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# $s$-CONVEX FUNCTIONS IN THE THIRD SENSE

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Abstract. In this paper, the concept of $s$-convex function in the third sense is given. Then fundamental characterizations and some basic algebraic properties of $s$-convex function in the third sense are presented. Also, the relations between the third sense $s$-convex functions according to the different values of $s$ are examined.

## 1. Introduction

Convex functions are one of the important structures which interconnect the branches of both pure and applied mathematics such as geometry, analysis and optimization. In this context, the new findings about these functions in one aspect of mathematics have generally impact on the others, even other sciences e.g. economics. To illustrate, the quasiconvex functions which can be thought as the extension of convex functions have led to new approaches in microeconomics, game theory and equilibrium theory [14,18]. Another generalizations of convexity, $\mathbb{B}$-convexity and $\mathbb{B}^{-1}$-convexity have made the progress in same theories for abstract economies [1–3, 10, 11, 15, 28, 29]. The abstract convex functions with respect to some elementary function families, which is another approach in generalization of these functions, have made substantial contributions to optimization theory [4, 25]. As it can be seen from the examples, the generalizations and extensions of the notion of convexity can yield to notable advances. So, they have always a place in pure and applied sciences. This study may be counted as one of them.

The classical definition of convex functions on a convex subset $A$ of vector space $X$ is the statement that $f : A \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in A$ and $\lambda \in [0, 1]$.

In [20], one of the most extensive generalizations of the definition above is introduced, namely, $k$-convex sets and $(k, h)$-convex functions, as follows.

Let $k : (0, 1) \to \mathbb{R}$ and $D$ be subset of $X$. If

$$k(\lambda)x + k(1-\lambda)y \in D \tag{1}$$

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for all $x, y \in D$ and $\lambda \in (0,1)$, then $D$ is called $k$-convex set. Let $D \subseteq X$ be a $k$-convex set and let $h : (0,1) \to \mathbb{R}$ and $f : D \to \mathbb{R}$. If for all $x, y \in D$ and $\lambda \in (0,1)$,

$$f(k(\lambda)x + k(1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y) \tag{2}$$

is satisfied, then $f$ is said to be $(k,h)$-convex function. The case of $k(\lambda) = \lambda$ and $h(\lambda) = \lambda$ gives the classical definition of convex set and function. In case of the choice of $k(\lambda)$ and $h(\lambda)$ as the powers of $\lambda$, this definition yields to special kinds of convexity which attract many researchers.

By taking $k(\lambda) = \lambda^{\frac{1}{p}}$ and $h(\lambda) = \lambda^{\frac{1}{p}}$ for $0 < p \leq 1$ in (1), (2), the concepts of $p$-convex set and $p$-convex function are obtained, which have already introduced in [7, 27].

Definition 1.1. [7] Let $U \subseteq \mathbb{R}^n$ and $0 < p \leq 1$. If for each $x, y \in U$, $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, $\lambda x + \mu y \in U$, then $U$ is called a $p$-convex set in $\mathbb{R}^n$.

It is clear that any interval of real numbers including zero or accepting zero as a boundary point is a p-convex set.

Definition 1.2. [27] Let $U \subseteq \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a function. If the set

$$epi f = \left\{(x, \alpha) \in \mathbb{R}^{n+1} : x \in U,\ \alpha \in \mathbb{R},\ f(x) \leq \alpha\right\}$$

is $p$-convex set, then $f$ is called a $p$-convex function.

The following theorem gives us a characterization of $p$-convex functions:

Theorem 1.1. [27] *Let $U \subseteq \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a function. Then, $f$ is a p-convex function if and only if $U$ is a p-convex set, for all $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$ and for each $x, y \in U$*

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \tag{3}$$

*is satisfied.*

The case $k(\lambda) = \lambda^{\frac{1}{s}}$ and $h(\lambda) = \lambda$ for $0 < s \leq 1$ in (2) corresponds to the following type of $s$-convexity, which was introduced by Orlicz in [22] and was used in the theory of Orlicz spaces [19, 21, 24].

