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REDUCED PROPERTY OVER IDEMPOTENTS

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Abstract. This article concerns the property that for any element a in a ring, if $a^{2n} = a^n$ for some $n \geq 2$ then $a^2 = a$. The class of rings with this property is large, but there also exist many kinds of rings without that, for example, rings of characteristic $\neq 2$ and finite fields of characteristic ≥ 3. Rings with such a property is called *reduced-over-idempotent*. The study of reduced-over-idempotent rings is based on the fact that the characteristic is 2 and every nonzero non-identity element generates an infinite multiplicative semigroup without identity. It is proved that the reduced-over-idempotent property pass to polynomial rings, and we provide power series rings with a partial affirmative argument. It is also proved that every finitely generated subring of a locally finite reduced-over-idempotent ring is isomorphic to a finite direct product of copies of the prime field $\{0, 1\}$. A method to construct reduced-over-idempotent fields is also provided.

1. Reduced-over-idempotent rings

Throughout this note every ring is an associative ring with identity unless otherwise stated. A nilpotent element is also said to be a *nilpotent* for short. Let R be a ring. We denote the center, the set of all nilpotents, the set of all idempotents, the group of all units, and the Jacobson radical of R by $Z(R)$, $N(R)$, $Id(R)$, $U(R)$, and $J(R)$, respectively. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). The characteristic of R is written by $Ch(R)$. Let $a \in R$. The right (resp., left) annihilator of a in R is denoted by $r_R(a)$ (resp., $l_R(a)$). a is called *right* (resp., *left*) *regular* if $r_R(a) = 0$ (resp., $l_R(a) = 0$); and a is called *regular* if a is both right and left regular. For $S \subseteq R$, $|S|$ denotes the cardinality of S. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$.

A ring is usually called *reduced* if it has no nonzero nilpotents. It is easily proved that a ring R is reduced if and only if $a^2 = 0$ for $a \in R$ implies $a = 0$. A ring is usually called *Abelian* if every idempotent is central. Reduced rings are easily shown

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to be Abelian, but there exist many non-reduced rings which are Abelian (e.g., $D_2(R)$ over a commutative ring R).

Recall that a ring is called *locally finite* [8] if every finite subset in it generates a finite semigroup multiplicatively. It is obvious that every locally finite ring is of finite characteristic. It is obtained by [7, Theorem 2.2(1)] that a ring is locally finite if and only if every subring generated by a finite subset is finite. Finite rings are clearly locally finite, and an algebraic closure of a finite field is locally finite but not finite. Note that if a ring R is locally finite, then for any $r \in R$ there exists $n = n(r) \geq 1$ such that $r^n \in Id(R)$ (see the proof of [8, Proposition 16]). Here, r need not be an idempotent. It is clear that for any ring A and $a \in A$, $a \in Id(A)$ implies $a^k \in Id(A)$ for all $k \geq 1$.

Based on these facts, we introduce a new ring property.

Definition 1.1. A ring R is said to be *reduced-over-idempotent* provided that for any $a \in R$, $a^n \in Id(R)$ for some $n \geq 1$ implies $a \in Id(R)$.

The following consists of basic properties of reduced-over-idempotent rings which are essential for our study.

Lemma 1.2. *For a reduced-over-idempotent ring R, we have the following assertions.*

(1) *R is reduced.*

(2) *$Ch(R) = 2$ and then R is an algebra over $\mathbb{Z}_2$.*

(3) *Every non-identity regular element in R forms an infinite multiplicative semigroup without identity.*

(4) *If R is locally finite, then R is Boolean.*

(5) *If R is locally finite, then $U(R) = \{1\}$.*

Proof. (1) Let $a^2 = 0$ for $a \in R$. Then $a \in Id(R)$ since R is reduced-over-idempotent, so that $a = a^2 = 0$. Thus R is reduced.

(2) Since R is reduced-over-idempotent, $(-1)^2 = 1$ implies $-1 \in Id(R)$, so that $-1 = (-1)^2 = 1$. Thus $Ch(R) = 2$.

(3) Let a be a non-identity regular element in R. Consider the multiplicative semigroup $S = \{a^n \mid n \geq 1\}$ generated by a. Assume $a^{k_1} = a^{k_2}$ for some $k_1 \neq k_2$. Then $a^h = 1$ for some $h \geq 1$ since a is regular. Here, since R is reduced-over-idempotent, we get $a \in Id(R)$ and hence the regularity of a implies $a = 1$, contrary to $a \neq 1$. Therefore S is an infinite multiplicative semigroup without identity.

