E, of Γ with respect to $G(\Gamma)$ is defined by

$$V(\Gamma^E) = \{\, a^g \in G(\Gamma) : a \in V(\Gamma),\ g \in G(\Gamma) \,\},$$
$$E(\Gamma^E) = \{\, \{a^g, b^h\} : a^g, b^h \in V(\Gamma^E),\ [a^g, b^h] \neq 1 \text{ in } G(\Gamma) \,\}.$$

It is clear that $\Gamma \leqslant \Gamma^E$. Extension graphs are usually infinite and locally infinite. Our work uses the extension graph theorem [8, Theorem 1.3] which states, under our convention, that for finite graphs Γ_1 and Γ_2, if $\Gamma_1 \leqslant \Gamma_2^E$ then $G(\Gamma_1) \leqslant G(\Gamma_2)$.

(a) P_5 (b) $K_{3,3}$ embeds into P_5^E.

FIGURE 2. $w = (a_1 a_2 \cdots a_5)(a_4 a_3 a_2 a_1)$

2. Proof of Theorem 1.2

For a finite graph Γ and $g \in G(\Gamma)$, the *support* of g, denoted by $\mathrm{supp}(g)$, is defined as the set of vertices a in Γ such that a or a^{-1} appears in a reduced word representing g. It is known that $\mathrm{supp}(g)$ is well-defined.

First we show that a complete bipartite graph embeds into the extension graph of a path graph.

PROPOSITION 2.1. *$K_{n,n} \leqslant P_{2n-1}^E$ for $n \geqslant 2$.*

Proof. Let $a_1, \ldots, a_{2n-1}$ denote the vertices of P_{2n-1} in this order as in Figure 2(a).

Choose any $w \in G(P_{2n-1})$ such that $\mathrm{supp}(a_i^w) = V(P_{2n-1})$ for each $i \in \{1, \ldots, 2n-1\}$. For instance, we may take $w = (a_1 a_2 \cdots a_{2n-1})(a_{2n-2} \cdots a_2 a_1)$.

Let Γ be the subgraph of P_{2n-1}^E induced by

$$\{a_1, a_3, \ldots, a_{2n-1}\} \cup \{a_1^w, a_3^w, \ldots, a_{2n-1}^w\}.$$

Now we will show that Γ is isomorphic to $K_{n,n}$, which completes the proof.

For distinct $i, j \in \{1, 3, 5, \ldots, 2n-1\}$, one has $[a_i, a_j] = 1$ and hence $[a_i^w, a_j^w] = 1$ in $G(P_{2n-1})$, which implies $\{a_i, a_j\}, \{a_i^w, a_j^w\} \notin E\big(P_{2n-1}^E\big)$. See Figure 2(b).

Meanwhile, it is well-known by the centralizer theorem of Servatius [12] that, for a finite graph Λ and $a \in V(\Lambda)$, if $g \in G(\Lambda)$ commutes with a, then each element of $\mathrm{supp}(g)$ commutes with a. Thus, for any $i, j \in \{1, 3, 5, \ldots, 2n-1\}$, one has $[a_i, a_j^w] \neq 1$ in $G(P_{2n-1})$ because $\mathrm{supp}(a_j^w) = V(P_{2n-1})$. Namely $\{a_i, a_j^w\} \in E\big(P_{2n-1}^E\big)$. See Figure 2(b). $\square$

Using the above proposition, we prove Theorem 1.2.

Proof of Theorem 1.2. (i) Let $k \geqslant 2n-1$. Since $P_{2n-1} \leqslant P_k$, we have $G(P_{2n-1}) \leqslant G(P_k)$. Since $K_{n,n} \leqslant P_{2n-1}^E$ by Proposition 2.1, we have $G(K_{n,n}) \leqslant G(P_{2n-1})$ by the extension graph theorem [8, Theorem 1.3]. Meanwhile, the condition $m \leqslant n$ implies $K_{n,m} \leqslant K_{n,n}$ and hence $G(K_{n,m}) \leqslant G(K_{n,n})$. Therefore $G(K_{n,m}) \leqslant G(P_k)$.

Conversely, assume that $G(K_{n,m}) \leqslant G(P_k)$.

It is well-known (e.g. [2] and [8, Lemma 2.3]) that, for a finite graph Γ, the maximum rank of a free abelian subgroup of $A(\Gamma)$ is the clique number of Γ, i.e. the maximum number of pairwise adjacent vertices in Γ. Since $G(\Gamma) = A(\bar{\Gamma})$, the maximum rank of a free abelian subgroup of $G(\Gamma)$ is the independence number of Γ, i.e. the maximum number of pairwise non-adjacent vertices in Γ.

Thus the maximum rank of a free abelian subgroup of $G(P_k)$ is $\lceil k/2 \rceil$. Since $G(K_{n,m})$ contains a free abelian subgroup of rank n, we have $\lceil k/2 \rceil \geqslant n$ and hence $k \geqslant 2n-1$.

(ii) Let $\ell \geqslant 2n$. Choose any k with $2n \leqslant k+1 \leqslant \ell$ (e.g. $k = \ell - 1$). Then $G(K_{n,m}) \leqslant G(P_k)$ by (i), and $G(P_k) \leqslant G(C_\ell)$ by [5, Theorem1.4(3)]. Therefore $G(K_{n,m}) \leqslant G(C_\ell)$.

Conversely, assume that $G(K_{n,m}) \leqslant G(C_\ell)$. Then the proof is similar to (i). More precisely, the maximum rank of a free abelian subgroup of $G(C_\ell)$ is $\lfloor \ell/2 \rfloor$ that is the independence number of C_ℓ. Since $G(K_{n,m})$ contains a free abelian subgroup of rank n, we have $\lfloor \ell/2 \rfloor \geqslant n$ and hence $\ell \geqslant 2n$. □

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