In the following definition, the concept of $s$-convex set is the same as the concept of $p$-convex set in Definition 1.1.

Definition 1.3. [22] Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a $s$-convex set. A function $f : U \to \mathbb{R}$ is said to be $s$-convex in the first sense if

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y) \tag{4}$$

for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$.

In the case $k(\lambda) = \lambda$ and $h(\lambda) = \lambda^s$ for $0 < s \leq 1$ in (2), the following type of $s$-convexity is obtained as follows:

Definition 1.4. [9] Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a convex set. A function $f : U \to \mathbb{R}$ is said to be $s$-convex in the second sense if the inequality

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y) \tag{5}$$

holds for all $x, y \in U$ and all $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$.

The classes of $s$-convex functions in first and second senses are denoted by $K_s^1$ and $K_s^2$ respectively. It can be easily seen that in the case $s = 1$, each type of $s$-convexity is reduced to the ordinary convexity of functions.

The origin of the $p$-convex sets involves the $p$-normed spaces [7,23]. Some properties involving convex hull, discrete sets and Caratheodory Theorem are found in [5–8, 17, 23], and the references therein for further reading. Also, there is a large number of studies on $s$-convex functions and their properties, relevant inequalities mainly including Hermite-Hadamard type inequalities ( [12, 13, 16, 22, 26] and the references therein).

In this study, replacing $k(\lambda) = \lambda^{\frac{1}{s}}$ and $h(\lambda) = \lambda^{\frac{1}{s^2}}$ in (2), we present a new type of convex function called the $s$-convex function in the third sense. Its basic characterizations, basic algebraic and functional properties and also relations between $s$-convex functions in the third sense with respect to different values of $s$ are examined.

## 2. $s$-Convex Function In The Third Sense And Their Properties

Definition 2.1. Let $s \in (0, 1]$ and $U \subseteq \mathbb{R}^n$ be a $s$-convex set. A function $f : U \to \mathbb{R}$ is said to be $s$-convex function in the third sense if the inequality

$$f(\lambda x + \mu y) \leq \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y) \tag{6}$$

is satisfied for all $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^s + \mu^s = 1$.

The inequality (6) is equivalent to the following inequalities:

$$f(\lambda^{\frac{1}{s}} x + (1-\lambda)^{\frac{1}{s}} y) \leq \lambda^{\frac{1}{s^2}} f(x) + (1-\lambda)^{\frac{1}{s^2}} f(y)$$

or

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^{\frac{1}{s}} f(x) + (1-\lambda^s)^{\frac{1}{s^2}} f(y)$$

where $\lambda \in [0, 1]$ and $x, y \in U$. The class of these functions is denoted by $K_s^3$. In this paper, $U \subseteq \mathbb{R}^n$ will be taken as a $s$-convex set.

Example 2.1. Let $s \in (0, 1]$ and $a, b, c \in \mathbb{R}$ with $b, c < 0$. The function

$$f(x) = \begin{cases} a, & \text{if } x = 0 \\ bx^{\frac{1}{s}} + c, & \text{if } x > 0 \end{cases}$$

is $s$-convex function in the third sense on $(0, \infty)$. By adding extra condition $c \leq a$, we can say $f$ is $s$-convex function in the third sense on $[0, \infty)$.