(4) and (5) Let R be locally finite. Then, for any $a \in R$, there exists $m \geq 1$ such that $a^m \in Id(R)$ by the proof of [8, Proposition 16]. Thus $a \in Id(R)$ because R is reduced-over-idempotent, showing that R is Boolean.

Next, for $u \in U(R)$, we must get $u = 1$ by the preceding argument, as desired. □

The class of reduced-over-idempotent rings is seated between Boolean rings and reduced rings by Lemma 1.2(1, 4). From Lemma 1.2(3), we obtain an equivalent condition of reduced-over-idempotent domains.

Theorem 1.3. (1) *Let R be a domain. Then R is reduced-over-idempotent if and only if every non-identity regular element forms an infinite multiplicative semigroup without identity.*

(2) *Every free algebra over $\mathbb{Z}_2$ is reduced-over-idempotent.*

(3) *Let R be a locally finite reduced-over-idempotent ring. Then every finitely generated subring of R is isomorphic to a finite direct product of copies of $\mathbb{Z}_2$.*

Proof. (1) It suffices to show the sufficiency by Lemma 1.2(3). Assume the necessity and let $0 \neq a \in R$ such that $a^n \in Id(R)$ for some $n \geq 1$. Then $a^n = 1$ since R is a domain, so that we must have $a = 1$ by assumption. Thus R is reduced-over-idempotent.

(2) Let R be a free algebra over $\mathbb{Z}_2$. Then R is a domain such that $U(R) = \{1\}$ and every non-identity regular element forms an infinite multiplicative semigroup without identity. So R is reduced-over-idempotent by (1).

(3) Let S be a finitely generated subring of R. Then S is finite since R is locally finite, and hence S is isomorphic to a finite direct product of $Mat_{n_i}(F_i)$'s for some finite fields F_i and positive integers n_i by the Wedderburn-Artin theorem. Moreover S is also reduced-over-idempotent by Proposition 1.5(1) below, and then S is reduced by Lemma 1.2(1). From this we see that S is isomorphic to a finite direct product of F_i's. But every F_i must coincide with $\mathbb{Z}_2$ by Lemma 1.2(5), and therefore S is isomorphic to a finite direct product of copies of $\mathbb{Z}_2$. □

The arguments below elaborate upon Lemma 1.2 and Theorem 1.3.

Remark 1.4. (1) Fields need not be reduced-over-idempotent. For example, consider the field $\mathbb{C}$ of complex numbers. Then $\mathbb{C}$ is not reduced-over-idempotent by Lemma 1.2(2), since $Ch(\mathbb{C}) = 0$. Moreover, it implies that every subring of $\mathbb{C}$ cannot be reduced-over-idempotent.

Assume that a field F is reduced-over-idempotent. If F is finite, then $F \cong \mathbb{Z}_2$ by Lemma 1.2(4), so that every finite field E with $|E| \geq 3$ cannot be reduced-over-idempotent; for example, the Galois field $GF(2^k)$ with $k \geq 2$.

(2) Let $R = \mathbb{Z}_2\langle X\rangle$ be a free algebra generated by a set X over $\mathbb{Z}_2$. Then R is a reduced-over-idempotent domain by Theorem 1.3(2). If $|X| = 1$, then $R \cong \mathbb{Z}_2[x]$. If $|X| \geq 2$, then $Z(R) = \mathbb{Z}_2$ by the proof of [2, Proposition 1.3(7)].

(3) Note that Boolean rings are obviously reduced-over-idempotent but not conversely. Indeed, let $R = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over $\mathbb{Z}_2$. Then R is reduced-over-idempotent by Theorem 1.3(2), but R is not Boolean clearly.

(4) Any of $Mat_n(R)$, $T_n(R)$ and $D_n(R)$, over any ring R for $n \geq 2$, cannot be reduced-over-idempotent because they are not reduced.

The following properties of reduced-over-idempotent rings do basic roles throughout this article.

Proposition 1.5. (1) *The class of reduced-over-idempotent rings is closed under subrings.*

(2) *For a family $\{R_\gamma \mid \gamma \in \Gamma\}$ of rings, the following statements are equivalent:*

(i) *R_γ is reduced-over-idempotent;*

(ii) *The direct product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ is reduced-over-idempotent;*

(iii) *The direct sum $\oplus_{\gamma\in\Gamma} R_\gamma$ of R_γ is reduced-over-idempotent.*

(3) *Let R be an Abelian ring and $e \in Id(R)$. Then R is reduced-over-idempotent if and only if both eR and $(1-e)R$ are reduced-over-idempotent.*

Proof. (1) Note that $Id(S) = Id(R) \cap S$ for any subring S of a ring R.

(2) The proof comes from (1) and the fact that $Id(\prod_{\gamma\in\Gamma} R_\gamma) = \prod_{\gamma\in\Gamma} Id(R_\gamma)$ and $Id(\oplus_{\gamma\in\Gamma} R_\gamma) = \oplus_{\gamma\in\Gamma} Id(R_\gamma)$.

(3) This follows (2), since $R \cong eR \oplus (1-e)R$. □

Related to Proposition 1.5(1), one may ask whether the class of reduced-over-idempotent rings is closed under homomorphic images. But the answer is negative as follows. We use the construction in [1, Example 4.8]. Consider the reduced-over-idempotent ring $R = \mathbb{Z}_2\langle a, b\rangle$ as in Remark 1.4(3). Let J be the ideal of R generated by b^2 and $\bar{r} = r + J$ for $r \in R$. Then R/J is not reduced-over-idempotent by Lemma 1.2(1) because it is not reduced; indeed, $\bar{b}^2 = \bar{0}$ but $\bar{b} \neq \bar{0}$.

On the other hand, there exists a ring whose nontrivial factor rings are reduced-over-idempotent, but the ring is not reduced-over-idempotent. Consider the ring $R = T_2(\mathbb{Z}_2)$ which is not reduced-over-idempotent by Remark 1.4(4). Note that $\mathbb{Z}_2$ is obviously reduced-over-idempotent, and hence $\mathbb{Z}_2\times\mathbb{Z}_2$ is also reduced-over-idempotent by Proposition 1.5(2). All nontrivial factor rings of R are $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $R/J \cong Z_2$, and $R/K \cong \mathbb{Z}_2$; hence these are reduced-over-idempotent, where $I = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$, $J = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$, and $K = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$.

A ring R is called a *subdirect product* of a family of rings $\{R_\gamma \mid \gamma \in \Gamma\}$ if there is a monomorphism $f : R \to \prod_{\gamma\in\Gamma} R_\gamma$ such that $\pi_\gamma \circ f$ is onto for all $\gamma \in \Gamma$, where $\pi_\gamma : \prod_{\gamma\in\Gamma} R_\gamma \to R_\gamma$ is the canonical epimorphism. The following is another application of Proposition 1.5(2).

Proposition 1.6. *A subdirect product of reduced-over-idempotent rings is reduced-over-idempotent.*

Proof. Let R be a subdirect product of a family $\{R_\gamma \mid \gamma \in \Gamma\}$ of reduced-over-idempotent rings. Then $f(Id(R)) \subseteq Id(\prod_{\gamma\in\Gamma} R_\gamma) = \prod_{\gamma\in\Gamma} Id(R_\gamma)$ clearly. Suppose that for $a \in R$ there exists $n \geq 1$ such that $a^n \in Id(R)$. Then $f(a)^n = f(a^n) \in Id(\prod_{\gamma\in\Gamma} R_\gamma)$. Since every R_γ is reversible-over-idempotent, $\prod_{\gamma\in\Gamma} R_\gamma$ is reversible-over-idempotent by Proposition 1.5(2). So $f(a)^n \in Id(\prod_{\gamma\in\Gamma} R_\gamma)$ implies $f(a) \in Id(\prod_{\gamma\in\Gamma} R_\gamma)$. There exists $e_\gamma \in Id(R_\gamma)$ for each $\gamma \in \Gamma$ such that $f(a) = (e_\gamma)_{\gamma\in\Gamma}$. Then $f(a^2) = (f(a))^2 = [(e_\gamma)_{\gamma\in\Gamma}]^2 = (e_\gamma)_{\gamma\in\Gamma} = f(a)$ and hence $a^2 = a$, since f is injective. Thus $a \in Id(R)$. Therefore R is reduced-over-idempotent. □

Recall that a ring R is called *local* if $R/J(R)$ is a division ring. A ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. One-sided Artinian rings are clearly semiperfect. Local rings are Abelian and semilocal.