Assume that $x, y \in (0, \infty)$. Then $\lambda x + \mu y > 0$ with $\lambda^s + \mu^s = 1$.

$$\begin{aligned} f(\lambda x + \mu y) &= b\ (\lambda x + \mu y)^{\frac{1}{s}} + c \\ &\leq b\left(\lambda^{\frac{1}{s}} x^{\frac{1}{s}} + \mu^{\frac{1}{s}} y^{\frac{1}{s}}\right) + c \\ &= b\left(\lambda^{\frac{1}{s}} x^{\frac{1}{s}} + \mu^{\frac{1}{s}} y^{\frac{1}{s}}\right) + c\,(\lambda^s + \mu^s) \\ &\leq b\left(\lambda^{\frac{1}{s}} x^{\frac{1}{s}} + \mu^{\frac{1}{s}} y^{\frac{1}{s}}\right) + c\left(\lambda^{\frac{1}{s}} + \mu^{\frac{1}{s}}\right) \\ &= \lambda^{\frac{1}{s}}\left(bx^{\frac{1}{s}} + c\right) + \mu^{\frac{1}{s}}\left(by^{\frac{1}{s}} + c\right) \\ &= \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y) \end{aligned}$$

By considering extra condition $c \le a$, we have two cases $x = 0$, $y \ne 0$ and $x = y = 0$. For $y > x = 0$ we have

$$\begin{aligned} f(\lambda 0 + \mu y) = f(\mu y) &= b\mu^{\frac{1}{s}} y^{\frac{1}{s}} + c = b\mu^{\frac{1}{s}} y^{\frac{1}{s}} + c\,(\lambda^s + \mu^s) \\ &\le b\mu^{\frac{1}{s}} y^{\frac{1}{s}} + c\left(\lambda^{\frac{1}{s}} + \mu^{\frac{1}{s}}\right) \\ &= \lambda^{\frac{1}{s}} c + \mu^{\frac{1}{s}}\left(b y^{\frac{1}{s}} + c\right) \\ &= \lambda^{\frac{1}{s}} c + \mu^{\frac{1}{s}} f(y) \\ &\le \lambda^{\frac{1}{s}} a + \mu^{\frac{1}{s}} f(y) \\ &= \lambda^{\frac{1}{s}} f(0) + \mu^{\frac{1}{s}} f(y) \end{aligned}$$

For $y = x = 0$ we have

$$f(\lambda 0 + \mu 0) = a \le a\left(\lambda^{\frac{1}{s}} + \mu^{\frac{1}{s}}\right) = \lambda^{\frac{1}{s}} f(0) + \mu^{\frac{1}{s}} f(0).$$

Example 2.2. Let

$$U = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n |\ x_1 + \cdots + x_n \ge 0\}$$

and $k \in \mathbb{R}_+$. It can be clearly seen that $U$ is a $s$-convex set. If we define $f : U \to \mathbb{R}$ such that $f(x_1, \cdots, x_n) = -k(x_1 + \cdots + x_n)$, then $f \in K_s^3$. Because, we have $\lambda, \mu \ge 0$ with $\lambda^s + \mu^s = 1$, it can be written

$$\begin{aligned} f(\lambda(x_1, \cdots, x_n) \ &+\ \mu(y_1, \cdots, y_n)) = -k(\lambda x_1 + \mu y_1 + \cdots + \lambda x_n + \mu y_n) \\ &=\ \lambda(-k)(x_1 + \cdots + x_n) + \mu(-k)(y_1 + \cdots + y_n) \\ &\le\ \lambda^{\frac{1}{s}}(-k)(x_1 + \cdots + x_n) + \mu^{\frac{1}{s}}(-k)(y_1 + \cdots + y_n) \\ &=\ \lambda^{\frac{1}{s}} f(x_1 + \cdots + x_n) + \mu^{\frac{1}{s}} f(y_1 + \cdots + y_n). \end{aligned}$$

So, it is obtained that $f \in K_s^3$.