Proposition 1.7. *A ring R is reduced-over-idempotent and semiperfect if and only if R is a finite direct product of local reduced-over-idempotent rings.*

Proof. Suppose that R is reduced-over-idempotent and semiperfect. Then R is Abelian because R is reduced by Lemma 1.2(1). Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [12, Proposition 3.7.2], say $R = \sum_{i=1}^n e_iR$ such that each e_iRe_i is a local ring. Since R is Abelian, each

e_iR is an ideal of R with $e_iR = e_iRe_i$. But each e_iR is also a reduced-over-idempotent ring by Proposition 1.5(3).

Conversely assume that R is a finite direct product of local reduced-over-idempotent rings. Then R is Abelian and semiperfect since local rings are semiperfect by [12, Corollary 3.7.1], and moreover R is reduced-over-idempotent by Proposition 1.5(2). □

We see an application of Proposition 1.7.

Corollary 1.8. *Let R be a reduced-over-idempotent ring. If R is right Artinian then R is a finite direct product of division rings.*

Proof. Let R be right Artinian. Then $J(R)$ is nilpotent, and hence $J(R) = 0$ because R is reduced by Lemma 1.2(1). Moreover R is a finite direct product of local reduced-over-idempotent rings by Proposition 1.7, $R = \sum_{i=1}^{n} R_i$. Note $J(R_i) = 0$ since R_i is right Artinian and R_i is reduced. This implies that there exist a finite number of division rings D_i's such that R is isomorphic to the direct product of D_i's. □

Corollary 1.8 can be obtained also by using the Wedderburn-Artin theorem.

2. Extensions

In this section, we study the reduced-over-idempotent ring property of several kinds of extensions, concentrating on polynomial rings and power series rings. $R[x; x^{-1}]$ means the *Laurent polynomial ring* with an indeterminate x over a ring R.

Lemma 2.1. (1) [10, Lemma 8] *For an Abelian ring R, we have that $Id(R) = Id(R[x]) = Id(R[[x]])$ and that both $R[x]$ and $R[[x]]$ are Abelian.*

(2) *Let R be a reduced ring. Then $Id(R[x; x^{-1}]) = Id(R)$.*

Proof. (2) Let $f(x) \in Id(R[x; x^{-1}])$ for $0 \neq f(x) = \sum_{i=m}^{n} a_i x^i \in R[x; x^{-1}]$, where $m \in \mathbb{Z}$, $a_m \neq 0$ and $a_n \geq 0$. If $m \leq -1$ then $a_m^2 \neq 0$ implies $f(x)^2 = a_m^2 x^{-2m} + \cdots \neq f(x)$, entailing $m \geq 0$. Next if $n \geq 1$ then $a_n^2 \neq 0$ implies $f(x)^2 = \cdots + a_n^2 x^{2n} \neq f(x)$, entailing $n = 0$. Consequently $f(x) = a_0$ and $a_0^2 = a_0$ follows. □

The preceding lemma does an essential role in the proposition and remark below.

Proposition 2.2. *For a ring R, the following conditions are equivalent:*

(1) *R is reduced-over-idempotent;*

(2) *$R[x]$ is reduced-over-idempotent;*

(3) *$R[x; x^{-1}]$ is reduced-over-idempotent.*

Proof. It suffices to show (1) ⇒ (3) by Proposition 1.5(1). Let R be reduced-over-idempotent. Then R is reduced by Lemma 1.2(1). Suppose that $f(x)^k \in Id(R[x; x^{-1}])$ for $0 \neq f(x) = \sum_{i=m}^{n} a_i x^i \in R[x; x^{-1}]$ and $k \geq 1$, where $m \in \mathbb{Z}$. Then $f(x)^k = e$ for some $e \in Id(R)$ by Lemma 2.1(2). By the reducedness of R, we must get $f(x) = a_0$. This entails $a_0^k = e$. But since R is reduced-over-idempotent, $a_0 \in Id(R)$ and $a_0 = e$ follows. Thus $R[x; x^{-1}]$ is reduced-over-idempotent. □

From Theorem 1.3(1) and Proposition 2.2, we can obtain reduced-over-idempotent fields. For example, let $F = \mathbb{Z}_2(x)$, the quotient field of $\mathbb{Z}_2[x]$, a reduced-over-idempotent domain by Proposition 2.2. Taking $f \in E$ such that $f \neq 1$ and $f \neq 0$, we have that $\{f^n \mid n \geq 1\}$ is an infinite multiplicative semigroup without identity. Thus E is reduced-over-idempotent by Theorem 1.3(1).

Considering the preceding proposition, one may ask whether the reduced-over-idempotent property also go up to power series rings. We do not know the complete answer, but we provide a partial one for this question as follows.

Remark 2.3. Let R be a reduced-over-idempotent ring. Then R is reduced (hence Abelian) and $Ch(R) = 2$ by Lemma 1.2(1, 2). We will use these facts and Lemma 2.1(1) freely in the following computation.

Let $0 \neq f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ be such that $f(x)^m \in Id(R[[x]])$ for some $m \geq 1$. Then $f(x)^m = e = a_0$ by the proof of Proposition 2.2. Write ${}_mC_k = \frac{m(m-1)\cdots(m-(k-1))}{k(k-1)\cdots 2} = \frac{m!}{(m-k)!k!}$ for $1 \leq k \leq m$. Note that ${}_mC_k$ is an integer and that there exist even m's such that ${}_mC_k$ is odd for some $1 \leq k \leq m-1$, for example, ${}_6C_2$, ${}_{14}C_2$ and ${}_{14}C_4$.

(i) Let $m = 2$. The coefficient of the term of degree 2 of $f(x)^2$ is $0 = 2a_0a_2 + a_1^2 = a_1^2$, so that $a_1 = 0$. From this we see that the coefficient of the term of degree 2^2 of $f(x)^2$ is $0 = 2a_0a_4 + a_2^2 = a_2^2$, so that $a_2 = 0$. Inductively assume that $a_1 = \cdots = a_{k-1} = 0$. Then the coefficient of the term of degree k^2 in $f(x)^2$ is

$$0 = 2a_0a_{2k} + a_k^2 = a_k^2,$$

so that $a_k = 0$. Therefore we now have that $a_i = 0$ for all $i \geq 1$, concluding $f(x) = a_0 \in Id(R[[x]])$.

(ii) Let $m = 3$. The coefficient of the term of degree 1 of $f(x)^3$ is $0 = 3a_0a_1 = a_0a_1$. The coefficient of the term of degree 2 of $f(x)^3$ is $0 = 3a_0a_2 + 3a_0a_1^2 = a_0a_2$. The coefficient of the term of degree 3 of $f(x)^3$ is $0 = 3a_0a_3 + 3a_0a_1a_2 + 3a_0a_2a_1 + a_1^3 = a_0a_3 + a_1^3$. Multiplying this equality by a_0, we get $0 = a_0a_3 + a_0a_1^3 = a_0a_3$. Inductively assume that $a_0a_i = 0$ for $i = 1, \ldots, k-1$. Then the coefficient of the term of degree k of $f(x)^3$ is

$$0 = 3a_0a_k + \sum_{s_1+s_2+s_3=k \text{ and } s_i<k} a_{s_1}a_{s_2}a_{s_3} = a_0a_k + \sum_{s_1+s_2+s_3=k \text{ and } s_i<k} a_{s_1}a_{s_2}a_{s_3}.$$

Multiplying this equality by a_0, we get

$$0 = a_0a_k + a_0 \sum_{s_1+s_2+s_3=k \text{ and } s_i<k} a_{s_1}a_{s_2}a_{s_3} = a_0a_k + \sum_{s_1+s_2+s_3=k \text{ and } s_i<k} a_0a_{s_1}a_{s_2}a_{s_3} = a_0a_k$$

by assumption. Hence $a_0a_i = 0$ for all $i \geq 1$.