Theorem 2.1. *Let $f : U \to \mathbb{R}_+$. If $f \in K_s^3$, then $f$ is a $s$-convex function (where, the concept of $s$-convex function is the same as the concept of $p$-convex function in Definition 1.2).*

*Proof.* Let $x, y \in U$ and $\lambda, \mu \ge 0$ such that $\lambda^s + \mu^s = 1$. Then, we have

$$f(\lambda x + \mu y) \le \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y) \le \lambda f(x) + \mu f(y).$$

□

Theorem 2.2. *Let $f : U \to \mathbb{R}_+$ be a function. If $f \in K_s^3$, then $f \in K_s^1$.*

*Proof.* Let $x, y \in U$ and $\lambda, \mu \ge 0$ such that $\lambda^s + \mu^s = 1$. Then, we have

$$f(\lambda x + \mu y) \le \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y) \le \lambda^s f(x) + \mu^s f(y).$$

□

Theorem 2.3. *Let $f : U \to \mathbb{R}_+$ be a function. If $f \in K_s^3$, then the inequality*

$$f(\frac{x + y}{2^{\frac{1}{s}}}) \le \frac{f(x) + f(y)}{2^{\frac{1}{s}}}$$

*is valid for all $x, y \in U$.*

*Proof.* From $s \in (0,1]$, it is clear that $\frac{1}{2^{\frac{1}{s^2}}} \leq \frac{1}{2^{\frac{1}{s}}}$. Thus, the inequalities

$$f(\frac{x+y}{2^{\frac{1}{s}}}) \leq \frac{f(x)+f(y)}{2^{\frac{1}{s^2}}} \leq \frac{f(x)+f(y)}{2^{\frac{1}{s}}}$$

are obtained. □

THEOREM 2.4. *Let $f : U \to \mathbb{R}$ be a function and the function $g : [0,1] \to \mathbb{R}$ define by $g(t) = f(tx + (1-t^s)^{\frac{1}{s}} y)$. If $g \in K_s^3$, then $f \in K_s^3$.*

*Proof.* Let $x, y \in U$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} f\left(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y\right) = g(\lambda) &= g\left(\lambda \cdot 1 + (1-\lambda^s)^{\frac{1}{s}} \cdot 0\right) \\ &\leq \lambda^{\frac{1}{s}} g(1) + (1-\lambda^s)^{\frac{1}{s^2}} g(0) \\ &= \lambda^{\frac{1}{s}} f(x) + (1-\lambda^s)^{\frac{1}{s^2}} f(y). \end{aligned}$$

□

THEOREM 2.5. *Let $f_i : U \to \mathbb{R}$ be functions and $f : U \to \mathbb{R}$ define by $f(x) = \sum_{i=1}^{m} a_i f_i(x)$ where $a_i \geq 0$. If $f_i \in K_s^3$ for $i = 1, 2, \ldots, m$, then $f \in K_s^3$.*

*Proof.* For $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^s + \mu^s = 1$, we have

$$\begin{aligned} f(\lambda x + \mu y) &= \sum_{i=1}^{m} a_i f_i(\lambda x + \mu y) \\ &\leq \sum_{i=1}^{m} a_i \left(\lambda^{\frac{1}{s}} f_i(x) + \mu^{\frac{1}{s}} f_i(y)\right) \\ &= \lambda^{\frac{1}{s}} \sum_{i=1}^{m} a_i f_i(x) + \mu^{\frac{1}{s}} \sum_{i=1}^{m} a_i f_i(y) \\ &= \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y). \end{aligned}$$

This shows that $f \in K_s^3$. □

THEOREM 2.6. *Let $f_i : U \to \mathbb{R}$ be functions and $f : U \to \mathbb{R}$ define by $f(x) = \max_{1 \leq i \leq m} \{f_i(x)\}$. If $f_i \in K_s^3$ for $i = 1, 2, \ldots, m$, then $f \in K_s^3$.*

*Proof.* For each $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^s + \mu^s = 1$,

$$\begin{aligned} f(\lambda x + \mu y) &= \max_{1 \leq i \leq m} \{f_i(\lambda x + \mu y)\} \\ &\leq \max_{1 \leq i \leq m} \left\{\lambda^{\frac{1}{s}} f_i(x) + \mu^{\frac{1}{s}} f_i(y)\right\} \\ &\leq \max_{1 \leq i \leq m} \left\{\lambda^{\frac{1}{s}} f_i(x)\right\} + \max_{1 \leq i \leq m} \left\{\mu^{\frac{1}{s}} f_i(y)\right\} \\ &\leq \lambda^{\frac{1}{s}} \max_{1 \leq i \leq m} \{f_i(x)\} + \mu^{\frac{1}{s}} \max_{1 \leq i \leq m} \{f_i(y)\} \\ &= \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y). \end{aligned}$$

Thus, $f = \max\limits_{1\le i\le m} \{f_i\} \in K_s^3$. □

THEOREM 2.7. *Let $f : U \to \mathbb{R}_+$ be a function. If $f \in K_s^3$, then any local minimum of $f$ is a global minimum.*

*Proof.* Let $x^*$ be a local minimum of $f$. Assume the contrary, that is, $f(y) < f(x^*)$, for some $y \in U$. Since $f \in K_s^3$, for all $\lambda, \mu \ge 0$ such that $\lambda^s + \mu^s = 1$, we have

$$\begin{aligned} f(\lambda x^* + \mu y) &\le \lambda^{\frac{1}{s}} f(x^*) + \mu^{\frac{1}{s}} f(y) \\ &\le \lambda^s f(x^*) + \mu^s f(y) \\ &\le (1-\mu^s) f(x^*) + \mu^s f(y) \\ &\le f(x^*) + \mu^s (f(y) - f(x^*)) \\ &< f(x^*) \end{aligned}$$

for some small $\mu > 0$. This contradicts with $x^*$ being a local minimum point. Hence every local minimum of $f$ is a global minimum. □

The Jensen inequality for $s$-convex functions in the third sense is given in the following theorem.

THEOREM 2.8. *Let $f : U \to \mathbb{R}$ be a function, $x_1, \ldots, x_m \in U$ and $\lambda_1, \ldots, \lambda_m \ge 0$ with $\lambda_1^s + \cdots + \lambda_m^s = 1$. If $f \in K_s^3$, then*

$$f(\lambda_1 x_1 + \cdots + \lambda_m x_m) \le \lambda_1^{\frac{1}{s}} f(x_1) + \cdots + \lambda_m^{\frac{1}{s}} f(x_m).$$

*Proof.* Induction on $m$ will be used in proof. The inequality is trivially true when $m = 2$. Assume that it is true when $m = k$, where $k > 2$. Now we show the validity for $m = k+1$. Let $x$ be defined by the equation

$$x = \lambda_1 x_1 + \cdots + \lambda_{k+1} x_{k+1}$$

where $x_1, \ldots, x_{k+1} \in U$, $\lambda_1, \ldots, \lambda_{k+1} \ge 0$ with $\lambda_1^s + \cdots + \lambda_{k+1}^s = 1$. At least one of $\lambda_1, \ldots, \lambda_{k+1}$ must be less than 1. Let us say $\lambda_{k+1} < 1$ and write

$$\lambda_1^s + \cdots + \lambda_k^s = 1 - \lambda_{k+1}^s.$$

One can find $\lambda_* < 1$ such that $\lambda_1^s + \cdots + \lambda_k^s = \lambda_*^s$. Using $\left(\frac{\lambda_1}{\lambda_*}\right)^s + \cdots + \left(\frac{\lambda_k}{\lambda_*}\right)^s = 1$ and the assumption of hypothesis, we get

$$f\left(\frac{\lambda_1}{\lambda_*} x_1 + \cdots + \frac{\lambda_k}{\lambda_*} x_k\right) \le \left(\frac{\lambda_1}{\lambda_*}\right)^{\frac{1}{s}} f(x_1) + \cdots + \left(\frac{\lambda_k}{\lambda_*}\right)^{\frac{1}{s}} f(x_k).$$