Next we will show that $a_i = 0$ for all i. From the equality $0 = a_0a_3 + a_1^3 = a_1^3$, we obtain $a_1 = 0$. The coefficient of the term of degree 6 of $f(x)^3$ is

$$0 = 3a_0a_6 + a_2^3 + \sum_{s_1+s_2+s_3=6 \text{ and } s_i<6} a_{s_1}a_{s_2}a_{s_3} = a_2^3 + \sum_{s_1+s_2+s_3=6 \text{ and } s_i<6} a_{s_1}a_{s_2}a_{s_3}.$$

But some s_i is either 0 or 1, hence $\sum_{s_1+s_2+s_3=6 \text{ and } s_i<6} a_{s_1}a_{s_2}a_{s_3} = 0$ by the results above, entailing $a_2^3 = 0$. Thus $a_2 = 0$.

Now inductively we assume that $a_i = 0$ for $i = 1, \ldots, k-1$. The coefficient of the term of degree $3k$ in $f(x)^3$ is

$$0 = 3a_0a_{3k} + a_k^3 + \sum_{s_1+s_2+s_3=3k \text{ and } s_i<3k} a_{s_1}a_{s_2}a_{s_3} = a_k^3 + \sum_{s_1+s_2+s_3=3k \text{ and } s_i<3k} a_{s_1}a_{s_2}a_{s_3}.$$

But some s_i is seated in $[0, k-1]$, hence $\sum_{s_1+s_2+s_3=3k \text{ and } s_i<3k} a_{s_1}a_{s_2}a_{s_3} = 0$ by assumption and the result that $a_0a_i = 0$ for all $i \geq 1$, entailing $a_k^3 = 0$. Thus $a_k = 0$. Then $a_i = 0$ for all $i \geq 1$. Consequently we now have $f(x) = a_0 \in Id(R[[x]])$.

Now we consider the case of $m \geq 4$. Note that the coefficient of degree vm of $f(x)^m$ is

$$
\begin{aligned}
&{}_mC_0a_v^m + {}_mC_{m-1}a_0^{m-1}a_{vm} + \sum_{i_1+i_2=vm \text{ and } i_t<vm} {}_mC_{m-2}a_0^{m-2}a_{i_1}a_{i_2} \\
&+ \sum_{j_1+j_2+j_3=vm \text{ and } j_p<vm} {}_mC_{m-3}a_0^{m-3}a_{j_1}a_{j_2}a_{j_3} \\
&+ \cdots + \sum_{s_1+s_2+\cdots+s_{m-2}=vm \text{ and } s_q<vm} {}_mC_2a_0^2a_{s_1}a_{s_2}\cdots a_{s_{m-2}} \\
&+ \sum_{t_1+t_2+\cdots+t_{m-1}=vm \text{ and } t_w<vm} {}_mC_1a_0a_{t_1}a_{t_2}\cdots a_{t_{m-1}} \\
&= a_v^m + {}_mC_1a_0a_{vm} + \sum_{i_1+i_2=vm \text{ and } i_t<vm} {}_mC_2a_0a_{i_1}a_{i_2} \\
&+ \sum_{j_1+j_2+j_3=vm \text{ and } j_p<vm} {}_mC_3a_0a_{j_1}a_{j_2}a_{j_3} \\
&5 + \cdots + \sum_{s_1+s_2+\cdots+s_{k-2}=vm \text{ and } s_q<vm} {}_mC_2a_0a_{s_1}a_{s_2}\cdots a_{s_{m-2}} \\
&+ \sum_{t_1+t_2+\cdots+t_{m-1}=vm \text{ and } t_w<vm} {}_mC_1a_0a_{t_1}a_{t_2}\cdots a_{t_{m-1}}, \quad (*)
\end{aligned}
$$

where we use $a_0 \in Id(R) \cap Z(R)$. Note that $\{i_1, i_2\} \cap [0, v-1] \neq \emptyset$, $\{j_1, j_2, j_3\} \cap [0, v-1] \neq \emptyset$ and $\{s_1, s_2, \ldots, s_{m-2}\} \cap [0, v-1] \neq \emptyset$.

(iii) Let m be an even integer such that ${}_mC_k$ is even for all $1 \leq k \leq m-1$, for example, $m = 4$. Then, for every $v \geq 1$, the coefficient of the term of degree vm of $f(x)^m$ is $a_v^m = 0$ by the preceding $(*)$, so that $a_v = 0$. Thus $f(x) = a_0 \in Id(R[[x]])$.

(iv) We do not know the computation of the general case that $m \geq 5$ and ${}_mC_k$ is odd for some $1 \leq k \leq m-1$, for example, $m = 6$.

Let R be a ring with an endomorphism σ. Recall that the skew polynomial ring $R[x;\sigma]$ is a ring of polynomial in x with coefficients in R and subject to the relation $xr = \sigma(r)x$ for $r \in R$. The skew Laurent polynomial ring $R[x, x^{-1};\sigma]$ is a localization of $R[x;\sigma]$ with respect to the set of powers of x.

For a ring R with a monomorphism σ, let $A(R,\sigma)$ be the subset $\{x^{-i}rx^i \mid r \in R$ and $i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1};\sigma]$. Note that for $j \geq 0$, $x^jr = \sigma^j(r)x^j$ implies $rx^{-j} = x^{-j}\sigma^j(r)$ for $r \in R$. This yields that for each $j \geq 0$ we have $x^{-i}rx^i = x^{-(i+j)}\sigma^j(r)x^{i+j}$. It follows that $A(R,\sigma)$ forms a subring of $R[x, x^{-1};\sigma]$ with the following natural operations: $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\sigma^j(r) + \sigma^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\sigma^j(r)\sigma^i(s)x^{i+j}$ for $r, s \in R$ and $i, j \geq 0$. Note that $A(R,\sigma)$ is an over-ring of R, and the map $\bar{\sigma} : A(R,\sigma) \to A(R,\sigma)$ defined by $\bar{\sigma}(x^{-i}rx^i) = x^{-i}\sigma(r)x^i$ is an automorphism of $A(R,\sigma)$. Jordan showed, with the use of left localization of the skew polynomial $R[x;\sigma]$ with respect to the set of powers of x, that for any pair (R, σ), such an extension $A(R, \sigma)$ always exists in [9]. This ring $A(R, \sigma)$ is usually said to be the *Jordan extension* of R by σ.

THEOREM 2.4. *Let R be an Abelian ring with a monomorphism σ. Then R is reduced-over-idempotent if and only if the Jordan extension $A = A(R, \sigma)$ of R by σ is reduced-over-idempotent.*

Proof. It is enough to show the necessity by Proposition 1.5(1). Suppose that R is reduced-over-idempotent and let $a^n \in Id(A)$ for some $n \geq 1$, where $a = x^{-i}rx^i \in A$ for $i, j \geq 0$. Then $a^n = x^{-ni}\sigma^{(n-1)i}(r^n)x^{ni} \in Id(A)$ implies $\sigma^{(n-1)i}(r^n) \in Id(R)$, because $Id(A) = \{x^{-i}rx^i \mid r \in Id(R) \text{ and } i \geq 0\}$ clearly. Note that $\sigma(Id(R)) = Id(R)$ since σ is a monomorphism. So $\sigma^{(n-1)i}(r^n) \in Id(R)$ yields $r^n \in Id(R)$, and thus $r \in Id(R)$ since R is reduced-over-idempotent. Therefore the Jordan extension A of R by σ is reduced-over-idempotent. □

A multiplicatively closed subset S of a ring R is said to satisfy the *right Ore condition* if for each $a \in R$ and $b \in S$, there exist $a_1 \in R$ and $b_1 \in S$ such that $ab_1 = ba_1$. It is shown, by [13, Theorem 2.1.12], that S satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring R_S of R with respect to S exists.

Recall that a ring R is called *right* (resp., *left*) p.p. if each principal right (resp., left) ideal of R is projective. It is well known that a ring R is right p.p. if and only if the right annihilator of each element of R is generated by an idempotent. A ring is called *p.p.* if it is both right and left p.p..

Following Goodearl [4], a ring R (possibly without identity) is called (*von Neumann*) *regular* if for every $a \in R$ there exists $b \in R$ such that $a = aba$. It is easily shown that $J(R) = 0$ if R is regular, and a ring R (possibly without identity) is called *strongly regular* if $a \in a^2R$ for every $a \in R$. A ring is strongly regular if and only if it is Abelian regular if and only if it is reduced regular, by [4, Theorems 3.2 and 3.5].