Thus,

$$\begin{aligned} f(x) &= f\left(\lambda_*\left(\tfrac{\lambda_1}{\lambda_*} x_1 + \cdots + \tfrac{\lambda_k}{\lambda_*} x_k\right) + \lambda_{k+1} x_{k+1}\right) \\ &\le \lambda_*^{\frac{1}{s}} f\left(\tfrac{\lambda_1}{\lambda_*} x_1 + \cdots + \tfrac{\lambda_k}{\lambda_*} x_k\right) + \lambda_{k+1}^{\frac{1}{s}} f(x_{k+1}) \\ &\le \lambda_1^{\frac{1}{s}} f(x_1) + \cdots + \lambda_{k+1}^{\frac{1}{s}} f(x_{k+1}) \end{aligned}$$

is obtained. This completes the proof by induction. □

THEOREM 2.9. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function and $g : U \to \mathbb{R}_+$ be a function. If $f, g \in K_s^3$, then $f \circ g \in K_s^3$.*

*Proof.* Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^s + \mu^s = 1$.

$$\begin{aligned}
(f \circ g)(\lambda x + \mu y) &= f(g(\lambda x + \mu y)) \\
&\leq f(\lambda^{\frac{1}{s}} g(x) + \mu^{\frac{1}{s}} g(y)) \\
&\leq f(\lambda g(x) + \mu g(y)) \\
&\leq \lambda^{\frac{1}{s}} f(g(x)) + \mu^{\frac{1}{s}} f(g(y)) \\
&= \lambda^{\frac{1}{s}} (f \circ g)(x) + \mu^{\frac{1}{s}} (f \circ g)(y).
\end{aligned}$$

Hence, $f \circ g \in K_s^3$. □

Theorem 2.10. *If $f \in K_s^3$ and $g : \mathbb{R} \to \mathbb{R}$ is an increasing linear function, then $g \circ f \in K_s^3$.*

*Proof.* Let $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^s + \mu^s = 1$.

$$\begin{aligned}
(g \circ f)(\lambda x + \mu y) &= g(f(\lambda x + \mu y)) \\
&\leq g(\lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y)) \\
&= \lambda^{\frac{1}{s}} g(f(x)) + \mu^{\frac{1}{s}} g(f(y)) \\
&= \lambda^{\frac{1}{s}} (g \circ f)(x) + \mu^{\frac{1}{s}} (g \circ f)(y).
\end{aligned}$$

Hence, $g \circ f \in K_s^3$. □

Theorem 2.11. *Let $g : U \to V$ be a linear transformation and $f : V \to \mathbb{R}$ be a function. If $f \in K_s^3$, then $f \circ g \in K_s^3$.*

*Proof.* Let $\lambda, \mu \geq 0$ such that $\lambda^s + \mu^s = 1$. Thus, we get

$$\begin{aligned}
(f \circ g)(\lambda x + \mu y) &= f(g(\lambda x + \mu y)) \\
&= f(\lambda g(x) + \mu g(y)) \\
&\leq \lambda^{\frac{1}{s}} f(g(x)) + \mu^{\frac{1}{s}} f(g(y)) \\
&= \lambda^{\frac{1}{s}} (f \circ g)(x) + \mu^{\frac{1}{s}} (f \circ g)(y)
\end{aligned}$$

for all $x, y \in U$. Hence, $f \circ g \in K_s^3$. □

Theorem 2.12. *Let $f : U \to \mathbb{R}$ be a function and $f \in K_s^3$. Then, the inequality* (6) *holds for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s \leq 1$ if and only if $f(0) \leq 0$.*