PROPOSITION 2.5. *Let S be a multiplicatively closed subset of an Abelian ring R.*

(1) *Suppose that S satisfies the right Ore condition. If the right quotient ring R_S of R with respect to S is reduced-over-idempotent, then so is R. Conversely, if R is locally finite reduced-over-idempotent, then R_S is strongly regular.*

(2) *Suppose that S consists of central regular elements and $Id(S^{-1}R) = \{u^{-1}e \mid e \in Id(R) \text{ and } u \in S\}$. Then R is reduced-over-idempotent if and only if $S^{-1}R$ is reduced-over-idempotent.*

Proof. (1) It is clear that R is reduced-over-idempotent when R_S is reduced-over-idempotent by Proposition 1.5(1), since R is a subring of R_S.

Conversely, suppose that R is locally finite reduced-over-idempotent. Then R is reduced regular by Lemma 1.2(1, 4) and so R is p.p. by [4, Theorem 1.1]. Moreover R_S is reduced by [10, Theorem 16]. We claim that R_S is also p.p.. Let $ab^{-1} \in R_S$. Since R is right p.p., $r_R(a) = eR$ for some $e \in Id(R)$. So $ab^{-1}e = aeb^{-1} = 0$ and $eR_S \subseteq r_{R_S}(ab^{-1})$ follows. For the converse, let $cd^{-1} \in r_{R_S}(ab^{-1})$. Then $ab^{-1}cd^{-1} = 0 \Rightarrow ab^{-1}c = 0 \Rightarrow cab^{-1} = 0$, since R_S is reduced. So $ca = 0 \Rightarrow ac = 0$ because R is reduced. Thus $c \in eR \Rightarrow c = ec$, and hence $cd^{-1} = ecd^{-1} \in eR_S$ and $r_{R_S}(ab^{-1}) \subseteq eR_S$. Consequently, we get $r_{R_S}(ab^{-1}) = eR_S$, and thus R_S is right p.p.. Moreover R_S is left p.p. by [6, Lemma 1(i)], since it is reduced. Therefore R_S is a reduced p.p. ring and so it is strongly regular by [5, Lemma 3.3].

(2) It is sufficient to show the necessity by Proposition 1.5(1). Assume that R is reduced-over-idempotent, and let $\alpha = u^{-1}a \in S^{-1}R$ be such that $\alpha^n \in Id(S^{-1}R)$ for some $n \geq 2$. Then $(u^n)^{-1}a^n \in Id(S^{-1}R)$, and so $a^n \in Id(R)$ by hypothesis. But R is reduced-over-idempotent, and hence $a \in Id(R)$. This implies $\alpha = u^{-1}a \in Id(S^{-1}R)$, concluding that $S^{-1}R$ is reduced-over-idempotent. □

Notice that there exist rings in which the hypothesis "$Id(S^{-1}R) = \{u^{-1}e \mid e \in Id(R)$ and $u \in S\}$" in Proposition 2.5(2) does not hold, by [11, page 1967], in general.

Let A be an algebra over a commutative ring S. Due to Dorroh [3], the *Dorroh extension* of A by S is the Abelian group $A \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. We use $A \times_{dor} S$ to denote the Dorroh extension of A by S.

Proposition 2.6. *Let R be a unitary algebra over a commutative ring S. Suppose that R is Boolean and S is reduced-over-idempotent. Then $D = R \times_{dor} S$ is reduced-over-idempotent.*

Proof. $Ch(R) = 2$ by Lemma 1.2(2), and note that $Id(D) = Id(R) \times Id(S)$. For, $(r, s) \in Id(D)$ if and only if $(r, s)^2 = (r, s)$ if and only if $(r^2, s^2) = (r, s)$ if and only if $(r, s) \in Id(R) \times Id(S)$. We freely use these facts throughout this proof.

Let $(r, s) \in D$ be such that $(r, s)^n \in Id(D)$ for some $n \geq 2$. Then $s^n \in Id(S)$. Since S is reduced-over-idempotent, $s \in Id(S)$. If $n = 2$ then the result is obvious, so suppose $n \geq 3$. Since R is Boolean, we have

$$(r, s)^n = (r^n + 2(2^{n-1} - 1)sr, s^n) = (r^n, s^n) = (r, s).$$

But $(r, s)^n \in Id(D)$ and $(r, s) \in Id(D)$ follows. Therefore D is reduced-over-idempotent. □

As an application of Proposition 2.6, let R be a direct product of $\mathbb{Z}_2$'s and consider $R \times_{dor} \mathbb{Z}_2$. Then this Dorroh extension is reduced-over-idempotent by Proposition 2.6.

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