*Proof.* Necessity is obvious by taking $\lambda = \mu = 0$. Therefore, assume that $x, y \in U$ and $\lambda, \mu \geq 0$ and $0 < \gamma < 1$ where $\gamma = \lambda^s + \mu^s$. Put $\alpha = \lambda \gamma^{-\frac{1}{s}}$ and $\beta = \mu \gamma^{-\frac{1}{s}}$. Then

$$\alpha^s + \beta^s = \frac{\lambda^s}{\gamma} + \frac{\mu^s}{\gamma} = 1$$

and hence we have sufficiency:

$$\begin{aligned} f(\lambda x+\mu y) &= f\left(\alpha\gamma^{\frac{1}{s}}x+\beta\gamma^{\frac{1}{s}}y\right) \\ &\le \alpha^{\frac{1}{s}}f(\gamma^{\frac{1}{s}}x)+\beta^{\frac{1}{s}}f(\gamma^{\frac{1}{s}}y) \\ &= \alpha^{\frac{1}{s}}f(\gamma^{\frac{1}{s}}x+(1-\gamma)^{\frac{1}{s}}.0)+\beta^{\frac{1}{s}}f(\gamma^{\frac{1}{s}}y+(1-\gamma)^{\frac{1}{s}}.0) \\ &\le \alpha^{\frac{1}{s}}\left[\gamma^{\frac{1}{s^2}}f(x)+(1-\gamma)^{\frac{1}{s^2}}f(0)\right]+\beta^{\frac{1}{s}}\left[\gamma^{\frac{1}{s^2}}f(y)+(1-\gamma)^{\frac{1}{s^2}}f(0)\right] \\ &= \alpha^{\frac{1}{s}}\gamma^{\frac{1}{s^2}}f(x)+\beta^{\frac{1}{s}}\gamma^{\frac{1}{s^2}}f(y)+(\alpha^{\frac{1}{s}}+\beta^{\frac{1}{s}})(1-\gamma)^{\frac{1}{s^2}}f(0) \\ &\le \alpha^{\frac{1}{s}}\gamma^{\frac{1}{s^2}}f(x)+\beta^{\frac{1}{s}}\gamma^{\frac{1}{s^2}}f(y) \\ &= \lambda^{\frac{1}{s}}f(x)+\mu^{\frac{1}{s}}f(y). \end{aligned}$$

□

THEOREM 2.13. *Let $0 < s_1 \le s_2 \le 1$, $f : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function, $g : U \to \mathbb{R}_+$ be a function and $g(0) = 0$. If $f \in K^3_{s_1}$ and $g \in K^3_{s_2}$, then $f \circ g \in K^3_{s_1}$.*

*Proof.* Let $x, y \in U$ and $\lambda, \mu \ge 0$ with $\lambda^{s_1} + \mu^{s_1} = 1$. According to Theorem 2.12 and using $\lambda^{s_2} + \mu^{s_2} \le \lambda^{s_1} + \mu^{s_1} = 1$, we have

$$\begin{aligned} (f\circ g)(\lambda x+\mu y) &= f(g(\lambda x+\mu y)) \\ &\le f(\lambda^{\frac{1}{s_2}}g(x)+\mu^{\frac{1}{s_2}}g(y)) \\ &\le f(\lambda g(x)+\mu g(y)) \\ &\le \lambda^{\frac{1}{s_1}}(f\circ g)(x)+\mu^{\frac{1}{s_1}}(f\circ g)(y), \end{aligned}$$

which means that $f \circ g \in K^3_{s_1}$. □

THEOREM 2.14. *Let $0 < s_1 \le s_2 \le 1$. If $f : U \to (-\infty, 0]$ and $f \in K^3_{s_2}$, then $f \in K^3_{s_1}$.*

*Proof.* Let $f \in K^3_{s_2}$, $x, y \in U$ and $\lambda, \mu \ge 0$ with $\lambda^{s_1} + \mu^{s_1} = 1$. Then we have

$$\lambda^{s_2} + \mu^{s_2} \le \lambda^{s_1} + \mu^{s_1} = 1.$$

Since $U$ is $s$-convex set, it includes the origin. According to Theorem 2.12, we have

$$f(\lambda x+\mu y) \le \lambda^{\frac{1}{s_2}}f(x)+\mu^{\frac{1}{s_2}}f(y) \le \lambda^{\frac{1}{s_1}}f(x)+\mu^{\frac{1}{s_1}}f(y).$$

That means $f \in K^3_{s_1}$. □

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